ABEL MAPS FOR STABLE CURVES

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Abstract. The Abel map embeds a nonsingular projective curve in a projective algebraic group, the so-called Jacobian variety of the curve. Using the group structure we can consider higher versions of the Abel map, which carry a lot of information about the projective geometry of the curve. If the curve varies in a family, so do its Jacobian variety and the Abel map. So it is natural to ask what happens when a curve degenerates to a singular curve, and more specifically, to a Deligne-Mumford stable curve. We will see in this talk how to construct an analogue of the Abel map that "nearly" embeds a stable curve in a generalization of the Jacobian variety. This is joint work with Caporaso (Roma 3) and Coelho (IMPA).

Let C be a smooth projective connected curve over $k = \overline{k}$. Let J_C be the variety parametrizing line sheaves of degree 0 on C up to isomorphism.

Given $P \in C$, then we get an Abel map $A\colon C \to J_C$ sending Q to $\mathcal{O}_C(P-Q)$. If $C \not\simeq \mathbb{P}^1$, then A is an embedding.

Higher Abel maps: for $d \geq 1$, we have

$$\begin{aligned} A_d\colon S^d(C) &\to J_C \\ Q_1 + \cdots + Q_d &\mapsto \mathcal{O}_C(dP - Q_1 - \cdots - Q_d). \end{aligned}$$

The fibers are complete linear systems:

$$A_d^{-1}([L]) = \mathbb{P}(H^0(C, L^*(dP))),$$

where L^* is the dual of L.

Question 0.1. Can we extend the Abel map to stable curves?

There is a generalized Jacobian J_C for a stable curve, but it is not projective. For example, if C is a nodal cubic with node P, then $J_C = C - \{P\} \simeq \mathbb{G}_m$. In order to embed the curve, we need to compactify J_C.

D'Souza (1974): If C is irreducible, $J_C \subset \overline{J}_C$, where

$$\begin{aligned} \overline{J}_C &= \{\text{torsion-free rank-1 sheaves of degree 0 on } C\}/\simeq \\ &= \{\mathcal{I}_{\Gamma/C} \otimes L : \Gamma \subset C \text{ finite subscheme}, L \text{ invertible}, \ell(\Gamma) = \deg L\} \end{aligned}$$

where $\mathcal{I}_{\Gamma/C}$ is the ideal sheaf, and $\ell(\Gamma)$ is the length of Γ. Here $\overline{J}_C$ is a projective fine moduli scheme.

Altman-Kleiman: If $P \in C^{\text{sm}}$, then

$$\begin{aligned} A\colon C &\to \overline{J}_C \\ Q &\mapsto \mathcal{I}_{Q/C} \otimes \mathcal{O}_C(P) \end{aligned}$$

is an embedding if $C \not\simeq \mathbb{P}^1$.

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Suppose C is nodal (possibly reducible). Then

$$J_C = \coprod_{|\underline{d}|=0} J_C^{\underline{d}},$$

where $\underline{d} = (d_1, \dots, d_n)$ and n is the number of components of C. If C is a triangle of lines in $\mathbb{P}^2$, then $C = \overline{J}_C$.

Oda-Seshadri (1979): Several compactifications of finite unions of the $J_C^{\underline{d}}$.

Caporaso (1994) and Pandharipande (1995): over $\overline{M}_g$. These are not fine.

Definition 0.2. If I is a torsion-free rank-1 sheaf of degree 0 on C, then I is called stable (resp. semistable) if for every subcurve $Y \subsetneq C$,

$$|\deg I_Y| < \frac{\delta_Y}{2}$$

(resp., $\leq$). Here I_Y is $I|_Y$/torsion. (If $I = I_{\Gamma/C} \otimes \mathcal{L}$, then $I_Y = I_{(\Gamma\cap Y)/Y} \otimes \mathcal{L}|_Y$, so $\deg I_Y = \deg(\mathcal{L}|_Y) - \ell(\Gamma \cap Y) = \chi(I_Y) - \chi(\mathcal{O}_Y)$. And $\delta_Y := \# Y \cap \overline{C - Y}$.

For example, if C is the union of smooth curves Y and Z crossing transversely at one point, then $\delta_Y = 1$. If C is the union of smooth curves Y and Z crossing transversely at two point, then $\delta_Y = 2$.

A semistable I has a non-unique filtration

$$I = I_0 \supset I_1 \supset \cdots \supset I_q = 0$$

with stable quotients. Define $\mathrm{Gr}(I) = \bigoplus I_i/I_{i+1}$. Define $I_1 \sim_S I_2$ if and only if $\mathrm{Gr}(I_1) \simeq \mathrm{Gr}(I_2)$.

