ARE ORTHOGONAL POLYNOMIALS RELATED TO ALGEBRA?

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1. Introduction

Orthogonal polynomials are polynomials with an inner product. On $\mathbb{R}$, let $\mu(dx)$ be a probability measure with $\int x^j \mu(dx) < \infty$ for $j = 1, 2, 3, \ldots$. Define

$$\langle p, q \rangle = \int p(x) q(x) \mu(dx).$$

Use Gram-Schmidt to orthnormalize $1, x, x^2, x^3, \ldots$.

Examples 1.1.

1. Let $\mu(dx) = \frac{e^{-x^2}}{\sqrt{\pi}}\, dx$. Then the orthogonal polynomials are called Hermite polynomials.
2. Let $\mu(dx) = (1-x)^\alpha (1+x)^\beta\, dx$. Then we get Jacobi polynomials.
3. Let $\mu(j) = \frac{\binom{r}{j}\binom{n-r}{k-j}}{\binom{n}{k}}$ for $0 \le j \le n$. Then we get Hahn polynomials.

Common properties of a family of orthogonal polynomials P_n:

1. $xP_n = a_n P_{n+1} + b_n P_n + c_n P_{n-1}$. Proof: Write $xP_n = \sum_{j=0} a_j P_j$. Then $a_j = \int x P_n P_j$, which is zero unless $|j - n| \le 1$.
2. Favard's theorem: If q_n satisfy a 3-term recurrence then they are orthogonal polynomials with respect to some μ.
3. The zeros of each P_n are real, and they intertwine.
4. The uniform probability measure $\frac{1}{n}\sum \delta_{\lambda_j^n}$ on the zeros of P_n converges weakly as $n \to \infty$ to the measure μ.

For more, see the following book: T. Chihara, Introduction to orthogonal polynomials.
Uses:

- "Gaussian quadrature": approximate $\int f(x)$ by $\sum_{j=0}^n a_j f(b_j)$.
- Representation theory of $SU(2)$. Matrix entries. Jacobi polynomials.
- Eigenfunctions of various operators.

Work with Laurent Saloff-Coste and graduate students K. Kahare and Zhou.

2. Background

Gibbs sampler, Glauber dynamics, heat bath algorithm: Given a probability distribution $f(\underline{x})$ on $\mathbb{R}^d$ (for example, it could be $ze^{-\beta H(\underline{x})}$ but the normalizing constant z might be unknown), the problem is to pick $\underline{x}$ from $f(\underline{x})$.

Algorithm: Start with $(x_1, \ldots, x_d)$, change the first coordinate (using f), change the second coordinate, …, change the d-th coordinate; this is one pass of Glauber cycle. Repeat

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this cycle until convergence. (How long does it take to converge? This is analogous to the question of how many shuffles it takes to make a deck of cards "random".)

Example: Let $f_\theta(\underline{x}) = \binom{n}{\underline{x}} \prod_{i=1}^d \theta_i^{x_i}$ be the multinomial distribution on $\Delta_{d-1,n} := \{(x_1, \ldots, x_n) : \sum_1^d x_i = n, x_i \geq 0, x_i \in \mathbb{Z}\}$. Let $\pi(d\theta)$ be the uniform measure on $\{(\theta_1, \ldots, \theta_n) : \sum d_i = 1, d_i \in \mathbb{R}_{\geq 0}\}$. Let $f(\underline{x}, \underline{\theta}) = f_\theta(\underline{x})$. Problem: sample from this.

Gibbs sampler:

- From $\underline{\theta}$ choose $\underline{x}'$ from $f_\theta(\cdot)$.
- From $\underline{x}'$ choose θ' from
$$\frac{\Gamma(n+d)}{\prod_1^d \Gamma(x_i)} \prod_1^d \theta_i^{x_i'}.$$
x-chain:
$$h(\underline{x}, \underline{x}') = \frac{(n+d)_{k-1}}{(2n+1)_{d-1}} \frac{\binom{n}{\underline{x}}\binom{n}{\underline{x}'}}{\binom{2n}{\underline{x}+\underline{x}'}}.$$
This is a $\binom{n+d-1}{d-1}$ by $\binom{n+d-1}{d-1}$ matrix.

Proposition 2.1.

