THE PICARD GROUP OF THE MODULI STACK OF ELLIPTIC CURVES

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0. Picard groups of stacks

Definition 0.1. An elliptic curve over a scheme T is a diagram

$$E \underset{e}{\overset{f}{\rightleftarrows}} T$$

where

(1) f is flat and proper,
(2) e is a section, and
(3) the fibers of f are smooth connected genus 1 curves.

Let S be a scheme. Then $\mathcal{M}_{1,1,S}$ is a category whose objects are $(T \to S, E/T)$ and whose morphisms are $(T' \to S, E'/T') \to (T \to S, E/T)$ where

$$\begin{array}{ccc} E' & \xrightarrow{\tilde{g}} & E \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ T' & \xrightarrow{g} & T \end{array}$$

commutes and

(1) $E' \xrightarrow{\sim} T' \times_T E$, and
(2) $\tilde{g} \circ e' = e \circ g$

Let $\overline{\mathcal{M}}_{1,1,S}$ be the same, but with the conditions on

$$E \underset{e}{\overset{f}{\rightleftarrows}} T$$

being

(1) f flat and proper,
(2) e has image in E^{sm},
(3) the fibers are either
    (a) smooth connected genus-1 curves, or
    (b) nodal cubic curves.

Definition 0.2. An *invertible sheaf* L on $\mathcal{M}_{1,1,S}$ consists of the following data:

(1) For all $(T, E/T) \in \mathcal{M}_{1,1,S}$, a line bundle $L_{E/T}$ on T, and
(2) For all morphisms $(g, \tilde{g}) \colon (T', E'/T') \to (T, E/T)$,

$$L(g, \tilde{g}) \colon g^* L_{E/T} \xrightarrow{\sim} L_{E'/T'}.$$

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(3) If

$$(T'', E''/T'') \xrightarrow{g'} (T', E'/T') \xrightarrow{g} (T, E/T),$$

then

$$\begin{array}{ccc} (g')^* g^* L_{E/T} & \xrightarrow{(g')^* L(g)} & (g')^* L_{E'/T'} \\ \downarrow \simeq & & \downarrow L(g') \\ (g \circ g')^* L_{E/T} & \xrightarrow{L(g \circ g')} & L_{E''/T''} \end{array}$$

Remark 0.3. The set of isomorphism classes of line bundles on $\mathcal{M}_{1,1,S}$ forms a group $\operatorname{Pic}(\mathcal{M}_{1,1,S})$ under $\otimes$.

Remark 0.4. One can also define $\operatorname{Pic}(\underline{\mathcal{M}}_{1,1,S})$.

Problem: Compute $\operatorname{Pic}(\mathcal{M}_{1,1,S})$ and $\operatorname{Pic}(\underline{\mathcal{M}}_{1,1,S})$.
In 1965 Mumford found that for $S = \operatorname{Spec} k$ with $\operatorname{char} k \neq 2, 3$, we have

$$\operatorname{Pic}(\mathcal{M}_{1,1,S}) = \mathbb{Z}/(12), \quad \operatorname{Pic}(\underline{\mathcal{M}}_{1,1,S}) = \mathbb{Z}.$$

How does this notion of Pic relate to the usual notion defined for schemes?

Example 0.5. Consider $\mathbb{A}^1_S$. For this talk, let $\mathcal{A}^1_S$ be the category whose objects are $(T \to S, T \xrightarrow{f} \mathbb{A}^1_S)$ and whose morphisms are $(T', f') \to (T, f)$ where

$$\begin{array}{ccc} T' & \xrightarrow{g} & T \\ & {\scriptstyle f'}\searrow \quad \swarrow{\scriptstyle f} & \\ & \mathbb{A}^1_S. & \end{array}$$

Then $\operatorname{Pic}(\mathcal{A}^1_S) = \operatorname{Pic}(\mathbb{A}^1_S)$, because $\mathcal{A}^1_S$ has an initial object $(\mathbb{A}^1_S \to S, \mathrm{id}\colon \mathbb{A}^1_S \to \mathbb{A}^1_S)$.