Motivation for this definition: One can have a family of nontrivial extensions of semistable sheaves

$$0 \to J \to I \to N \to 0$$

for $t \neq 0$ degenerating to a trivial extension I_0 at $t = 0$. To get a separated moduli space, we need to identify I and I_0.

Let $P \in C^{\mathrm{sm}}$. A semistable sheaf I on C is P-quasistable if $\deg I_Y > -\delta_Y/2$ for every proper subcurve Y of C containing P.

Theorem 0.3. $\overline{J}_C^P = \{P\text{-quasistable}\}/\simeq$ *is a complete fine moduli scheme.*

In the relative case, it will be an algebraic space whose fibers are schemes.

There is a surjection

$$\overline{J}_C^P \to \overline{P}_C^0$$

where $\overline{P}_C^0$ is Caporaso's notation, and it identifies S-equivalent sheaves. Here $\overline{P}_C^0$ is a projective coarse moduli space for a different functor.

Consider the Abel map

$$\begin{aligned} A \colon C &\to \overline{J}_C^P \\ Q &\mapsto \mathcal{I}_{Q/C} \otimes \mathcal{O}_C(P). \end{aligned}$$

Theorem 0.4. *A is well-defined if C has no separating nodes. In addition A is an embedding if $C \not\simeq \mathbb{P}^1$.*

Proof. $\deg\left(\left(I_{Q/C}\otimes\mathcal{O}_C(P)\right)_Y\right)\geq -1$ with equality only if $Q\in Y$ and $P\notin Y$.

$$\deg(A(Q)_Y)\geq -\frac{\delta_Y}{2}$$

unless $Q\in Y$, $P\notin Y$, and $\delta_Y=1$ (in which case there is a separating node). Each nonempty fiber $A^{-1}(A(Q))$ is open and closed in $\mathbb{P}(\mathrm{Hom}(I_{Q/C},\mathcal{O}_C))$, so it equals $\mathbb{P}(\mathrm{Hom}(I_{Q/C},\mathcal{O}_C))$. So it is either a point or $\mathbb{P}^1$. Suppose it is $\mathbb{P}^1$, call it E, and let distinct Q_1,Q_2 on E. Since $\mathcal{I}_{Q_1/C}\cong\mathcal{I}_{Q_2/C}$, both Q_1 and Q_2 are simple points of C, and the isomorphism is given by a rational function h on C. This function is constant on all components of C except on E, where it has a unique zero (at Q_2) and a unique pole (at Q_1), and hence is injective there. However, if $C\not\cong\mathbb{P}^1$, and C has no separating nodes, then E is part of a loop in C. Since h is constant along the loop (away from E), there is at most two points of E where h takes the same value. This is impossible. ☐

What if C has separating nodes?

Definition 0.5. Let $Z\subset C$ be a tail, that is, a subcurve such that $\delta_Z=1$. Then there is a unique invertible sheaf, denoted $\mathcal{O}_C(Z)$, such that $\mathcal{O}_C(Z)|_Z\simeq\mathcal{O}_Z(-N)$ and $\mathcal{O}_C(Z)|_{Z^C}=\mathcal{O}_C(N)$.

The notation is justified as follows: If $\mathcal{C}$ is a regular model of C over $B=\mathrm{Spec}\mathbb{C}[[t]]$, then

$$\mathcal{O}_{\mathcal{C}}(Z)|C\simeq\mathcal{O}_C(Z).$$

Theorem 0.6. *There is a well-defined map*

$$\tilde{A}\colon C\to\overline{J}_C^P$$

such that if Q is not a separating node, then $\tilde{A}(Q)=\mathcal{I}_{Q/C}\otimes\mathcal{O}_C(P)\otimes\mathcal{O}_C(-\sum Z)$ where the sum is over tails Z with $Q\in Z$ and $P\notin Z$.

The map $\tilde{A}$ collapses trees of separating lines; i.e., $Y\subseteq C$ of arithmetic genus 0 such that $Y\cap\overline{C-Y}$ consists of separating nodes. If Q is not on the separating tree of lines, the differential $d_Q\tilde{A}$ is injective. If Y is a maximal tree of separating lines, then the subspaces $\{\mathrm{im}(d_Q\tilde{A}):Q\in Y\cap Y^c\}$ are linearly independent in $T_{\tilde{A}(Y)\overline{J}_C^P}$. The curve $\tilde{A}(C)$ has the same arithmetic genus as C.