TORIC VECTOR BUNDLES AND THE RESOLUTION PROPERTY

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Abstract. Is every coherent sheaf on an algebraic variety the quotient of a locally free sheaf of finite rank? I will discuss an investigation of this question via equivariant vector bundles on toric varieties, and will give examples of complete (singular, nonprojective) toric threefolds with no nontrivial equivariant vector bundles of rank less than or equal to 3. It is not known whether these varieties have any nontrivial vector bundles at all.

1. The resolution property

Let X be an algebraic variety over k. Say that X has the *resolution property* if every coherent sheaf on X is a quotient of a locally free sheaf of finite rank. In this case, one can get an infinite resolution of the coherent sheaf by locally free sheaves.

Totaru's question: Does every algebraic variety have the resolution property? (He actually asked more generally about certain stacks.)

Known cases: X has the resolution property if

1. X is quasi-projective (twist by $\mathcal{O}(n)$ for large n to get a globally generated sheaf)
2. X is smooth (or $\mathbb{Q}$-factorial)
3. X is a normal surface (Schroër-Vezzosi 2004)

In the first two cases, we are resolving by a direct sum of line bundles. This is not necessarily so in the third case.

The resolution property can fail for non-separated schemes. Voisin: The resolution property can fail for Kähler 3-folds (tori).

Theorem 1.1 (Payne)**.** *There are complete toric 3-folds with no nontrivial toric vector bundles of rank* ≤ 3.

Open problems:

1. Do these varieties have the resolution property?
2. Do these varieties have any nontrivial vector bundles.

Lemma 1.2. *If X is complete and $K^0X = \mathbb{Z}$, then X does not have the resolution property.*

Proof. Take $x \in X$ smooth. The Hilbert syzygy theorem gives a resolution of $\mathcal{O}_x$:

$$0 \to E_n \to \cdots \to E_0 \to \mathcal{O}_x \to 0.$$

Then

$$[E_0] - [E_1] + \cdots + (-1)^n [E_n] \notin \mathbb{Z}.$$

□

Theorem 1.3 (Merkurjev)**.** *The natural map $K^0_T X \to K^0 X$ from equivariant K-theory to ordinary K-theory is surjective.*

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Open problem: Is every rank-2 bundle on $\mathbb{P}^2$ a sum of line bundles for $n \geq 5$? Hartshorne's conjecture implies YES for $n \geq 7$.

Theorem 1.4 (Van de Ven 1970s). *A PGL_{n+1}-equivariant vector bundle of rank $< n$ splits as a sum of line bundles.*

Theorem 1.5 (Klyachko 1990). *Same for T-equivariant vector bundles, where $T \subset \mathrm{PGL}_{n+1}$ is a maximal torus.*

2. LINE BUNDLES ON TORIC VARIETIES

Definition 2.1. A *toric variety with torus T* is a normal variety with T-action such that T acts freely on a dense orbit.

Notation: Let M be the lattice of characters of T, so $T = \operatorname{Spec} k[M]$. Let $N = \operatorname{Hom}(M, \mathbb{Z})$ be the dual lattice, the lattice of 1-parameter subgroups. Let $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$.

A toric variety X with torus T corresponds to a fan Δ in $N_{\mathbb{R}}$, where a fan is a finite collection of rational polyhedral cones meeting along shared faces. The variety X is covered by T-invariant affine opens U_σ, and these correspond to cones $\sigma \in \Delta$. The variety X is complete if and only if $|\Delta| := \bigcup_{\sigma \in \Delta} \sigma = N_{\mathbb{R}}$.

Example: Take $N = \mathbb{Z}^2$. Divide $N_{\mathbb{R}}$ into three pieces by the positive x- and y-axes and the negative half of the line $x = y$. The 3 maximal cones correspond to 3 copies of $\mathbb{A}^2$. The 0-dimensional cone corresponds to the torus T in $\mathbb{P}^2$.

Define $M_\sigma = M/(\sigma^\perp \cap M)$. A T-equivariant line bundle L on X corresponds to a piecewise linear function ψ_L on $|\Delta|$, i.e., a function $\psi_L \colon |\Delta| \to \mathbb{R}$ such that $\psi|_\sigma \in M_\sigma$ for $\sigma \in \Delta$.

The underlying line bundle L is trivial if and only if ψ_L is globally linear. Thus $\operatorname{Pic}(X) = \operatorname{PL}(\Delta)/M$.

