# EFFECTIVE DIVISORS ON MODULI SPACES AND SYZYGIES

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Let $g \geq 3$. Let $\mathcal{M}_g$ be the moduli space of genus $g$ curves. $\overline{\mathcal{M}}_g - \mathcal{M}_g = \Delta_0 \cup \Delta_1 \cup \cdots \cup \Delta_{\lfloor g/2 \rfloor}$, where $\Delta_0$ parametrizes genus $g-1$ curves with two points identified, and for $i \geq 1$, $\Delta_i$ parametrizes curves of genus $i$ and $g-i$ intersecting at a point. Let $\delta_i$ be the class of $[\Delta_i]$ in $\operatorname{Pic} \overline{\mathcal{M}}_g$.

Hodge class $\mathbb{E} \to \overline{\mathcal{M}}_g$, $\mathbb{E}[C] = H^0(\omega_C)$, $c_1(\mathbb{E}) = \lambda$. Then $\operatorname{Pic}(\overline{\mathcal{M}}_g)$ is freely generated by $\lambda_1, \delta_0, \ldots, \delta_{\lfloor g/2 \rfloor}$.

If $D \in \operatorname{Eff}(\overline{\mathcal{M}_g})$ and $\Delta_i \subseteq D$, we can write

$$D \equiv a\lambda - \sum_{i \geq 0} b_i \delta_i$$

with $a, b_i \geq 0$. The *slope* of $D$ is defined as

$$s(D) := \frac{a}{\min\{b_i\}} \geq 0.$$

Also $s(\overline{calM}_g) := \inf_{D \in \operatorname{Eff}(\overline{\mathcal{M}}_g)} s(D) \geq 0$.

**Conjecture 0.1** (Slope conjecture of Harris and Morrison)**.** $s(D) \geq 6 + \frac{12}{g+1}$ for all $D \in \operatorname{Eff}(\overline{\mathcal{M}}_g)$, with equality if and only if $D$ is a combination of Brill-Noether divisors.

For $r, d \geq 1$, the number

$$\rho = g - (r+1)(g - d + r)$$

is the expected dimension of the variety of $g^r_d$'s on a fixed genus $g$ curve. In cases where $\rho = -1$,

$$\mathcal{M}^r_{g,d} := \{[C] \in \mathcal{M}_g : C \text{ has a } g^r_d\}$$

is an irreducible divisor whose class in $\operatorname{Pic}(\overline{\mathcal{M}}_g)$ is

$$c\left((g+3)\lambda - \frac{g+1}{6}\delta_0 - \sum_{i \geq 1} i(g-i)\delta_i\right)$$

where $c$ is some positive constant depending on $r, g, d$. Then

$$s(\overline{\mathcal{M}}^r_{g,d}) = 6 + \frac{12}{g+1}.$$

Applications:

(1) Geometric solution to Schottky problem.
(2) Kodaira dimension of $\overline{\mathcal{M}}_g$:

$$K_{\overline{\mathcal{M}}_g} \equiv 13\lambda - 2\delta_0 - 3\delta_1 - 2\delta_2 - \cdots - 2\delta_{g/2}.$$

(3) $\overline{\mathcal{M}}_g$ is of general type if $s(\overline{\mathcal{M}}_g) < \frac{13}{2} = s(K_{\overline{\mathcal{M}}_g})$ ($\lambda$ is big and nef).

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We have $6 + \frac{12}{g+1} > \frac{13}{2}$ if and only if $g < 23$. If the slope conjecture were true, then the Kodaira dimension $\kappa(\overline{\mathcal{M}}_g)$ would be $-\infty$ for $g < 23$.

**Theorem 0.2** (Farkas, M. Popa)**.** *The slope conjecture fails for $\overline{\mathcal{M}}_{10}$ for $D = k_{10} := \{[C] \in \mathcal{M}_{10} : C$ sits on a K3 surface$\}$.*

In fact we have

$$\overline{k}_{10} \equiv 7\lambda - \delta_0 - 5\delta_1 - 9\delta_2 - 12\delta_3 - 14\delta_4 - 15\delta_5$$

and $s(\overline{k}_{10}) = 7 < 6 + \frac{12}{11}$, violating the slope conjecture.

Questions:

(1) Are there more counterexamples to the slope conjecture?
(2) Can we use other counterexamples to find other $\mathcal{M}_g$ of general type?
(3) Is there a weak slope counterexample?

