THADDEUS FLIPS FOR COMPLEX PROJECTIVE K3 SURFACES

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Abstract. Serre first observed that adjoint linear series on a smooth curve C yield families of rank-two vector bundles on C (with nowhere-vanishing section) via the identification of $|K_C + L|$ with lines in $\mathrm{Ext}^1(L, \mathcal{O}_C)$. Thaddeus refined this observation by constructing a one-(real)-parameter family of slope functions on the stack of pairs (E, s) that allowed him to construct "flips" of $|K_C + L|$ along the secant varieties of the embedded curve via GIT. At first sight this looks impossible to generalize to surfaces, since the points of $|K_S + L|$ correspond to lines in $\mathrm{Ext}^2(L, \mathcal{O}_S)$. In the derived category of a surface, however, $\mathrm{Ext}^2(L, \mathcal{O}_S) = Ext^1(L, \mathcal{O}_S[1])$, and at least in case S is a K3 surface, the recent slope functions introduced by Tom Bridgeland allow us to construct precise analogues of Thaddeus flips on an appropriate stack of pairs (where E is replaced by a "derived" object). This leads to some very pretty projective geometry of embedded K3 surfaces. It is joint work with Daniele Arcara.

1. Adjoint linear series

Let X be a smooth projective variety over $\mathbb{C}$ of dimension n. Let L be an ample line bundle on X. The adjoint linear series is

$$\phi = \phi_{|K_X+L|} \colon X \dashrightarrow |K_X + L| = \mathbb{P}(H^0(X, K_X + L)).$$

By Serre duality,

$$H^0(X, K_X + L)^\vee = H^n(X, L^\vee) = \mathrm{Ext}^n_{\mathcal{O}_X}(L, \mathcal{O}_X).$$

For example, given $Z \subseteq X$ of finite length, then the linear span (or secant plane) $\overline{\phi(Z)}$ is $\mathbb{P}(\mathrm{im}(\mathrm{Ext}^n_{\mathcal{O}_X}(L \otimes \mathcal{O}_Z, \mathcal{O}_X) \to \mathrm{Ext}^n_{\mathcal{O}_X}(L, \mathcal{O}_X)))$. Then

$$\mathrm{Sec}^d(X) \text{“="} \bigcup_{Z \in X^{[d]}} \overline{\phi(Z)}$$

where $X^{[d]}$ is the Hilbert scheme. We will assume that $n = 1$ or 2.

Serre: $|K_X + L| = \mathbb{P}(\mathrm{Ext}^1_{\mathcal{O}_C}(L, \mathcal{O}_C)) = \frac{\{0 \to \mathcal{O}_C \to E \to L \to 0\} - \{0\}}{\sim}$. The isomorphism type of E is preserved under equivalence.

Corollary 1.1. *There is a family of vector bundles of rank 2, determinant L, parameterized by $E_{|K_X+L|}$ with $\Phi\colon |K_X + L| \to \mathcal{M}_{\mathcal{O}_C}(2, L)$ where $\mathcal{M}_{\mathcal{O}_C}(2, L)$ denotes the stack of bundles. There is an open subscheme $M_{\mathcal{O}_S}(2, L)$ (gerbe over the projective moduli space of stable bundles) of $\mathcal{M}_{\mathcal{O}_C}(2, L)$ (if $\deg L$ is odd). So there is a rational map*

$$|K_X + L| \to M_{\mathcal{O}_S}(2, L).$$

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2. Stability

Definition 2.1. *Slope* $\mu(E) = (\deg E)/(\operatorname{rk} E)$. So $\mu\colon \pi_0(\mathcal{M}_{\mathcal{O}_C}(\bullet,\bullet) \to \mathbb{Q}$.

Define E to be *stable* if $F \subset E$ implies $\mu(F) < \mu(E)$.

One could use alternate slopes: $b\mu$ for $b > 0$, or $b\mu + a$ with $a \in \mathbb{R}$.

Higher-dimensional analogue: Fix an ample class H, and let $\mu_H(E) = \frac{c_1(E).H^{n-1}}{\operatorname{rk}(E)}$.