Example 0.6. Let Δ be a finite group. Then $B\Delta_S$ is the category whose objects are S-schemes T, and whose morphisms $T' \to T$ are pairs (g, δ) where $g\colon T' \to T$ and $\delta \in \Delta$. Define composition as follows

$$T'' \xrightarrow{(g', \delta')} T' \xrightarrow{(g, \delta)} T, \qquad T'' \xrightarrow{(g \circ g', \delta\delta')} T$$

1. Mumford's computation

Example 1.1. The Hodge bundle $\lambda \in \operatorname{Pic}(\mathcal{M}_{1,1,S})$ is defined on

$$E \underset{e}{\overset{f}{\rightleftarrows}} T$$

as $f_* \Omega^1_{E/T}$.

Proposition 1.2. $\lambda^{\otimes 12}$ *is canonically trivial.*

Proof. Let E/T be an elliptic curve. Zariski locally on T, it is the projectivization of the equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

with $a_i \in \mathcal{O}_T$, equipped with the section $e = [0 : 1 : 0]$. A basis for the differentials $\pi = \frac{dx}{2y+a_1x+a_3}$. One can also define

$$\begin{aligned} b_2 &= a_1^2 + 4a_2 \\ b_4 &= a_1a_3 + 2a_4 \\ b_6 &= a_3^2 + 4a_6 \\ b_8 &= -a_1a_3a_4 - a_4^2 + a_1^2a_6 + a_2a_3^2 \\ \Delta &= -b_2^2b_8 - 8b_4^3 - 27b_6^2 + 9b_2b_4b_6 \in \mathcal{O}_T^\times. \end{aligned}$$

Here Δ is called the *discriminant.*

Two choices of coordinates differ by a change of variable

$$x' = u^2x + r, \quad y' = u^3y + su^2x + t$$

where $u \in \mathcal{O}_T^\times$ and $r, s, t \in \mathcal{O}_T$. Then

$$\pi' = u^{-1}\pi, \quad \Delta' = u^{12}\Delta,$$

so

$$\Delta\pi^{\otimes 12} \in H^0(E, \Omega^1_{E/T}).$$

□

Theorem 1.3 (Mumford)**.** *Let $S = \operatorname{Spec} k$, where k is a field with $\operatorname{char} k \neq 2, 3$. Then*

$$\mathbb{Z}/(12) \xrightarrow{\lambda} \operatorname{Pic}(\mathcal{M}_{1,1,s})$$

is an isomorphism.

2. New results

If $(T, E/T) \in \mathcal{M}_{1,1,S}$, we get $j(E) \in \mathcal{O}_T$ defined as $(b_2^2 - 24b_4)^3/\Delta$. This defines

$$j\colon \mathcal{M}_{1,1,S} \to \mathbb{A}^1_S$$

(i.e., to $\mathcal{A}^1_S$). If $\mathcal{L}$ is a line bundle on $\mathbb{A}^1_S$, then $j^*\mathcal{L} \in \operatorname{Pic}(\mathcal{M}_{1,1,S})$ is defined so that its value on $(T, E/T)$ is

$$(j^*\mathcal{L})_{E/T} = j(E)^*\mathcal{L}.$$

Theorem 2.1. *The map*

$$\mathbb{Z}/(12) \times \operatorname{Pic}(\mathbb{A}^1_S) \xrightarrow{\lambda\times j^*} \operatorname{Pic}(\mathcal{M}_{1,1,S})$$

is an isomorphism if either

(1) *S is a $\mathbb{Z}[1/2]$-scheme, or*
(2) *S is reduced.*

Theorem 2.2. *The map*

$$\mathbb{Z} \times \operatorname{Pic}(S) \xrightarrow{\lambda\times j^*} \operatorname{Pic}(\overline{\mathcal{M}}_{1,1,S})$$

is an isomorphism.

