LINEAR SERIES ON MODULI SPACES OF VECTOR BUNDLES ON CURVES

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Let X be a smooth projective curve over $\mathbb{C}$ of genus $g(X) = g \geq 2$. A vector bundle E is *semistable* if for every proper subbundle F, we have

$$\frac{\deg F}{\operatorname{rk} F} \leq \frac{\deg E}{\operatorname{rk} E}.$$

We call E *stable* if the same holds with $<$ in place of $\leq$.

Two moduli spaces:

1. Let $U_X(r, d)$ be the moduli space of semistable vector bundles on X of rank r and degree d.
2. For all $L \in \operatorname{Pic}^d X$, $SU_X(r, L)$ is the same as $U_X(r, d)$ except that we take only vector bundles with fixed determinant L.

These are normal irreducible projective varieties with rational singularities. The singular locus is the locus of strictly semistable bundles, with one exception: $g = 2$, $r = 2$.

Let $h = (r, d)$ (the greatest common divisor). If $h = 1$, then all semistable bundles are stable (and the moduli space is a fine moduli space and is smooth).

In the late 1980s and early 1990s:

- physics (conformal field theory): Verlinde formula
- systematic use of generalized theta divisors: nonabelian analogues of the theta divisor on Jacobians

Here we'll see that two more techinques can be added to the study of linear series on these spaces:

- moduli spaces of stable maps
- Fourier-Mukai transform for sheaves on abelian varieties

1. Generalized theta divisors

Fix one of the two moduli spaces SU_X or U_X, and consider E in one of these. Let $h = (r, d)$, $r_1 = r/h$, $d_1 = d/h$. For all $m \geq 1$, fix

$$F \in U_X(mr_1, m(r_1(g-1) - d_1)).$$

Then $\mu(E \otimes F) = g - 1$ (equivalent by Riemann-Roch to $\chi(E \otimes F) = 0$). For a general such F, have a generalized theta divisor

$$\Theta_F = \{E \in (S)U_X(r, d) : h^0(E \otimes F) \neq 0\}.$$

One could define E to be semistable if and only if for some such complementary F, $h^0(E \otimes F) = 0$, which is easily seen to be equivalent to the definition given at the beginning.

Theorem 1.1 (Drezet, Narasimhan). *$(S)U_X(r, d)$ are locally factorial and*

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(1) $\operatorname{Pic} SU_X(r,d) \simeq \mathbb{Z} \cdot \mathcal{L}$, *where the ample generator* $\mathcal{L}$ *is called the* determinant line bundle.

(2) $U_X(r,d) \to \operatorname{Pic}^d X \simeq J(X)$ *(the Jacobian). Let* det *be the composition. Then* $\operatorname{Pic} U_X(r,d) \simeq \mathbb{Z} \cdot \mathcal{O}(\Theta_F) + \det^*(\operatorname{Pic} J(X))$, *where the* Θ_F *is as above for* $m = 1$.

If we fix m, then $\Theta_F \in |\mathcal{L}^m|$ on SU_X.

Conjecture 1.2 (Strange Duality)**.** The Θ_F span the linear series $|\mathcal{L}^m|$ for all m.

The conjecture is known only for the case in which $SU_X(r,0)$ and $m = 1$ (a theorem of Beauville, Narasimhan, Ramanan). In this case it is equivalent to the fact that there exists a canonical isomorphism

$$H^0(SU_X(r,0), \mathcal{L})^* \simeq H^0(J(X), \mathcal{O}_J(r\Theta)).$$

We can prove positive results using generalized theta divisors (Faltings).

Theorem 1.3.

(1) *(Popa)* $|\mathcal{L}^m|$ *on* $SU_X(r,d)$ *is basepoint-free for* $m \geq \lfloor r^2/4 \rfloor$.

(2) *(Esteves, Popa)* $|\mathcal{L}^m|$ *is very ample for* $m \geq r^2 + r$, *if* $(r,d) = 1$ *(otherwise on smooth locus).*

The best lower bound replacing $\lfloor r^2/4 \rfloor$ must grow as $r \to \infty$ (proved by Popa in his thesis).