21st century view of μ:

$$\frac{d}{r} = -\cot(\arg(-d + ir)).$$

Since $-\cot$ is an increasing function, we could use $\arg(-d + ir)$ as a slope.

$$-d + ir = -\int_C e^{-iH}\operatorname{ch}(E)$$

where $H = [\mathrm{pt}]$. Alternative:

$$-\int_C e^{-(a+ib)H}\operatorname{ch}(E).$$

The advantage of writing it this way is so that it will generalize to the physics.

3. Thaddeus flips

Let $\mathcal{P}(r, d)$ be the stack of pairs $(\mathcal{O}_C \xrightarrow{s} E)$. Then $\mathcal{P}(r, d)$ has many "projective" realizations:

$$\mu_t(\mathcal{O}_C \xrightarrow{s} E) = \begin{cases} d - t/r, & \text{if } s \neq 0 \\ d + t/r, & \text{if } s = 0 \end{cases}$$

for $t > 0$.

Theorem 3.1 (Thaddeus)**.** *For "generic" (off a discrete set) $t > 0$, the open substacks $\mathcal{P}_t(r, d) \subset \mathcal{P}(r, d)$ of t-stable pairs are projective.*

If $r = 2$ and determinant $d = L$, then the walls are at $t = d$, $t = d - 2$, $t = d - 4$, ... Between $d-2$ and d we have $P_{t_0}(2, L) = |K_C + L|$, between $d-4$ and $d-2$ we have $P_{t_1}(2, L)$, and so on, and there are explicit flips between these moduli spaces.

4. K3 surfaces

Let S be a K3 surface of genus g: this means $\pi_1(S) = 1$, $K_S = \mathcal{O}_S$. Assume $\operatorname{Pic}(S) = \mathbb{Z} \cdot L$ where $L^2 = 2g - 2$, where $g \geq 2$.

Remark 4.1. As $g \to \infty$, the line bundles L become "more and more ample". For example, if $g \geq 3$, then L is very ample. If $g \geq 4$, then $S \hookrightarrow \mathbb{P}^n$ is cut out by quadrics.

Serre idea again: We have $|K_S + L| = \mathbb{P}(\operatorname{Ext}^2_{\mathcal{O}_S}(L, \mathcal{O}_S))$. Derived category trick: $\mathbb{P}(\operatorname{Ext}^2_{\mathcal{O}_S}(L, \mathcal{O}_S)) = \mathbb{P}(\operatorname{Ext}^1_{\mathcal{D}^b(\mathcal{O}_S)}(L, \mathcal{O}_S[1]))$, but $\mathcal{D}^b(\mathcal{O}_S)$ is not an abelian category.

Theorem 4.2 (Happel, Reiten, Smald)**.** *There is an abelian category (heart of a t-structure) $\mathcal{A}$ such that*

(1) *torsion sheaves on S are objects in $\mathcal{A}$.*
(2) *torsion-free degree-stable sheaves such that $\mu_H(E) > g - 1$ are in $\mathcal{A}$.*

(3) *torsion-free degree-stable sheaves such that* $\mu_H(E) \leq g-1$ *are in* $\mathcal{A}$ *after a shift:* $E[1] \in \mathcal{A}$.

Theorem 4.3 (Lieblich)**.** *There is an Artin stack of flat families of objects of* $\mathcal{A}$*, for a suitable notion of "flat".*

5. Bridgeland stability

$$Z(E) := -\int_S e^{-(a+bi)L}\,\mathrm{ch}(E)\sqrt{\mathrm{td}(S)}.$$

in $\mathcal{A}$. Then $\mu := \arg Z(E)$. For $a = 1/2$, we can write

$$Z(E) := (\text{real part}) + ib(c_1(E).L - \frac{1}{2}rL^2).$$

and

$$c_1(E).L - \frac{1}{2}rL^2 \geq 0$$

for all objects of $\mathcal{A}$.

$$0 \to \mathcal{O}[1] \to E \to L \to 0$$

Theorem 5.1 (Arcara, Bertram)**.** *For each* $b = \sqrt{\frac{\alpha}{g-1}}$ *with* $\alpha > \frac{g+3}{36}$*, the open substack* $M^{\alpha}_{\mathcal{A}}(0, L, L^2/2)$ *of* $\mathcal{M}_{\mathcal{A}}(0, L, L^2/2)$ *of* α*-stable objects of* $\mathcal{A}$ *is projective.*

$|K_S + L|$ in $M_{\mathcal{A}}(0, L, L^2/2)$ is Lagrangian.