(1) *The eigenvalues are* $\beta_0 = 1$ *and*
$$\beta_\alpha = \frac{n(n-1)\cdots(n-|\alpha|+1)}{(n+2)(n+3)\cdots(n+|\alpha|+1)}.$$
In particular, $\beta_1 = 1 - \frac{2}{n+2}$.

(2) $h(\underline{x}, \underline{x}')$ *operates on polynomials* $Kp(x) = \sum_{x'} h(x,x')p(x')$ *of the same degree.*

(3) *Eigenfunctions are orthogonal polynomials for*
$$m(x) := \frac{1}{2\binom{n+d-1}{d-1}},$$
for $x \in \Delta_{d-1,n}^1$.

(4) $\chi^2(\ell) = \sum_{x'} \frac{(h^\ell(x,x')-m(x'))^2}{m(x)} = \sum_{\alpha \neq 0} \beta_\alpha^{2\ell} P_\alpha^2(x)$.

The P_α *are orthogonal polynomials for* $m(x)$.

Question: From a starting point x, and $n = 100$, and $d = 5$, how large must ℓ be so that $\chi^2(\ell) < \epsilon$.

3. Orthogonal polynomials in several variables

How do we discuss P_α, where α is now a vector? There is a pretty good book *Orthogonal polynomials of several variables* by Charles F. Dunkl and Yuan Xu.

On $\mathbb{R}^d$, for $\alpha \in \mathbb{N}^d$, $s_\alpha \in (0, \infty)$, define $\ell(x^\alpha) = s_\alpha$ and $\langle p, q \rangle = \ell(p \cdot q)$.

Example 3.1. If $\mu(dx)$ is a probability measure on $\mathbb{R}^d$ with $\operatorname{supp}(\mu)$ open, then $s_\alpha = \int x^\alpha \mu(dx)$.

Definition 3.2. A polynomial P_α is orthogonal with respect to ℓ if

- $\langle x^\alpha, P_\beta \rangle = 0$ if $|\alpha| < |\beta|$
- The $\binom{d+\ell-1}{d-1}$ by $\binom{d+\ell-1}{d-1}$ matrix $\langle P_\alpha, P_\beta \rangle$ indexed by (α, β) with $|\alpha| = |\beta| = \ell$ is invertible.

The orthogonal polynomials with respect to ℓ form a basis for the space of all polynomials.

Theorem 3.3.

- *There is a 3-term recurrence: If*
$$P_h = \begin{pmatrix} P_{\alpha^1} \\ \vdots \\ P_{\alpha^{nh}} \end{pmatrix}$$
where $|\alpha| = h$, then there exist A_{ni}, B_{ni}, C_{ni} such that
$$x_i P_n = A_{ni} P_{n+1} + B_{ni} P_n + C_{ni} P_{n-1}.$$
- *There is a Favand's theorem.*

4. LAST TOPIC

Let V be a finite or countable discrete set in $\mathbb{R}^d$. Let $I(V)$ be the set of polynomials P such that $P(x) = 0$ for all $x \in V$. Let $R[V] = \mathbb{R}[x_1, \ldots, x_d]/I(V)$.

Easy fact: If Λ is the set of all α such that x^α is not a leading term of any element of $I = I(V)$, then any polynomial $P \in \mathbb{R}[V]$ can be written as
$$P = \sum_{\alpha \in \Lambda} c_\alpha x^\alpha \pmod{I}.$$

Choose a measure $\mu(x) > 0$ defined for $x \in V$. Let $\ell(P) = \sum_{x \in V} P(x)\mu(x)$.

Definition 4.1. A sequence of polynomials $\{P_\alpha\}$ is orthogonal with respect to μ if

- $\ell(x^\alpha P_\beta) = 0$ for all $\alpha, \beta \in \Lambda$ with $|\alpha| < |\beta|$, and
- The matrix $\langle P_\alpha, P_\beta \rangle$ indexed by $\alpha, \beta \in \Lambda$ with $|\alpha| = |\beta| = h$ is invertible.

See Y. Xu, On discrete orthogonal polynomials of several variables.

Reference: Diaconis, P., Khare, K. and Saloff-Coste, L. Gibbs Sampling, Exponential Families and Orthogonal Polynomials. (2006)

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