# COMPACT MODULI SPACES FOR SURFACES OF GENERAL TYPE

BRENDAN HASSETT

## 1. Deligne-Mumford compactification (review)

Fix $g \geq 2$. Let $\mathcal{M}_g$ be the moduli stack of smooth curves of genus $g$. Let $\overline{\mathcal{M}}_g$ be the moduli stack of stable curves of genus $g$.

The families of curves over $B$ are the elements of $\mathcal{M}_g(B) = \mathrm{Hom}(B, \mathcal{M}_g)$; these are smooth proper $\pi\colon C \to B$ with geometrically connected fibers of genus $g$.

For $\overline{\mathcal{M}}_g$, it is the same, except "flat" replaces "smooth", and "geometrically connected" is replaced by "stable". Call $C$ *stable* if

(1) $C$ has only nodes/normal crossings as singularities $\{xy = 0\}$, and
(2) the dualizing sheaf $\omega_C$ is ample.

Good properties:

- $\overline{\mathcal{M}}_g$ is functorially defined
- $\overline{\mathcal{M}}_g$ is *proper*: one parameter limits exist (valuative criterion applies), and $\overline{\mathcal{M}}_g$ is of finite type (bounded)
- $\overline{\mathcal{M}}_g - \mathcal{M}_g = \delta_0 \cup \cdots \cup \delta_{\lfloor g/2 \rfloor}$ is a union of normal crossing divisors. Here $\delta_i$ is the closure of the space of curves obtained by glueing a curve of genus $i$ to a curve of genus $g-i$ at one point (except $i=0$ is slightly different). So $\overline{\mathcal{M}}_{g-i,1} \times \overline{\mathcal{M}}_{i,1}$ for $0 < i < g/2$. Note that one encounters moduli of pointed stable curves, even if one is interested only in (un-pointed) moduli spaces to begin with.
- Can do ($\mathbb{Q}$-valued) intersection theory on $\overline{\mathcal{M}}_g$ (it is a smooth stack).

## 2. Review of classification theory

Let $k$ be an algebraically closed field of characteristic 0. Let $X$ be a smooth projective variety over $k$.

**Definition 2.1.** $X$ is of *general type* if

$$\dim \Gamma(X, \omega_X^{\otimes n}) \sim n^{\dim X}$$

for $n \gg 0$. Let $R(X) := \bigoplus_{n \geq 0} \Gamma(X, \omega_X^{\otimes n})$. The $k$-algebra $R(X)$ is a birational invariant.

Fundamental conjecture of birational geometry: $R(X)$ is finitely generated.

This is known for $\dim X \leq 3$, by work of Mori (and classical constructions for $\dim X \leq 2$), and for $\dim X = 4$ of general type (Shokurov, McKernan, Hacon: not yet published).

Assuming the fundamental conjecture, one can define $X^{\mathrm{can}} := \mathrm{Proj}\, R(X)$. The variety $X^{\mathrm{can}}$ is birational to $X$. Let $K_{X^{\mathrm{can}}} := [\omega_{X^{\mathrm{can}}}]$, which is an ample $\mathbb{Q}$-Cartier divisor. By definition, $X^{\mathrm{can}}$ has *canonical singularities* (in dimension 2, these are rational double points).

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The canonical stack is defined as the stack quotient

$$\mathcal{X}^{\text{can}} := \left[\frac{\operatorname{Spec} R(X) - \{0\}}{\mathbb{G}_m}\right]$$

where the $\mathbb{G}_m$-action is given by the grading on $R(X)$. We have the invertible sheaf $\omega_{\mathcal{X}^{\text{can}}}$.

## 3. One parameter limits

Context: $X \to \Delta = \operatorname{Spec} DVR$, flat proper. Think of $\Delta$ as $\{t \in \mathbb{C} : |t| < 1\}$. Assume that when restricted to $\Delta^* := \Delta - \{0\}$, it is smooth with fibers of general type. Suppose $X$ is smooth and $X_0 = \pi^{-1}(0)$ has reduced normal crossings.

(Remark: in characteristic $p$, resolution of singularities is open for $\dim(X/\Delta) \geq 4$, and semistable reduction is open even for $\dim(X/\Delta) = 2$. But we will work in characteristic 0, so we have both.)