Idea of proof: For all $E \in SU_X(r,d)$, we want to find a Θ_F such that $E \notin \Theta_F$, i.e., such that $h^0(E \otimes F) = 0$. If $0 \neq H^0(E \otimes F)$, then we have a map $E^\vee \to F$; let G be the image, and let H be the cokernel. If the dimension of the set of F's in such diagrams is $< (mr_1)^2(g-1) + 1$ (the dimension of the moduli space of bundles of rank mr_1), then we can find an F as desired.

General problem: Let E be any vector bundle of rank r, degree e. Let $\operatorname{Quot}_{k,d}(E)$ be the moduli space of $E \to Q \to 0$ with $\operatorname{rk} Q = k$ and $\deg Q = d$. Can we get a good upper bound for $\dim \operatorname{Quot}_{k,d}(E)$?

$\operatorname{Quot}_{k,d}(E)$ "compactifies" the same open space (the space of locally free quotients) as the space of sections σ_Q of $\mathbb{G}(E,k) \to X$, i.e., the space $\overline{\mathcal{M}}_g(\mathbb{G}(E,k), \beta_d)$ of stable maps, where $\beta_d = \sigma_Q([X])$.

Theorem 1.4 (Popa, Roth)**.** $\dim \operatorname{Quot}_{k,d}(E) \leq \dim \overline{\mathcal{M}}_g(\mathbb{G}(E,k), \beta_d) \leq k(r-k) + (d-d_k)r$, *where* $d_k = \min\{\deg Q : E \to Q \to 0, \operatorname{rk} Q = k\}$.

This result allows one to perform the computation above and find effective bounds for m.

Now look at $|m\Theta_F|$ on $U_X(r,d)$, and decide whether they are basepoint-free, or very ample. Idea: We have the composition $\det\colon U_X(r,d) \to \operatorname{Pic}^d(X) \to J(X)$. Call

$$E_m := \det{}_* \mathcal{O}(m\Theta_F)$$

the *Verlinde vector bundle on* $J(X)$. Its fibers are the spaces $H^0(SU_X(r,L), \mathcal{L}^m)$. If $\mathcal{L}^m$ is globally generated on SU_X and E_m is globally generated on $J(X)$, then $|m\Theta_F|$ is globally generated. If $\mathcal{L}^m$ is very ample on SU_X and $E_m \otimes \mathcal{I}_{\{a\}}$ is globally generated for all $a \in J(X)$, then $|m\Theta_F|$ is very ample. We have

$$\begin{tikzcd}
SU_X(r,d) \times J(X) \arrow[r] \arrow[d, "p_2"] & U_X(r,d) \arrow[d, "\det"] \\
J(X) \arrow[r, "\phi_r"'] & J(X)
\end{tikzcd}$$

where ϕ_r is multiplication-by-r, and the top map maps (E, ξ) to $E \otimes \xi$. Then $\phi_r^* E_m \simeq \bigoplus \mathcal{O}_J(mrr_1\Theta)$.

Theorem 1.5 (Pareschi,Popa)**.** *If $\mathcal{F}$ is a coherent sheaf on an abelian variety A satisfying*

$$H^i(\mathcal{F} \otimes \alpha) = 0$$

for all $i > 0$ and all $\alpha \in \operatorname{Pic}^0 A$, then $\mathcal{F} \otimes L$ is globally generated for all L ample on A.

The proof uses the Fourier-Mukai transform, and Mukai's duality theorem.

E_m is globally generated if and only if $m \geq h+1$ (where $h = (r,d)$). Also, $E_m \otimes \mathcal{I}_{\{a\}}$ is globally generated for all a if and only if $m \geq 2h+1$.

Using the Theorem above, the idea is that we have

$$H^0\mathcal{F} \to \mathcal{F}(x) \to H^1(\mathcal{F} \otimes \mathcal{I}_x).$$

$E_m \otimes \mathcal{I}_{\{a\}}$ is globally generated if $H^1(E_m(-\Theta_\beta) \otimes \mathcal{I}_{\{a\}}) = 0$, which holds if $E_m(-\Theta_\beta)$ is globally generated for all β, which holds if $H^1(E_m(-\Theta_\beta)(-\Theta_\gamma)) = 0$ for all $\beta, \gamma, \ldots$

Conjecture 1.6.

(1) $|(r+1)\Theta_F|$ is globally generated on $U_X(r,0)$ for all r.
(2) $|(2r+1)\Theta_F|$ is very ample on $U_X(r,0)$ for all r.

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