REPRESENTING TROPICAL LINEAR SPACES BY CIRCUITS

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ABSTRACT. Tropical linear spaces are the tropical analogues of ordinary linear spaces, and they are finite intersections of tropical hyperplanes. However, other similarities to ordinary linear algebra are not as clear. For example, what is the relationship between the tropical rank of a matrix and the tropical space its rows cut out? I will discuss this question briefly and then move on to tropical bases. A tropical basis is more like a Groebner basis than a vector space basis, in that it is a generating set of sorts, and need not be minimal.

The talk will focus on finding minimal tropical bases of realizable tropical linear spaces. We will consider these tropical linear spaces by the circuits of their associated matroids, and will find their minimal tropical bases using combinatorial methods. This is joint work with Josephine Yu.

Let $\mathbb{T} = (\mathbb{R} \cup \{\infty\}, \oplus, \odot)$ where $\oplus$ is min, and $\odot$ is $+$.

Let K be the field of Puiseux series with complex coefficients. Tropicalize

$$a_1x_1 + \cdots + a_nx_n$$

with $a_i \in K$ by taking the lowest exponent in each coefficient:

$$f = c_1 \odot x_1 \oplus \cdots \oplus c_n \odot x_n.$$

The tropical hyperplane is

$$\mathcal{T}(f) = \{x \in \mathbb{T}^n : \text{ min in } f \text{ occurs at least twice}\}.$$

Given an ideal I generated by linear forms over K the *associated tropical linear space* is the intersection of all the tropical hyperplanes of the tropicalizations of all linear forms in I. A *tropical basis* is a finite set of linear forms whose tropicalizations cut out the tropical linear space. There always exists a tropical basis. A *circuit* is the tropicalization of a minimal-support linear form.

Fact (Ardila/Klivans): The set of all circuits form a (non-minimal) tropical basis for any linear space. (And there are only finitely many circuits up to scaling, so they give rise to only finitely many tropical hyperplanes.)

Constant-coefficient case ($a_i \in \mathbb{C}$): The degrees (i.e., coefficients of tropical linear forms) are 0 or ∞.

Matroid: generalizes combinatorial information about vectors in a vector space. For instance

$$3x_1 - \frac{1}{4}x_3 + x_5 = 0$$

gives rise to a set $\{1, 3, 5\}$ and then to

$$0 \odot x_1 \oplus \infty \odot x_2 \oplus 0 \odot x_3 \oplus \infty \odot x_4 \oplus 0 \odot x_5.$$

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Construct a matroid whose dependent sets are supports of the ordinary linear forms in the ideal. A *circuit* in a matroid is a minimal dependent set. (This notion corresponds to our previous one.)

Given a matroid $\mathcal{M}$, let $\mathcal{C}$ be the set of circuits. For each $c \in \mathcal{C}$, let

$$\mathcal{T}(c) := \{x \in \mathbb{R}^n : \min\{x_i : i \in c\} \text{ is attained at least twice}\}.$$

Given $S \subset \mathcal{C}$, let

$$\mathcal{T}(S) := \bigcap_{c \in S} \mathcal{T}(c).$$

The set $\mathcal{T}(\mathcal{C})$ is called the *tropical variety* or *Bergman fan* of $\mathcal{M}$.

Let $U_{k,n}$ be the matroid corresponding to n generic vectors in k-space.

Example: Consider $U_{2,4}$. Label the elements of $U_{2,4}$ as 1 through 4. Its circuits are 123, 124, 134, 234. The point $(0, 1, 0, 0)$ belongs to the tropical variety. But $(0, 1, 0, 1)$ does not belong.

Goal: find a minimal tropical basis for any matroid.

Lemma 1 (Yu,Yuster)**.** *When checking if B is a tropical basis, it is sufficient to check that $\mathcal{T}(B) \cap \{0, 1\}^n = \mathcal{T}(\mathcal{C}) \cap \{0, 1\}^n$.*

Proof. We will show that if there is a point in $\mathcal{T}(B)$ outside $\mathcal{T}(\mathcal{C})$, then there is a point in $\mathcal{T}(B) \cap \{0, 1\}^n$ outside $\mathcal{T}(\mathcal{C})$. Suppose $x \in \mathcal{T}(B) - \mathcal{T}(\mathcal{C})$. Then there exists a circuit $c \in \mathcal{C} - B$ that excludes x. So the set $\{x_i : i \in c\}$ has unique minimum m. Construct v such that

$$v_i = \begin{cases} 1, & \text{if } x_i > m \\ 0, & \text{if } x_i \leq m. \end{cases}$$

Then v is excluded by C. So $v \notin \mathcal{T}(\mathcal{C})$. But $v \in \mathcal{T}(B)$. □

A *graphic matroid* is one for which the elements are edges in a graph, and the circuits are cycles in the graph (closed paths in which all vertices have degree 2 in the cycle).

(An *induced subgraph* of a graph is obtained by choosing a subset of the vertices and taking all edges in the original graph going between vertices in the subset.)

Theorem 2 (Yu,Yuster)**.** *The unique minimal tropical basis of a graphic matroid consists of the induced cycles, that is, the induced subgraphs which are cycles.*

A graph is 3-connected if removing any 2 edges yields a connected graph. A *graph cut* in a graph is a set of edges whose removal yields a disconnected graph. A *cographic matroid* is one for which the elements are edges in a 3-edge connected graph and the circuits are minimal graph cuts.

Theorem 3 (Yu,Yuster)**.** *The unique minimal tropical basis of a cographic matroid consists of minimal graph cuts that split the graph into two 2-edge-connected components.*

Back to $U_{2,4}$. Any three of 123, 124, 134, 234 form a tropical basis. Thus minimal tropical bases are not always unique. It turns out that minimal tropical bases need not even have the same cardinality: this happens for $U_{2,5}$, for example.

Question 4. Does there exist a unique minimal tropical basis for any regular matroid?

Question 5. Which matroids have a unique minimal tropical basis?