**Theorem 0.3.** *We showed that $k_{10}$ has four other incarnations:*

(1) *$k_{10}$ is the set of $[C] \in \mathcal{M}_{10}$ such that there exists a rank-2 vector bundle $E$ on $C$ with $\det E = k_C$ and $h^0(E) \geq 7$.*
(2) *$42k_{10}$ is the set of $[C] \in \mathcal{M}_{10}$ such that there exists an $g^4_{12}$ embedding $C \hookrightarrow \mathbb{P}^4$ such that $C$ lies on a quadric.*
(3) *$42k_{10}$ is the set of $[C] \in \mathcal{M}_{10}$ such that there exists an $g^4_{12}$ embedding $C \hookrightarrow \mathbb{P}^4$ such that $C$ fails property $(N_0)$.*
(4) ...

Note: $g^4_{12} = k_C(-g^1_6)$.

A new stratification of $\mathcal{M}_g$: Suppose $g = 2k-2$. The class $[C] \in \mathcal{M}_g$ has $\frac{(2k-2)!}{k!(k-1)!}$ pencils $g^1_k$. Pick a $g^1_k$ $A$. Let $L = k_C - A$. Then $|L|\colon C \hookrightarrow \mathbb{P}^{k-2}$ is a $g^{k-2}_{3k-6}$. Let $Z_{g,i}$ be the set of $[C] \in \mathcal{M}_g$ such that there exists a $g^1_k$ $A$ such that $|L|\colon C \hookrightarrow \mathbb{P}^{k-2}$ is a $g^{k-2}_{3k-6}$ fails $(N_i)$. Property $(N_i)$ means that for a free $S$-resolution

$$\cdots \to F_2 \to F_1 \to I_C \to 0,$$

$F_j$ is a direct sum of copies of $S(-j-1)$ for all $j \leq i$. Then

$$Z_{g,0} \subseteq Z_{g,1} \subseteq \cdots .$$

Define the locus $\mathcal{N}_{g,i}$ of $[C] \in \mathcal{M}_g$ such that $[k_C]\colon C \hookrightarrow \mathbb{P}^{g-1}$ fails $(N_i)$.

Green's conjecture can be formulated as:

$$\mathcal{N}_{g,i} = \mathcal{M}^1_{g,i+2} = \{[C] : \exists C \hookrightarrow \mathbb{P}^1 \text{ of degree} \leq i+2\}.$$

Examples: $\mathcal{N}_{g,0}$ is the hyperelliptic locus, and

$$Z_{g,0} = k_g = \{[C] \in \mathcal{M}_g : \ C \text{ sits on a K3 surface}\},$$

which is of dimension $k + 19$ (if $g \geq 13$).

Fix $L = k_C - A$ and $|L|\colon C \hookrightarrow \mathbb{P}^{k-2}$ as before. Then we have

$$0 \to M_L \to H^0(L) \otimes \mathcal{O}_C \to L \to 0$$

where $M_L$ is a stable vector bundle on $C$ of rank $k-2$ and slope $-3$. We have a Hurwitz space $\mathcal{G}^1_{g,k} \xrightarrow{\pi} \mathcal{M}_g$ parameterizing $(C, A)$ where $A$ is a $g^1_k$ on $C$: here $\pi$ is finite. There exist two vector bundles $\mathcal{A}$ and $\mathcal{B}$ on $\mathcal{G}^1_{g,k}$:

$$\mathcal{A}(C,A) := H^0(\bigwedge^i M_{\mathbb{P}^{k-2}}(2))$$
$$\mathcal{B}(C,A) := H^0(\bigwedge^i M_L(2)).$$

And there is a map $\phi\colon \mathcal{A} \to \mathcal{B}$. For $g = 6i+10$, $\operatorname{rk}\mathcal{A} = \operatorname{rk}\mathcal{B}$, and $Z_{g,i}$ is $\pi_*$ of the degeneracy locus of $\phi$. We expect this to be a divisor.

