# THE CREMONA TRANSFORM IN GROMOV-WITTEN THEORY

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## 1. Gromov-Witten theory

Let $X$ be a smooth projective variety over $\mathbb{C}$. Let $\beta \in H_2(X, \mathbb{Z})$. Let $\gamma_1, \ldots, \gamma_n \in H^*(X, \mathbb{Z})$.

Let $\overline{M}_{g,n}(X, \beta)$ be the stack of stable maps to $X$: its objects are stable maps

$$[f\colon (\Sigma, p_1, \ldots, p_n) \to X]$$

where $\Sigma$ is a possibly nodal marked curve and $f_*[\Sigma] = \beta$.

This is a proper DM stack with a perfect obstruction theory. This defines a virtual fundamental class $[\overline{M}_{g,n}(X, \beta)]^{\mathrm{vir}} \in H_{\mathrm{vd}}(\overline{M}_{g,n}(X, \beta))$, where the virtual dimension vd is

$$(\dim X - 3)(1 - g) - K_X.\beta + n.$$

In order to gain some insight into the nature of the virtual class, consider the following special cases.

- The moduli space is unobstructed. This happens precisely when $h^1(\Sigma, f^*T_X) = 0$ for each stable map $f$. In this case, $\overline{M}_{g,n}(X, \beta)$ has an honest fundamental class, which coincides with the virtual fundamental class.
- The moduli space is smooth. In this case, the virtual fundamental class is an Euler class.

Gromov-Witten invariants are defined to be integrals over this virtual fundamental class. The *genus-$g$, class $\beta$ Gromov-Witten invariant of $X$ with insertions $\gamma_i$* is defined by:

$$\langle \gamma_1, \ldots, \gamma_n \rangle_{g,\beta}^X = \int_{[\overline{M}_{g,n}(X,\beta)]^{\mathrm{vir}}} \prod_i \mathrm{ev}_i^* \gamma_i.$$

where the evaluation maps are

$$\overline{M}_{g,n}(X, \beta) \xrightarrow{\mathrm{ev}_i} X$$
$$[f, (\Sigma, p_1, \ldots, p_n)] \mapsto f(p_i).$$

**Example 1.1.** Quintic 3-fold. Consider the set $Q$ of $(x_0 : \cdots : x_4) \in \mathbb{P}^4$ such that

$$x_0^5 + \cdots + x_4^5 - k(x_0, \ldots, x_4) = 0.$$

This is a Calabi-Yau 3-fold: this means that $c_1(T_Q) = 0$.

How many lines are there in $Q$? The answer is 2875, as was discovered by H. Schubert in 1886.

How many conics are there in $Q$? In 1986, Sheldon Katz showed that there are 609250 of them.

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In 1991, a group of physicists (Candelas et al) got the number of degree $d$ genus-0 curves in $Q$ in the sense that they computed the Gromov-Witten invariants:

$$\langle\rangle^Q_{0,[\text{line}]} = 2875.$$

$$\langle\rangle^Q_{0,2[\text{line}]} = 609250 + \frac{2875}{8}.$$

The $\frac{2875}{8}$ counts double covers of a $\mathbb{P}^1$. As such, it is called a multiple cover contribution, or a *local Gromov-Witten invariant*.

Let $i\colon Z \hookrightarrow Y$ is a closed subvariety of a Calabi-Yau 3-fold $Y$. Let $\beta \in H_2(Z, \mathbb{Z})$. Define a substack

$$M_Z = \{[f] \in \overline{M}_g(Y, i_*\beta) : \operatorname{im}(f) \subseteq Z\}.$$

If $M_Z$ is a union of path connected components of $\overline{\mathcal{M}}_g(Y, i_*\beta)$, then we get a virtual fundamental zero-cycle $[M_Z]^{\text{vir}}$. Define the *genus-$g$ degree-$d$ local Gromov-Witten invariant of* $Z \subset Y$ to be:

$$N^g_d(Z \subset Y) := \int_{[M_Z]^{\text{vir}}} 1.$$

This depends on a formal or analytic neighborhood of $Z$.

The local theory of rational curves in a CY3 $X$ has a particularly rich history.

- (Aspinwal-Morrison) Let $C \simeq \mathbb{P}^1$ be a curve in $X$ with $N_{C/X} \simeq \mathcal{O}(-1) \oplus \mathcal{O}(-1)$. Then $N^0_d(C) = 1/d^3$.
- (Faber-Pandharipande) The genus-$g$ invariants of such a $(-1,-1)$ curve are given by $N^g_d(C) = \frac{|(2g-1)B_{2g}|}{(2g)!} d^{2g-3}$, where $B_{2g}$ are Bernoulli numbers.
- (Bryan-Katz-Leung) The local invariants of ADE configurations of rational curves are determined. For example, consider the $E_6$ Dynkin diagram, but interpret each segment as a curve, and each vertex as an intersection of curves.
- (Bryan-Karp) Let $C = C_1 \cup C_2 \cup C_3$ be the closed topological vertex, i.e. a chain of three rational curves meeting in a single triple point. Then

$$N^g_{d_1,d_2,d_3}(C) = \begin{cases} 0 & \text{if } d_i \neq d_j \\ N^g_{d,0,0} & \text{otherwise} \end{cases}$$

- (Karp-Liu-Mariño) Similarly, for three chains of lengths $d_1, d_2, d_3 > 0$ meeting at a point, the local invariants of these configurations are determined.

