MODULAR, LOG CANONICAL, AND TROPICAL COMPACTIFICATIONS

JENIA TEVELEV (WITH HACKING, KEEL)

Abstract. The moduli space of stable curves of Grothendieck and Knudsen has a straightforward generalization in all dimensions: the moduli space of stable pairs of Kollar, Shepherd-Barron, and Alexeev. Its construction relies on Mori theory and its boundary is very difficult to describe. Using the classical example of a cubic surface with 27 lines, I'll describe joint work with Hacking and Keel where we construct the moduli space of stable Del Pezzo surfaces and describe its boundary using non-archimedean amoebas. This is one instance of the beautiful connection between Mori theory and tropical geometry.

Goal: boundary strata and geometry of Kollár–Shepherd-Barron–Alexeev moduli space of stable pairs. Example: compact functorial moduli of cubic surfaces.

Dim 1: Grothendieck-Knutsen-Mumford: The moduli space of $\overline{M}_{g,n}$ of smooth genus g stable marked curves is a smooth DM stack with normal crossings.

Dim 2: Surfaces (S, B) should be semi log canonical models (Mori theory). (Semi means semi-normal surfaces, and B is a divisor.)

KSBa is Kollár–Shepherd-Barron–Alexeev but also (K_S, surface, boundary, log discrepancies).

Assume (S, B) has semilogcanonical singularities, and $K_S + B$ is ample. If S is normal (and in particular, irreducible), then K_S is a Weil divisor. Condition: $K_S + B$ is $\mathbb{Q}$-Cartier.

If $S' \xrightarrow{\pi} S$ is a resolution of singularities, so $\pi^{-1}(S)$ has normal crossings, then

$$K_{S'} + \tilde{B} = \pi^*(K_S + B) + \sum a_i E_i$$

where $\tilde{B}$ is the strict transform of B and the E_i are the exceptional divisors. Semilogcanonical singularities means all $a_i \geq -1$.

Seminormal means only double normal crossings singularities in codimension 1.

Ampleness of $K_S + B$: on each irreducible component S_1 of S, $K_{S_1} + B + R$ is ample.

The moduli space $\overline{M}$ of these exists and is a projective DM stack.

How can we compute this even in simple examples? E.g., one-parameter limits?

The machinery works if $S - B$ is sufficiently open.

1. Cubic surfaces

Let $S \subseteq \mathbb{P}^3$ be a smooth cubic surface (degree 3). Then S is Del Pezzo, so $-K_S$ is ample. Unfortunately S is not of general type.

Theorem 1.1 (Cayley, Salmon). *S has 27 lines $B_1, \ldots, B_{27}$.*

Look at the moduli of pairs $(S, B_1 + \cdots + B_{27})$. For these pairs, $K_S + B$ is ample, so these are of log general type. Any stable cubic surface will have the same configuration of 27 "lines".

Date: October 31, 2006.

Let's compute some limits of smooth cubic surfaces in the KSBA world.

There are 45 tritangent planes (planes containing three of the lines). Sometimes the three lines in a plane pass through a single point, which is then called an Eckhart point, but these are not allowable pairs in the KSBA world. In a family of smooth cubic surfaces degenerating to a smooth cubic surface with an Eckhart point, if one blows up the Eckhart point not on the special fiber, but in the whole family, one gets a special fiber containing a $\mathbb{P}^2$ glued to the blowup of the cubic surface along a line.

Theorem 1.2 (Clebsch). *S is the blowup of $\mathbb{P}^2$ in 6 distinct points, no 3 on a line, no 5 on a conic. The 27 lines on S are: the 6 exceptional curves E_i, and the strict transforms of the lines ℓ_{ij} through pairs of the 6 points, and the conics c_{abcde} through 5 of the 6 points.*

Another limit: Let 5 of the 6 points degenerate onto a conic. What is the right limit? It has 8 irreducible components.

Basic strategy: Let Y^6 be the moduli space of smooth cubics ($\dim Y^6 = 4$) marked with the blowup map to $\mathbb{P}^2$. Let $(S, B) \to Y^6$ be the universal family over Y^6, where B is the union of the 27 lines (not normal crossings). Then $S - B$ is the moduli space Y^7 of degree 2 del Pezzos.

(1) Find an effective way of compactifying $Y_7 \to Y_6$ to $\overline{Y}^7_{\ell c} \to \overline{Y}^6_{\ell c}$.
(2) Effective flattening. $(\tilde{S}, \tilde{B}) \to \overline{Y}^6_{ss}$ mapping to $\overline{Y}^7_{\ell c} \to \overline{Y}^6_{\ell c}$. Raynaud flattening stratification. Then $(\tilde{S}, \tilde{B}) \to \overline{Y}^6_{ss}$ will be the answer, the KBSA space.

We want the fibers of $(\tilde{S}, \tilde{B}) \to \overline{Y}^6_{ss}$ to be canonical models. Why is $K_S + B$ ample on fibers? (For a fiber F of $T \to S$, we have $K_T|_F \simeq K_F$.) So we could ask for $K + B$ to be ample on the whole family $\overline{Y}^7_{\ell c}$. So $Y^7_{\ell c}$ has to be a log canonical model of Y^7.

Theorem 1.3. *This program works: Y^6_{ss} is smooth, simple normal crossing boundary, 5 types of boundary divisors up to $W(E_6)$.*

2. Tropical compactifications

Starting from $Y_7 \to Y_6$, we tried to compactify and flatten.

If instead we had a homomorphism of algebraic tori $\tilde{T} \to T$, we could compactify in terms of the fans $\tilde{F}$ and F by using $\tilde{\mathbb{P}}(\tilde{F}) \to \mathbb{P}(F)$, and there is an effective flattening: $\tilde{C} \subset \tilde{F}$ maps onto the cone $C \subset F$.

Exercise: There is a unique minimal flattening.

In our case, there are compatible closed immersions $Y_7 \hookrightarrow \mathbb{G}_m^{35}$ and $Y_6 \hookrightarrow \mathbb{G}_m^{15}$, defined by units on the surface: $M^{35} = \mathcal{O}^*(Y_7)/k^*$ and so on. Given a closed Y in a torus T, find the toric variety $\mathbb{P}(F)$ of T such that $\overline{Y}$ in $\mathbb{P}(F)$ sits in the best possible may. Look at $\overline{Y} \to \tilde{\mathbb{P}}(F)/T$ or $\overline{Y} \times T \to \mathbb{P}(F)$. Ideally you want smooth ψ. It is possible for Y^6, Y^7.

Theorem 2.1 (Tevelev). *It is always possible to make ψ flat and surjective by choosing an appropriate toric variety.*

Then the support of F is unique, an invariant of Y called its nonarchimedean amoeba.

Y has a shadow $\mathbb{P}(F)$. After flattening toric varieties, we get the moduli space.

University of Massachusetts, Amherst