Another version of the previous theorem:

Theorem 2.3. $\mathbb{Z} \times B\mathbb{G}_m \to Pic(\overline{\mathcal{M}}_{1,1,S})$ *is an equality of algebraic stacks.*

3. Idea of proof when $6 \in \mathcal{O}_S^\times$

Let E_0 over $\mathbb{Z}[1/6]$ be the elliptic curve $y^2 + y = x^3$, which has $\Delta = -27$ and $j = 0$. Then $\mathbf{Aut}(E_0) = \mu_2 \times \mu_3$ where $(-1, 1)$ acts as $(x, y) \mapsto (x, -y-1)$ and $(1, \zeta)$ (where ζ is a cube root of 1) acts as $(x, y) \mapsto (\zeta x, y)$.

Let E_{1728} over $\mathbb{Z}[1/6]$ be the elliptic curve $y^2 = x^3 + x$, which has $\Delta = -64$ and $j = 1728$. Then $\mathbf{Aut}(E_{1728}) = \mu_4$, with a 4^{th}-root of unity ζ acting as $(x, y) \mapsto (\zeta^2 x, \zeta y)$.

Let $\mathcal{L}$ be a line bundle on $\mathcal{M}_{1,1,S}$. Let $\mathcal{L}_0$ be its value on $(S, E_0 \times_{\mathbb{Z}[1/6]} S)$; this is a $\mu_2 \times \mu_3$-representation. Let $\mathcal{L}_{1728}$ be its value on $(S, E_{1728} \times_{\mathbb{Z}[1/6]} S)$; this is a μ_4-representation.

To give such representations is to give an element $\chi(\mathcal{L}) \in \mathbb{Z}/(4) \times \mathbb{Z}/(6)$, since $\mathbb{Z}/(4)$ is the character group of μ_4, and $\mathbb{Z}/(6)$ is the character group of μ_6.

Lemma 3.1. $\chi(\mathcal{L}) \in \mathbb{Z}/(12) \subset \mathbb{Z}/(4) \times \mathbb{Z}/(6)$.

Lemma 3.2. $\chi(\lambda) = 1 \in \mathbb{Z}/(12)$

So we have a map

$$\mathbb{Z} \times \text{Pic}(\mathbb{A}^1_S) \to \text{Pic}(\mathcal{M}_{1,1,S}).$$

Given an element $\mathcal{L}$ of $\text{Pic}(\mathcal{M}_{1,1,S})$, we can twist by a power of λ to assume that $\chi(\mathcal{L}) = 0$, and then we hope to show that $\mathcal{L}$ comes from $\text{Pic}(\mathbb{A}^1_S)$.

Now that $\chi(\mathcal{L}) = 0$, the stabilizer group at every geometric point of $\mathcal{M}_{1,1,S}$ acts trivially: to see this, one uses that $\mathcal{M}_{1,1,S}$ is connected. Finally, the result follows from applying the following to $j\colon \mathcal{M}_{1,1,S} \to \mathbb{A}^1_S$:

Lemma 3.3. *Given a tame DM stack $\mathcal{X}$, and $j\colon \mathcal{X} \to X$ the coarse space, if $\mathcal{L}$ is an invertible sheaf on $\mathcal{X}$ such that for all geometric points $\bar{x} \to \mathcal{X}$, the action of the stabilizer group on $\mathcal{L}(\bar{x})$ is trivial, then $j_*\mathcal{L}$ is an invertible sheaf on X and $j^*j_*\mathcal{L} \to \mathcal{L}$ is an isomorphism.*

4. Characteristic 2, non-reduced case

By deformation theory, we end up having to compute $H^1(\mathcal{M}_{1,1,k}, \mathcal{O}_{\mathcal{M}_{1,1,k}})$, which by a Leray spectral sequence is equal to $H^0(\mathbb{A}^1, R^1 j_* \mathcal{O}_{\mathcal{M}_{1,1,k}})$. Similarly, $H^1(\overline{\mathcal{M}}_{1,1,k}, \mathcal{O}_{\mathcal{M}_{1,1,k}}) = H^0(\mathbb{P}^1, R^1 j_* \mathcal{O}_{\mathcal{M}_{1,1,k}})$.

Proposition 4.1. *Let k be a field of characteristic 2. Then $R^1 j_* \mathcal{O}_{\overline{\mathcal{M}}_{1,1,k}}$ on $\mathbb{P}^1_k$ is a line bundle of negative degree.*

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