

# PREHISTORY OF ALGEBRAIC GEOMETRY

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The title is entirely due to David Eisenbud. This talk is not really about algebraic geometry. Rather it is about the history of imaginary (complex) numbers.

- (1) D. Flament, *Histoire des nombres complexes*, CNRS Editions (2003).
- (2) C. Wessel, On the analytical representation of direction (1797, in Danish, remained unknown for about 100 years), English translation (1999).
- (3) R. Argand, *Essai sur une mani'ere de repr'esenter les quantit'es imaginaires* (1806, in a private version), reprinted (1971).
- (4) B. Mazur, *Imagining numbers* (particularly  $\sqrt{-15}$ ) (2003).

G. Cardano published *Ars Magna* in 1545. This is an early book on algebra. He was particularly interested in the solution to the cubic equation, the solution to the quadratic equation being well known.

The solution method in a special case was discovered by S. Ferro, who taught it to Tartaglia. Cardano learned this from Tartaglia, and then figured out the general case.

The notion of a number has changed over time. To the Greeks, a number was one of 2, 3, 4, ...

Quantity/magnitude is used to measure the length of an interval, time, etc. One can take ratios of these. Numbers (in the Greek sense) were some examples of ratios.

Cardano would write a cubic equation as

$$x^3 = 6x + 40.$$

(He did not have 0 or negative numbers, so there were many cases.) Also he did not use a letter for variables. Rather he would do it by example.

Steps:

- (1)  $\frac{1}{3}(6) = 2$ , cube it to get 8.
- (2)  $\frac{1}{2}40 = 20$ , square it to get 400.
- (3) subtract to get  $400 - 8 = 392$ .
- (4) Then  $x = \sqrt[3]{20 + \sqrt{392}} + \sqrt[3]{20 - \sqrt{392}}$ .

In modern notation, the solution to  $x^3 = ax + b$  is

$$x = \sqrt[3]{\frac{b}{2} + \sqrt{\left(\frac{b}{2}\right)^2 - \left(\frac{a}{3}\right)^3}} + \sqrt[3]{\frac{b}{2} - \sqrt{\left(\frac{b}{2}\right)^2 - \left(\frac{a}{3}\right)^3}}.$$

*Proof.* If we can find  $u, v$  such that  $uv = a/3$  and  $u^3 + v^3 = b$ , then  $x = u + v$  is a solution, since

$$x^3 = u^3 + v^3 + 3uv(u + v) = b + ax.$$

The numbers  $u, v$  can be found by finding  $u^3, v^3$  as roots of a quadratic equation.  $\square$

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What does Cardano say about the case where  $\left(\frac{a}{3}\right)^3 > \left(\frac{b}{2}\right)$ , where the formula requires taking a square root of a negative number? He says that the formula is not correct, and says the problem is “false”. At the end of his book, in Chapter 37, to find  $a, b$  such that  $a + b = 10$  and  $ab = 40$ , he gets  $5 + \sqrt{-15}$  and  $5 - \sqrt{-15}$ . He says you will have to “imagine”  $\sqrt{-15}$ , and then verifies that it formally satisfies the equation. This is “sophisticated” (meaning false). Cardano considers this useless.

Bombelli (1572) published a book called *Algebra*. Actually he published only three chapters, saying the others were not proper yet, though they were published later, in 1920. Bombelli was studying a cubic equation in which  $\left(\frac{a}{3}\right)^3 > \left(\frac{b}{2}\right)$ , namely  $x^3 = 15x + 4$ . Get  $x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$ . He wrote

$$\text{R.c.L.2p.dim.R.q.121}]$$

to mean  $2 + \sqrt{-121}$ . He guesses

$$\sqrt[3]{2 + \sqrt{-121}} = x + \sqrt{-y},$$

cubes both sides to get equations

$$x^3 - 3xy = 2, \quad (3x^2 - y)\sqrt{-y} = \sqrt{-121}.$$

He then guesses  $y = 1$ , and tries  $x$ 's so as to make  $x^3 - 3xy = 2$ , by taking  $x = 2$ . Thus

$$\sqrt[3]{2 + \sqrt{-121}} = 2 + \sqrt{-1}.$$

An early use of the word “imaginary” occurs in Descartes (1637), in a large manual on how to think correctly, in a section on geometry. E.g., maybe he has a polynomial of degree 5 with 3 real roots, and then imagines that there are two more?

Euler, in 1748 and *Algebra* (1770). Already in 1748 he wrote

$$e^{ix} = \cos x + i \sin x.$$

Also, Euler stated de Moivre’s formula

$$(\cos x + i \sin x)^n = \cos nx + i \sin nx$$

and  $e^{i\pi} + 1 = 0$ . Euler realized that  $\log(\text{negative number})$  should be interpreted not as a single value, but as a set of values.

Now let us discuss the period in which complex numbers were put on a rigorous foundation: Wessel, Argand, Gauss, Cauchy, Hamilton.

Wessel (1797) was a surveyor, who wanted to have an arithmetic of length+direction, just as Descartes had an arithmetic of lengths. He argues that the proportion  $1 : a$  should equal the proportion  $b : ab$ . This defines  $ab$ . He observes that if  $\epsilon$  is a length of 1 at a right angle to 1, then  $\epsilon^2 = -1$ .

Argand (1806) tries to convince the reader, by analogy and generalization, that addition and multiplication ought to be the way he describes. He develops this by principles that he assumes (such as vector addition, and multiplication of lengths with addition of angles). But he does not claim to *define* the operations.

Gauss was looking at the Fundamental Theorem of Algebra

$$x^n + a_1x^{n-1} + \cdots + a_n = 0$$

with  $a_i \in \mathbb{R}$ . He published four proofs; in the first proof he shows that any polynomial can be factored into linear and quadratic problems (about 1796, in his thesis). In 1832 while studying biquadratic residues, he discusses “integral complex numbers”  $a + b\sqrt{-1}$ , with  $a, b \in \mathbb{Z}$ .

Cauchy essentially had  $\mathbb{C} := \mathbb{R}[x]/(x^2 + 1)$ .

Hamilton defined  $\mathbb{C}$  as  $\mathbb{R}^2$  with operations.

Von Staudt published *Geometrie der Lage* in 1847. He presents projective geometry purely synthetically, without coordinates. In supplements (Beiträge) during 1856–1860, he starts with the geometry and constructs the field of coefficients (though he thought he was just working with the real numbers). The other thing he did was to construct the  $\mathbb{C}\mathbb{P}^3$  out of  $\mathbb{R}\mathbb{P}^3$ . He is giving an existence proof.

Poncelet pretended that curves that did not meet at real points did meet, and deduced things using this.

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