The proof of in the last two cases uses the Cremona transform. The Cremona transformation is the rational map

$$\mathbb{P}^n \dashrightarrow \mathbb{P}^n$$

$$(x_0 : \cdots : x_n) \mapsto (x_1 x_2 \cdots x_n : \cdots : \prod_{i \neq j} x_i : x_0 x_1 \cdots x_{n-1}).$$

On $U = \{x_i \neq 0\}$, it is given by

$$(x_0 : \cdots : x_n) \mapsto (x_0^{-1} : \cdots : x_n^{-1}).$$

**Example 1.2.** On $\mathbb{P}^2$, the Cremona transform admits a resolution $\tau\colon X \to X$ where $X$ is the blowup of $\mathbb{P}^2$ along 3 $T$-fixed points. Let $\tilde{\ell}_i$ be the strict transforms of the lines $\ell_i$ through pairs of them. Let $e_i$ be the exceptional curves.

Note: Gromov-Witten invariants are invariant under isomorphisms. So we get the following observation of Bryan-Leung: If an automorphism sends $\beta$ to $\beta'$, then

$$\langle \mathrm{pt}^{\otimes n}\rangle^X_{0,\beta} = \langle \mathrm{pt}^{\otimes n}\rangle^X_{0,\beta'}.$$

We have $H_2(X,\mathbb{Z}) = \langle h, e_1, e_2, e_3\rangle$ and $\beta = dh - \sum a_i e_i$ with $d \in \mathbb{Z}_{>0}$, $a_i \in \mathbb{Z}$, and $\beta' = d'h - \sum a'_i e_i$, where $d' = 3d - 2(a_1 + a_2 + a_3)$, $a'_1 = d - (a_2 + a_3)$, $a'_2 = d - (a_1 + a_3)$, $a'_3 = d - (a_1 + a_2)$.

Consider $\mathbb{P}^3$. There is a resolution of the Cremona map on $\mathbb{P}^3$

$$\tau \colon \hat{X} \to \hat{X}$$

where $\hat{X}$ is the blowup of $\mathbb{P}^3$ at more than just points.

**Theorem 1.3** (Bryan, Karp)**.** *Let $X$ be the blowup of $\mathbb{P}^3$ at $m \geq 4$ points $p_i$. So $H_2(X) = \langle h, e_1, \ldots, e_m\rangle$, where $e_i$ is a line in the exceptional divisor $E_i$ over point $p_i$. Let $\beta = dh - \sum a_i e_i$ where $2d = \sum a_i$ (so $K_X.\beta = 0$). Then*

$$\langle\rangle^X_{g,\beta} = \langle\rangle^X_{g,\beta'}$$

*where $\beta' = d'h - \sum a'_i e_i$ where $d' = 3d - 2(a_1 + \cdots + a_4)$, $a'_1 = d - (a_2 + \cdots + a_4)$, $\ldots$, $a'_4 = d - (a_1 + \cdots + a_3)$, $a'_5 = a_5$, $\ldots$, $a'_m = a_m$.*

Does this argument extend to $\mathbb{P}^n$? Let $X_0$ be the blowup of $\mathbb{P}^n$ at $n+1$ points. Let $X_i$ be the blowup of $\mathbb{P}^n$ at the proper transforms of the $i$-dimensional $T$-invariant subvarieties of $\mathbb{P}^n$. Let $\hat{X} = X_{n-2}$. Then the polytope of $\hat{X}$ is the permutohedron, the convex hull of the points whose coordinates are a permutation of a fixed set of $n$ distinct real numbers.

**Conjecture 1.4.** $\langle\gamma\rangle^{X=X_0}_{0,\beta} = \langle\gamma'\rangle^{X=X_0}_{0,\beta'}$ where $\beta = dh - \sum a_i e_i$ and $\beta' = d'h - \sum a'_i e_i$ with $d' = nd - (n-1)(a_1 + \cdots + a_{n+1})$, $a'_1 = d - (a_2 + \cdots + a_{n+1})$, $\ldots$, $a'_{n+1} = d - (a_1 + \cdots + a_n)$, $a'_{n+2} = a_{n+2}$

**Example 1.5.** There exists a unique rational curve in $\mathbb{P}^n$ of degree $n$ through $n+3$ generic points: see Theorems 1.18 and 18.9 in Harris (it is the rational normal curve).

$$\langle\rangle^{X=\widetilde{\mathbb{P}^n}}_{0,nh-e_1-\cdots-e_{n+3}} = \langle\rangle^{X=\widetilde{\mathbb{P}^n}}_{0,h-e_{n+2}-e_{n+3}} = 1,$$

since there is a unique line through two points!

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