RATIONAL SURFACES IN SMOOTH HYPERSURFACES

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Abstract. It is conjectured that the class of rationally connected varieties is larger than the class of unirational varieties in dimensions greater than 2. However, no example of a rationally connected variety has been proven to be non-unirational. One type of candidates are general hypersurfaces of degree n in $\mathbb{P}^n$, $n > 3$.

Considering the above conjecture, some questions arise on rational surfaces contained in hypersurfaces of degree n in $\mathbb{P}^n$. I will talk about these questions and some results in this direction. This is joint work with Jason Starr.

We work over $\mathbb{C}$. Let X be a smooth projective variety of dimension m. Say that X is *unirational* if there exists a dominant morphism $f\colon \mathbb{P}^m \dashrightarrow X$.

Rationality implies unirationality.

Unirationality implies rationality in dimension 1 (Riemann-Hurwitz) and in dimension 2 (Castelnuovo's criterion): S is a rational if and only if $h^0(S, \Omega_S) = h^0(S, K_S^{\otimes 2}) = 0$. There are counterexamples in dimension ≥ 3:

1. Griffiths-Clemens: any smooth cubic 3-fold Y in $\mathbb{P}^4$. Proving unirationality is easy. To rule out rationality, they use the intermediate Jacobian $J(Y)$. They show that birational smooth 3-folds have isomorphic intermediate Jacobians. But $J(Y)$ is nonzero, whereas the intermediate Jacobian of $\mathbb{P}^3$ is trivial.
2. Manin-Iskovskih: any smooth quartic 3-fold X in $\mathbb{P}^4$ is non-rational. They showed that X is "super-rigid", which implies that any birational automorphism of X is biregular. They also find some smooth quartic 3-folds X in $\mathbb{P}^4$ that are unirational. It is not known whether every smooth quartic 3-fold X in $\mathbb{P}^4$ is unirational.
3. Artin-Mumford: a double cover of $\mathbb{P}^3$ branched along a quartic. They show that $H^3(X, \mathbb{Z})_{\text{tors}}$ is birational invariant.

Question 0.1. For what d, n is a general hypersurface X of degree d in $\mathbb{P}^n$ rational/unirational.

If $n \gg d$, then X is unirational. There is no hypersurface with $d \geq 4$ known to be rational.

Definition 0.2. X is *rationally connected* if two general points can be joined by a rational curve.

It is equivalent to require that any two points on X can be joined by a rational curve.

Properties:

- open and closed condition
- local criterion: X is rationally connected if and only if there exists a map $f\colon \mathbb{P}^1 \to X$ such that $f^*T_X = \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_n)$ with $a_i \geq 1$.
- $-K_X$ ample implies that X is rationally connected
- conjectural numerical criterion: X is rationally connected if and only if $h^0(X, \Omega_X^{\otimes n}) = 0$ for all $m \geq 1$. (Only one direction is known.)

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Unirationality implies rational connectivity. The converse is true for dimension 1 and 2. For each dimension ≥ 3, it is not known whether the converse is true.

Candidates for counterexamples:

- general hypersurface of degree n in $\mathbb{P}^n$ for large n. It is known that every smooth hypersurface of degree n in $\mathbb{P}^n$ is super-rigid and hence not rational.
- double covers of $\mathbb{P}^n$ branched along general hypersurfaces of degree $2n$ (for n large)
- general hypersurfaces of bidegree $(2, d)$ in $\mathbb{P}^2 \times \mathbb{P}^2$ for d large. These are conic bundles over $\mathbb{P}^2$, hence rationally connected.

Conjecture 0.3. A general hypersurface of degree n in $\mathbb{P}^n$ is not unirational for $n \geq 4$.

If X is unirational of dimension m, we have $f\colon PP^m \dashrightarrow X$. A general $\mathbb{P}^{m-1} \subset \mathbb{P}^m$ is mapped by f generically injectively to X. So X is covered by rational subvarieties of dimension k for any $1 \leq k \leq \dim X - 1$.

Conjecture 0.4. A general hypersurface of degree n in $\mathbb{P}^n$ is not covered by rational surfaces for $n \geq 5$.