The fundamental conjecture gives $Y = (X/\Delta)^{\text{can}} \to \Delta$: so $Y$ is defined as $\mathbf{Proj}\,(\bigoplus \pi_* \omega_\pi^{\otimes n})$. Then the fiber $Y_t$ equals $X_t^{\text{can}}$ for $t \neq 0$, and $Y_0$ has (by definition) *semilogcanonical* (SLC) singularities. The invertible sheaf $K_{Y_0}$ is ample $\mathbb{Q}$-Cartier, and $Y_0$ has normal crossings/nodes in codimension 1, and satisfies $S_2$. One-parameter limit procedure looking for moduli functor. The stable limit $Y_0$ is an invariant of $X^* \to \Delta^*$. We want $Y_0 = \lim_{t \to 0} X_t$ "in some compactification". These are called stable varieties.

**Definition 3.1.** A variety $S$ is *stable* if

(1) $S$ has SLC singularities, and
(2) $K_S$ is ample.

Let

$$\mathcal{Y} = \left[\frac{\mathbf{Spec}\,(\bigoplus \pi_* \omega_X^{\otimes n}) - \{0\}}{\mathbb{G}_m}\right].$$

This is a stack with coarse moduli space $Y$.

Remark: not all stable surfaces arise as limits of surfaces with rational double points.

### 3.1. Invariants of stable surfaces.

These are invariants that are preserved under a one-parameter degeneration from a canonical model:

- Euler characteristic $\chi(\mathcal{O}_S)$.
- $(K_S, K_S) = K_S^2$.

## 4. How do we define families of stable surfaces

Let $B$ be a base scheme. What is a family of stable surfaces $\mathcal{S} \to B$?

Issues:

(1) We can construct flat proper $S \to \Delta$ with stable fibers such that $K_{S_t}^2$ is not constant.
(2) Not every stable surface deforms to a nice surface: there exist components of the moduli space consisting of non-smooth (even non-normal) surfaces.
(3) We don't know how to take one-parameter limits in the weird components parameterizing non-smooth surfaces.

**Definition 4.1** (Definition A, by Viehweg / Hassett-Kovács using work of Shepherd-Barron, Kollár, Alexeev)**.** Fix invariants $K^2$ and $\chi(\mathcal{O}_S)$; then for $N \gg 0$ (depending on these invariants), call $S \xrightarrow{\pi} B$ (with those invariants) *stable* if it is a flat proper morphism with stable fibers such that there exists a line bundle $\mathcal{L}$ on $S$ such that $\mathcal{L}|_{S^{\mathrm{Gor}}} = \omega_\pi^{\otimes N}$, where $S^{\mathrm{Gor}}$ is the (open) Gorenstein locus. (This implies that $(K_{S_t})^2$ is constant.)

**Definition 4.2** (Definition B, by Kollár, Hacking, Abramovich-Hassett)**.** For each $N > 0$, there exists a sheaf $\mathcal{F}_N$ on $S$ such that

$$\mathcal{F}_N|_{S^{\mathrm{Gor}}} = \omega_\pi^{\otimes N}$$
$$\mathcal{F}_N|_{S_b} = \left(\omega_{S_b}^{\otimes N}\right)^{**}$$

where the $**$ denotes reflexive hull. Equivalently, $S \to B$ comes from a family $\mathcal{S} \to B$ of stacky stable surfaces.

Anything satisfying Definition B will satisfy Definition A for any $N$.

**Theorem 4.3.** *Let $\mathcal{S} \to B$ be a flat proper morphism with fibers that are stacky stable surfaces. Then the coarse moduli space $S \to B$ satisfies Definition B. Conversely, given a Definition B family,*

$$\mathcal{S} := \left[\frac{\mathbf{Spec}\,(\bigoplus \pi_* \omega_\pi^{\otimes n}) - \{0\}}{\mathbb{G}_m}\right]$$

*is a family of stacky stable surfaces.*

Motivation for all this: define a virtual fundamental class for the moduli of surfaces that is well adapted to functoriality.

Rice University