**Theorem 0.4.** *Let $\pi\colon \overline{\mathcal{G}}^1_{g,k} \to \overline{\mathcal{M}}_g$ be the admissible covering compactification. There exist extensions $\overline{\mathcal{A}}, \overline{\mathcal{B}}$ of $\mathcal{A}, \mathcal{B}$ over $\overline{\mathcal{G}}^1_{g,k}$ and $\overline{\phi}\colon \overline{\mathcal{A}} \to \overline{\mathcal{B}}$ such that $\overline{Z}_{g,i}$ is $\pi_*$ of the degeneracy locus of $\overline{\phi}$. Also,*

$$\pi_*(c_1(\overline{\mathcal{B}}) - c_1(\overline{\mathcal{A}})) = a\lambda - b_0\delta_0 - \cdots - a_{3i+5}\delta_{3i+5}$$

*for explicitly given constants, so*

$$s(\pi_* c_1(\overline{\mathcal{B}} - \overline{\mathcal{A}})) = \frac{3(4i+7)(6i^2+19i+12)}{(i+2)(12i^2+31i+18)} < 6 + \frac{12}{g+1}.$$

**Corollary 0.5.** *If $\overline{Z}_{g,i}$ is a divisor, we get a counterexample to the slope conjecture.*

Examples:

$i=0$: $Z_{10,0} = k_{10}$.

$i=1$: $Z_{16,1}$ is the set of $[C] \in \mathcal{M}_{16}$ such that there exists a $g^7_{21}$ from $C \hookrightarrow \mathbb{P}^7$ such that $C$ is not cut out by quadrics. This is a divisor, and

$$s(\overline{Z}_{16,1}) = \frac{407}{61} = 6.672\cdots < 6 + \frac{12}{17} = 6.705\ldots.$$

$i=2$: $Z_{22,2}$ is the set of $[C] \in \mathcal{M}_{22}$ such that there exists a $g^{10}_{30}$ from $C \hookrightarrow \mathbb{P}^{10}$ such that $(N_2)$ fails. We get

$$s(\overline{Z}_{22,2}) = 6.50039\ldots < 6 + frac1223.$$

We have counterexamples for every $g < 23$.

What about the question about Kodaira dimension?

**Theorem 0.6.** *The moduli space $\overline{\mathcal{M}}_{22}$ is of general type. (Equivalently, there exist $D \in \operatorname{Eff}(\overline{\mathcal{M}}_{22})$ with $s(D) < 6.5$.)*

*Proof.* Let $D_I$ be the set of $[C] \in \mathcal{M}_{22}$ such that there exists a $g^6_{25}$ $|L|\colon C \hookrightarrow \mathbb{P}^6$ such that $C$ sits on a quadric. Then a Riemann-Roch calculation suggest that $D_I$ is a divisor. If so, then $s(D_I) = \frac{17121}{2636} = 6.4961\ldots < 6.5$. If not, then three different conjectures would be violated! □

If $C$ is a curve and $L \in \operatorname{Pic}^d(C)$, then we have the multiplication map $\mu_L\colon \operatorname{Sym}^2 H^0(L) \to H^0(L^2)$. For $\rho \in \ker\mu_L$, define $\operatorname{ord}_P(\rho) \geq k$ if and only if $\rho = \sum a_{ij}\sigma_i \cdot \sigma_j$ with $\operatorname{ord}_P(\sigma_i) + \operatorname{ord}_P(\sigma_j) \geq k$.

Define $D_{II}$ to be the set of $[C] \in \mathcal{M}_{22}$ such that there exists a $g^7_{27}$ $L$ with a cusp at some point $P \in C$ such that there exists $0 \neq \rho \in \ker\mu_L$ with $\operatorname{ord}_P(\rho) \geq 7$. Then $D_{II}$ is a divisor on $\mathcal{M}_{22}$ and $s(D_{II}) < 6.5$.

Let $W \leq \operatorname{Sym}^2 H^0(L)$ be the set of $\rho$ with $\operatorname{ord}_P(\rho) \geq 7$. Let $\mathcal{G}^7_{27,\text{cusp}}$ be the set of $(C, L, P)$ such that $P$ is a cusp for $L$. Then $\mathcal{G}^7_{27,\text{cusp}}$ is irreducible and there is a finite map $\mathcal{G}^7_{27,\text{cusp}} \to \mathcal{M}_{22}$. Above it, $W$ and $H^0(L^2 - 7P)$ are 26-dimensional.

**Conjecture 0.7.** There exists $\epsilon > 0$ independent of $g$ such that $s(D) > \epsilon$ for all $D \in \operatorname{Eff}(\overline{\mathcal{M}}_g)$ for all $g$.

Maybe $\epsilon = 6$? I have an approach that might yield $\epsilon = 4$.

One can also define slope for the moduli space of abelian varieties. It is conjectured that

$$\lim_{g \to \infty} s(\overline{\mathcal{A}}_g) = 0.$$

Thus there would be lots of (non-explicit) modular forms vanishing on $\overline{\mathcal{M}}_g$.

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