AMPLE AND EFFECTIVE CONES ON THE SPACES OF STABLE MAPS

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The Kontsevich moduli space of stable maps $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ compactifies the space parameterizing maps f from $\mathbb{P}^1$ with n marked points to $\mathbb{P}^r$ such that the pullback of $\mathcal{O}(1)$ is $\mathcal{O}(d)$. It parameterizes the following objects: a tree of $\mathbb{P}^1$'s with n marked points with a map f such that if f is constant on any component C_0, then C_0 has at least 3 special points (marked points or nodes).

This is a smooth Deligne-Mumford stack. It has an associated coarse moduli space.

Question: What are ample divisors on this space? What are effective divisors on this space? (For these questions, it does not matter whether we consider the stack or its coarse moduli space.)

Let $\mathcal{H}$ be the class of the locus of maps whose image intersects a fixed $\mathbb{P}^{r-2}$ in $\mathbb{P}^r$.

Let $\{A, B\}$ be a partition of $\{1, 2, \ldots, n\}$. Let $\Delta_{(A,i),(B,d-i)}$ be the boundary divisor parameterizing two $\mathbb{P}^1$'s glued at one node with points in A on the first and points in B on the second, and such that the $\mathcal{O}(1)$ on $\mathbb{P}^r$ pulls back to $\mathcal{O}(i)$ on the first $\mathbb{P}^1$ and $\mathcal{O}(d-i)$ on the second.

Let $\mathcal{L}_i = \operatorname{ev}_i^* \mathcal{O}^{\mathbb{P}^r}(1)$ (i.e., the i-th points maps to a fixed hyperplane).

Theorem 0.1 (Pandharipande)**.** *These divisor classes generate* $\operatorname{Pic}\left(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)\right) \otimes \mathbb{Q}$.

If $n = 0$, then $\Delta_{i,d-i}$ with $i = 1, \ldots, \lfloor d/2 \rfloor$ and $\mathcal{H}$ freely generate $\operatorname{Pic}\left(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)\right) \otimes \mathbb{Q}$.

Define $\operatorname{Amp}(n, r, d)$ and $\operatorname{Eff}(r, d)$ (for $n = 0$).

Proposition 0.2.

- $\operatorname{Amp}(n, r, d) = \operatorname{Amp}(n, 2, d)$ *for all* $r \geq 2$.
- $\operatorname{Eff}(r, d) \subseteq \operatorname{Eff}(r + 1, d)$.
- $\operatorname{Eff}(r, d) = \operatorname{Eff}(r + 1, d)$ *if* $r \geq d$.

A divisor D is *nef* if $D.C \geq 0$ for every curve C in the space. Kleiman's criterion: The nef cone Nef is the closure of the ample cone, and the ample cone is the interior of the nef cone.

To prove $\operatorname{Amp}(n, r, d) = \operatorname{Amp}(n, 2, d)$, embed $\mathbb{P}^2$ in $\mathbb{P}^r$ to get one inequality. For the other inequality, use a suitable projection $\mathbb{P}^r \dashrightarrow \mathbb{P}^2$.

What are some nef divisors? The divisor $\mathcal{H}$ is nef (can use an automorphism of $\mathbb{P}^r$ to avoid the image of any particular map). Similarly, the divisor $\mathcal{L}_i$ is nef.

Let T be the tangency divisor: the class of the locus of maps where $f^{-1}(H)$ is not d distinct points, where H is a hyperplane. This also is nef. (One can express T explicitly in terms of the generators.)

There is a rational map $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d) \dashrightarrow \overline{\mathcal{M}}_{0,n+d}/\mathfrak{S}_d$, where $\mathfrak{S}_d$ is the permutation group acting on the last d points. To define this, fix a hyperplane $H \subseteq \mathbb{P}^r$; add to the n marked points the intersection of the image curve with H. We get

$$\nu\colon \operatorname{Pic}\left(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)\right)_{\mathbb{Q}}^{\mathfrak{S}_d} \to \operatorname{Pic}\left(\overline{\mathcal{M}}_{0,n+d}\right)_{\mathbb{Q}}.$$

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Theorem 0.3. *A divisor D on $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,d)$ is nef if and only if $D = a\nu(D_1) + b\mathcal{H} + cT + \sum d_i\mathcal{L}_i$ where D_1 is nef on $\overline{\mathcal{M}}_{0,n+d}/\mathfrak{S}_d$ and $a,b,c,d_i \in \mathbb{Q}_{\geq 0}$.*

$\overline{\mathcal{M}}_{0,n}$ has a topological stratification.

F-conjecture: D is ample if and only if D pairs positively with all 1-dimensional strata.

Theorem 0.4. *There exists a morphism*

$$\mathrm{cont}_{r,d}\colon \overline{\mathcal{M}}_{0,0}(\mathbb{P}^r,d) \to Y$$

such that

(1) *There exists an ample divisor on Y that pulls back to*

$$\mathcal{H} + \sum_{k=1}^{\lfloor d/2 \rfloor} k^2 \Delta_{k,d-k}.$$

(2) *It is an isomorphism on $\mathcal{M}_{0,0}(\mathbb{P}^r,d)$.*

(3) *On each boundary component $\overline{\mathcal{M}}_{0,1}(\mathbb{P}^r,i) \times_{\mathbb{P}^r} \overline{\mathcal{M}}_{0,1}(\mathbb{P}^r,d-i)$ it factors through the second projection.*

Let us now describe the effective cone. We will restrict to the case $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^d,d)$. Some effective divisors:

- $\Delta_{i,d-i}$
- D_{deg} (deg stands for "degenerate") is the class of the locus of maps where the image of f does not span $\mathbb{P}^d$. (This can be expressed explicitly in terms of the generators $\mathcal{H}$ and $\Delta_{k,d-k}$.)

Theorem 0.5. *The effective cone is generated by D_{deg} and the $\Delta_{i,d-i}$ for $i = 1,\dots,\lfloor d/2 \rfloor$.*

Let C be a moving curve and D is effective, then $D.C \geq 0$. Thus each moving curve gives a condition on the effective cone. To prove the theorem, we need to exhibit enough moving curves to cut out the cone generated by D_{deg} and the $\Delta_{i,d-i}$.

Let $d = 6$. Then

$$D_{\mathrm{deg}} \sim H - \frac{5}{7}\Delta_{1,5} - \frac{8}{7}\Delta_{2,4} - \frac{9}{7}\Delta_{3,3}.$$

Take the del Pezzo surface obtained by blowing up $\mathbb{P}^2$ in 7 points. Then $-2K$ embeds the surface as a degree 8 surface. This explains the 8/7.

Take the del Pezzo surface obtained by blowing up $\mathbb{P}^2$ in 8 points. Now $-3K$ embeds the surface as a degree 9 surface in $\mathbb{P}^6$.

Take $j(d+1)$ general points on $\mathbb{P}^1 \times \mathbb{P}^1$. Let

$$L(j) = dF_1 + j\frac{k(k+1)}{2}F_2 - \sum_{i=1}^{j(d+1)} kE_i,$$

where F_i are the pullbacks of the generators of $\mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1)$ and the E_i are the exceptional curves.

Assuming that $L(j) - F_2$ is non-special, we are done.

Instead if one defines

$$\tilde{L}(j) = dF_1 + j\frac{k(k+1)}{2}F_2 - \sum_{i=1}^{j(d+1)-n(k,d)} kE_i - \text{impose simple point condition},$$

where $n(k,d)$ is small, one can prove that $\tilde{L}(j) - F_2$ is non-special, and one gets the result by letting $j \to \infty$.

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