INVERTIBLE MATRICES OVER MONOMIAL ALGEBRAS

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Let k be a field. Gaussian elimination on a matrix in $\mathrm{SL}_n(k)$ can be done by about n^2 elementary transformations.

For $i \neq j$ and $\lambda \in k$, let e_{ij}^{λ} be the matrix equal to the identity but with λ in the i,j-position. Let $E_n(k)$ be the subgroup of elementary matrices, generated by all the e_{ij}^{λ}. By Gaussian elimination, $\mathrm{SL}_n(k) = E_n(k)$, and we have the uniformly bounded word property.

Also, one has $\mathrm{SL}_n(\mathbb{Z}) = E_n(\mathbb{Z})$, and we have the uniformly bounded word property. From now on, $n \geq 3$. More generally, if $\mathcal{O}$ is the ring of integers of a number field K, then $\mathrm{SL}_n(\mathcal{O}) = E_n(\mathcal{O})$, as proved by Keller and Carter.

Consider $k[t]$ where k is a field. One has a division algorithm, so $\mathrm{SL}_n(k[t]) = E_n(k[t])$. If moreover k is finite, we have the uniformly bounded word property. But for $k = \mathbb{C}$, the uniformly bounded word property fails (van der Kallen).

One has $\mathrm{SL}_n(R) = E_n(R)$ whenever R is a Euclidean domain. This is not always true for a principal ideal domain R. In fact, there exists a principal ideal domain R and an $\alpha \in \mathrm{SL}_2(R)$ such that

$$\begin{pmatrix} \alpha & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

with $n-2$ ones is not in $E_n(R)$ for any n: Grayson gave an example with $R = S^{-1}\mathbb{Z}[T]$ where $S = \{T^n - 1 : n \in \mathbb{N}\}$.

Example (P. Cohn): The element $\alpha := \begin{pmatrix} 1 - t_1t_2 & t_1^2 \\ -t_2^2 & 1 + t_1t_2 \end{pmatrix}$ is in $\mathrm{SL}_2(k[t_1,t_2]) - E_2(k[t_1,t_2])$, but the 3×3 matrix $\begin{pmatrix} \alpha & \\ & 1 \end{pmatrix}$ is in $E_3(k[t_1,t_2])$. One says that α is *stably elementary.*

Theorem 0.1 (Suslin)**.** *For $n \geq 3$ and $r \in \mathbb{N}$,*

$$\mathrm{SL}_n(k[t_1,\dots,t_r]) = E_n(k[t_1,\dots,t_r]).$$

This is the K_1-analogue of the Serre problem on projective modules over polynomial rings.

Theorem 0.2 (Park-Woodburn)**.** *Suslin's theorem yields an algorithm for factorization of an element of $\mathrm{SL}_n(k[t_1,\dots,t_r])$ into elementary matrices.*

This has applications to signal processing.

How about a "sparse" version of this theorem? I.e., over monomial algebras.

In P. Cohn's example, $\alpha \in \mathrm{SL}_2(R)$ where $R := k[t_1^2, t_1t_2, t_2^2]$. Over R, α is not even stably elementary (Srinivas, Gubeladze).

Observation: Let $R' = k[t_1^4, t_1^2t_2^2, t_2^4]$. So $R' \subset k[t_1^2, t_2^2] \subset R$, and abstractly $R' \simeq R$.

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Suslin's theorem plus this observation shows that under the isomorphism $R \simeq R'$, if α maps to β (i.e., β is obtained by squaring all monomials in α), then $\begin{pmatrix} \beta & \\ & 1 \end{pmatrix}$ is elementary over R.

Here is our main theorem.

Theorem 0.3. *Let R be any regular ring of dimension d. Let $n \geq \dim R + 3$. Let $c \in \mathbb{Z}_{\geq 2}$. Let $r \in \mathbb{Z}_{\geq 0}$. Let $\alpha \in \mathrm{SL}_n(R[t_1, \ldots, t_r])$. Let A be a monomial algebra in $R[t_1, \ldots, t_r]$, not necessarily finitely generated or normal. The R-algebra homomorphism $R[t_1, \ldots, t_r] \to R[t_1, \ldots, t_r]$ sending each t_i to t_i^c restricts to an endomorphism of A, mapping each monomial m to m^c. We have an induced homomorphism $c_*\colon \mathrm{GL}_n(A) \to \mathrm{GL}_n(A)$. Then $(c_*)^j(\alpha)$ is equivalent modulo elementary matrices to the matrix of constant terms $\alpha(0)$ for all $j \gg 0$. Here $(c_*)^j$ is the j-th iterate of c_*. Moreover, when R is a field, there is an algorithm that finds such j and reduces $(c_*)^j(\alpha)$ to $\alpha(0)$.*

Conjecture 0.4. $j \geq 1$ suffices.

Proof of Theorem. (Sketch) Let us consider the special case where R is a field and A is a normal affine monomial algebra. (Affine means finitely generated.)

Two steps:

1. (main step) Consider the cone generated by the monomials in A. Without loss of generality, we may assume that all lattice points in the cone are actual monomials in A. Take a cross-section defined by a rational hyperplane, giving a polytope P. Let $\xi \in \mathrm{int}(P)$. Let P_θ be the polytope obtained by contracting P by a factor of θ towards ξ. Let $A_\theta \subseteq A$ be the subalgebra generated by monomials corresponding to P_θ. Then there exists $\theta \in \mathbb{Q}$ with $\theta < 1$, depending only on P such that $(c_*)^j(\alpha) \sim \beta \in \mathrm{GL}_n(A_\theta)$.
2. Iterate Step 1. Then we can sandwich a simplex Δ between P and a P' and hence sandwich a polynomial ring between A and A'. Since $(c_*)^j(\alpha)$ is trivialized over the polynomial ring, it is trivialized also over A'.

□

K-theory. Let A be a ring as above. We have classical K-groups $K_0(A)$, $K_1(A)$, $K_2(A)$, and higher K-theory $K_3(A)$, $K_4(A)$, ... Our theorem is about $K_1(A)$.

Nilpotence theorem for all K-groups, proved only when R is a characteristic 0, stably: For any $x \in K_i(A)$ there exists j such that $(c_*)^j(x) \in K_i(k) \subseteq K_i(A)$.

Our main theorem extends the K_1-slice in two ways: k is extended to any regular local ring, and stable is replaced by unstable.

Actually, our real main theorem is about $K_2(A)$, but we need the K_1-result to prove it.

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