3. TORIC VECTOR BUNDLES

History:

1. Kaneyama (1975): combinatorial data (splitting type on affine pieces), linear algebra data, elements of GL_V for each pair of cones (satisfying cocycle condition)
2. Klyachko (1990): k-vector spaces with filtrations via 1-parameter subgroups satisfying some compatibility conditions.
3. Perling (2005): use the isotypical decomposition $\Gamma(U_\sigma, \mathcal{E}) = \bigoplus_{u \in M} \Gamma(U_\sigma, \mathcal{E})_u$.

Theorem 3.1 (Gubeladze). *Every vector bundle on an affine toric variety is trivial.*

Theorem 3.2 (Kaneyama 1975). *Every toric vector bundle on an affine toric variety is equivariantly trivializable (there exists an equivariant isomorphism to a direct sum of equivariant trivial line bundles).*

Proof. Let $O_\sigma \subseteq U_\sigma$ be a minimal T-orbit. Choose $x \in O_\sigma$. Let $s_1, \ldots, s_r$ be T-eigenvectors giving a basis for $\mathcal{E}_x$. The locus where $s_1, \ldots, s_r$ do not give a basis is closed, T-invariant, and does not contain O_σ, so it is empty. ☐

So if X is a toric variety, and E is a toric vector bundle on X, and $U_\sigma \subseteq X$, then $E|_{U_\sigma} \simeq \bigoplus_{u \in u(\sigma)} L_u$ where $u(\sigma) \subset M_\sigma$ is a multiset.

Theorem 3.3 (Payne)**.** *There is a natural isomorphism from the equivariant Chow ring* $A_T^*(X)$ *to the space* $\mathrm{PP}^*(\Delta)$ *of piecewise polynomial maps, i.e., the space of* $\psi\colon |\Delta| \to \mathbb{R}$ *such that* $\psi|_\sigma \in \operatorname{Sym} M_\sigma$ *for* $\sigma \in \Delta$*. We have* $c_1^T(L) \mapsto \psi_L$, $c_i^T(E) \mapsto \psi_E^i$ *where* $\psi_E^i|_\sigma = e_i(u(\sigma))$, *where* e_i *is the* i*-th elementary symmetric function.*

Think of $\{u(\sigma)\}_{\sigma\in\Delta}$ as giving a branched cover $\Delta_E \to \Delta$ with a piecewise linear function ψ_E on $|\Delta_E|$.

"multifans" (Hattori-Masuda 2003)
"systems of fans" (A'Campo-Neuen, Hausen 2001)
"rational polyhedral cone complexes with integral structure" (KKMSD 1973)

Example 3.4. $T\mathbb{P}^2$ on $\mathbb{P}^2$. If $U_\sigma \simeq \mathbb{A}^2$, then $TU_\sigma \simeq \mathcal{O}(D_1) \oplus \mathcal{O}(D_2)$, where D_1 and D_2 are the coordinate axes. [Picture drawn]

Strategy for computing all rank r toric bundles on a fixed toric variety X:

1. List all degree r branched covers of Δ, and compute piecewise linear functions on these branched covers.
2. Study the moduli of toric bundles with fixed equivariant Chern class.

The maximal branched covers of complete 3-dimensional toric varieties correspond to holomorphic branched covers of $\mathbb{P}^1(\mathbb{C})$ with specified branch points.

Example 3.5. Eikelberg's 3-fold. The fan combinatorially equivalent to the fan over the faces of the triangular prism with vertices $(1,0,\pm1), (0,1,\pm1), (-1,-1,\pm1)$, but $(0,1,1)$ replaced with $(0,1,2)$. One finds that $\operatorname{Pic}(X) = 0$. One can write down many nontrivial rank-2 toric vector bundles.

Example 3.6. Fulton's 3-fold. The fan combinatorially equivalent to the fan over the faces of the cube with vertices $(\pm1,\pm1,\pm1)$, but with $(1,1,1)$ replaced with $(1,2,3)$. Then $\operatorname{Pic}(X) = 0$.

Rank 2. What are the degree-2 branched covers? There are 18 of them. One need only consider those where the two vertices are opposite on a face. One checks that none of them have nontrivial piecewise linear functions. Hence the equivariant Chern class of a rank-2 toric bundle is trivial. One finds that there are no nontrivial rank-2 bundles.

Rank 3. There are 6^7 monodromy representations. Use a computer. One finds 12 monodromy representations giving degree 3 branched covers that have nontrivial piecewise-linear functions. One can write down many nontrivial rank 3 toric vector bundles.

If one also moves $(1,-1,1)$ to $(1,-1,2)$ then one get no nontrivial piecewise linear functions on degree-3 branched covers. No nontrivial rank-3 toric bundles.