Heuristic dimension counting predicts this. In fact, dimension counting predicts that for $d^2 > n$, no hypersurface of degree d in $\mathbb{P}^n$ is covered by rational surfaces.

Let X be a general hypersurface of degree n in $\mathbb{P}^n$.

Theorem 0.5 (Beheshti, Starr). *The images of regular morphisms from del Pezzo surfaces to X sweep out a subvariety of codimension ≥ 2.*

Theorem 0.6 (Beheshti, Starr). *The images of rational maps $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow S \subset X$ that map a general fiber $\{t\} \times \mathbb{P}^1$ isomorphically onto an $(n-1)$-normal curve in X sweep out a subvariety of codimension ≥ 2.*

- $C \subseteq \mathbb{P}^n$ is k-normal if $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k)) \twoheadrightarrow H^0(C, \mathcal{O}_{\mathbb{P}^n}(k)|_C)$.
- A non-degenerate rational curve of degree $\leq 2n-2$ is $(n-1)$-normal.

Idea of the proof:

First approach:

Lemma 0.7. *If Y is covered by rational surfaces, then there is a smooth rational surface S with $f\colon S \to Y$ such that if*

$$0 \to T_S \to f^*T_Y \to N_f \to 0$$

with $\operatorname{rk} = \dim Y - 2$, *then*

$$h^0(S, \bigwedge^k N_f/\mathit{torsion}) > 0$$

for all $1 \leq k \leq \dim T - 2$. If a divisor in Y is covered by rational surfaces, the same is true for $1 \leq k \leq \dim Y - 3$.

Apply this to X with $\dim X = n-1$. For S a Del Pezzo surface, we can compute

$$h^0(S, \bigwedge^{n-4} N_f/\text{torsion})$$

and show that it is zero.

Second approach:

Let $\overline{\mathcal{M}}(X,e)$ be the moduli stack of genus zero stable maps of degree e to X. Let $\overline{M}(X,e)$ be the associated coarse moduli space.

Theorem 0.8. *If M is a subscheme of $\overline{M}(X,e)$ and if*

- *curves parametrized by M cover X or a divisor in X, and*
- *a general point of M parametrizes an embedded smooth $(n-1)$-normal curve,*

then M is not uniruled.

Theorem 3 implies Theorem 2.

Idea of proof of theorem 3:

Let $\tilde{M}$ be a desingularization of M. Let $\mathcal{C}$ be the universal curve:

$$\begin{tikzcd}
 & \mathcal{C} \arrow[r] \arrow[d, "\pi"'] & X \\
 & \overline{\mathcal{M}}(X,e) \arrow[d] & \\
\tilde{M} \arrow[r] & \overline{M}(X,e) &
\end{tikzcd}$$

Goal: $H^0(\tilde{M}, \Omega^{n-3}_{\tilde{M}}) \neq 0$.

$H^p(X, \Omega^q_X) \to H^p(\mathcal{C}, \Omega^q_{\mathcal{C}})$, and the Leray spectral sequence maps this to $H^{p-1}(\overline{\mathcal{M}}(X,e), \Omega^{q-1}_{\overline{\mathcal{M}}(X,e)})$.

Now $H^1(X, \Omega^{n-2}_X)$ maps to $H^0(\overline{\mathcal{M}}(X,e), \Omega^{n-3}_{\overline{\mathcal{M}}(X,e)})$ which has a trace map to $H^0(\tilde{M}, \Omega^{n-3}_{\tilde{M}})$.

Let $[C]$ be a general point on $\tilde{M}$. The diagram

$$\begin{tikzcd}
H^1(X, \Omega^{n-2}_X) \arrow[r] \arrow[d] & H^0(\tilde{M}, \Omega^{n-3}_{\tilde{M}}) \arrow[d] \\
\vdots \arrow[d] & \Omega^{n-3}_{\tilde{M}}|_{[C]} \\
H^0(C, \bigwedge^{n-3} N_{C/X}) \arrow[ur] &
\end{tikzcd}$$

is used to show that

$$H^1(X, \Omega^{n-2}_X) \to \Omega^{n-3}_{\tilde{M}}|_{[C]}$$

is nonzero.

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