

ASYMPTOTIC IDEAL THEORY AND CASTELNUOVO-MUMFORD REGULARITY

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Let $S = \mathbb{C}[x_0, \dots, x_n]$. Let $P = (x_0, \dots, x_n)$. Let $I \subset S$ be a homogeneous ideal. Goal: Understand I^m for $m \gg 0$.

Consider the case $n = 1$:

Let $F_0, \dots, F_t \in \mathbb{C}[x_0, x_1]$ be forms of degree d . Let $I = (F_0, \dots, F_t) \subset P^d$. So $I^m \subset P^{md}$. We may harmlessly reduce to the special case where $\gcd(F_0, \dots, F_t) = 1$; in this 2-variable case this is equivalently, by the Nullstellensatz, to assuming that I contains a power of P . One measure of the difference between I^m and P^{md} is the smallest integer, call it b_m , such that I_m contains P^{md+b_m} .

How does b_m depend on m ?

Example 0.1. (In the case $n = 1$.) Take $F_0 = x_0^d$ and $F_1 = x_1^d$, so $I = (x_0^d, x_1^d)$, and $I^m = (x_0^{dm}, x_0^{d(m-1)}x_1^d, \dots)$, and it is easy to see that $\dim(P^{md}/I^m)_d = m(d-1)$ — note that this number grows linearly with m . Also $I \supset P^{d+(d-1)}$, and $I^m \supset P^{md+(d-1)}$; i.e., $b_m = d-1$ for all $m \geq 1$, so the smallest power of P contained in I^m also grows linearly with m .

It's not hard to see directly that the same behavior would hold for any pair of relatively prime forms of degree d ($t = 1$).

Riddles: What is b_m for large m if $t = 2$ and F_0, F_1, F_2 are “general” forms of degree d ? What about $t = 3$ or $t > 3$? By the end of this talk, the answers will be clear. . . .

First, let's generalize the problem, and return to the case of arbitrary $n \geq 1$. Define $d(I)$ be the maximum degree of a generator (where the generators are defined as follows: we take a basis for the forms in I of lowest degree, then add forms of the next lowest degree as necessary). Alternatively, $d(I)$ is the maximum d such that $(I/PI)_d \neq 0$. We extend the definition from the case of ideals to the case of finitely generated graded modules.

We next define the (Castelnuovo-Mumford) *regularity* of a homogeneous ideal $I \subset S$.

$$0 \rightarrow F_r \rightarrow \dots \rightarrow F_1 \rightarrow F_0 \rightarrow I \rightarrow 0$$

be a minimal graded free resolution of I : that is, this sequence is an exact complex where each F_i is a graded free module, the maps all have degree 0, and each F_i has the minimum possible number of generators. (By the Hilbert syzygy theorem, such a resolution has finitely many terms before reaching 0 on the left.) The resolution is in fact unique up to isomorphism. Define

$$\operatorname{reg} I = \max_j (d(F_j) - j).$$

Thus $\operatorname{reg} I \geq d(I)$. One should think of $\operatorname{reg} I$ as an approximation to $d(I)$ with “more stable” properties.

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There is also a cohomological description of $\text{reg } I$. For example, if I is a saturated ideal ($I = I^{\text{sat}}$), then $\text{reg } I$ is the smallest r such that $H^j(\mathcal{I}(r-j)) = 0$ for all $j > 0$, where \mathcal{I} is the ideal sheaf on \mathbb{P}^n corresponding to I . It is perhaps the tension between these two descriptions that gives the notion of regularity its power. The next three Propositions exploit these descriptions, and may give the reader some sense of what regularity means:

Example 0.2. Suppose $f_1(x), \dots, f_s(x)$ are non-homogeneous 1-variable polynomials. If they are relatively prime, then there exist $a_i(x)$ such that

$$\sum a_i(x)f_i(x) = 1.$$

Proposition 0.3. *One can choose the a_i such that $\max \deg a_i \leq \text{reg } I$, where I is the ideal generated by the homogenizations of the f_i .*

Proposition 0.4. *Let $I \subset S$ (for general n). If I contains some power of P , then $\text{reg } I$ is the smallest m such that I contains P^m .*

Let V be a finite subset of \mathbb{R}^n . Let $J(V)$ be the ideal of V . Let $R[V] = \mathbb{R}[\underline{x}]/J(V)$. Define the interpolation degree of V as the least d such that \mathbb{R}^V is induced by the polynomials of degree $\leq d$. This is largest, namely $n-1$, when the points are on a line, but is much less when the points are in general position.

Proposition 0.5. *If $V \subset \mathbb{R}^n \subset \mathbb{P}_{\mathbb{R}}^n$, and $I(V)$ is the homogeneous ideal of V , then the interpolation degree of V equals $\text{reg } I(V) - 1$.*

Suppose n is general, and $S \supset I$. Suppose I contains some power of P . The regularity tells us what power of P is contained in I . If I contains P^m , then I^2 contains P^{2m} . So $\text{reg } I^{md} \leq m \text{reg } I^d$ for all d, m . Karen Chandler: This is still true for the homogeneous ideal of a zero-dimensional scheme in \mathbb{P}^n . But Kodiyalam observed, using an example of Sturmfels, that it fails for higher-dimensional schemes. Nevertheless, high powers behave well:

Theorem 0.6 (Cutkosky-Herzog-Trung-Kodiyalam). $\text{reg } I^m = am + b$ for $m \gg 0$.

Kodiyalam: $a = \min_J d(J)$, where J is an ideal such that I is contained in the integral closure of J . Alternatively (by a result of Ein-Lazarsfeld-Cutkosky), $a = \lim d(I^m)/m$.

The value of b is still mysterious in all cases except one, which is now to be discussed.

The “other” point of view: Let $I = (F_0, \dots, F_t)$ with $\deg F_i = d$. Suppose $I \supset P^m$ for some m . Then I^m defines

$$\begin{aligned} \phi: \mathbb{P}^m &\rightarrow \mathbb{P}^* \\ (x) &\mapsto (F^\alpha(x)). \end{aligned}$$

Theorem 0.7 (Eisenbud, Harris, Huneke). *Suppose $I \subset S$, and $I \supset P^m$ for some m , and I be generated by forms of degree d . Then $\text{reg } I^m = dm + b$ where $b = \max_{y \in \mathbb{P}^t} \text{reg } I(\phi^{-1}(y)) - 1$.*

This result allows us to answer the riddles from the beginning of the talk:

Example 0.8. Let $n = 1$. For (F_0, F_1) , we have $b = d - 1$; i.e., $\text{reg } I^m = md + d - 1$.

Example 0.9. Let $n = 2$. Let F_0, F_1, F_2 be general. Then $\mathbb{P}^1 \rightarrow \mathbb{P}^2$ is general, and we get a curve with only nodes as singularities. Then $b = 2 - 1 = 1$. Thus $I^m \supset P^{md+1}$.

Example 0.10. Let $n = 1, t \geq 3$. Let $F_0, F_1, F_2, \dots, F_t$ be general. The image of $\mathbb{P}^1 \rightarrow \mathbb{P}^t$ will be smooth, so $I^m = P^{md}$.

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