MODULI SPACES OF CURVES WITH LINEAR SERIES AND THE SLOPE CONJECTURE

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Describe the general curve of genus g. For $g = 2$, have $y^2 = f(x)$. For $g = 3, 4, 5$, describe canonical curve in $\mathbb{P}^{g-1}$ as a complete intersection. For $g = 6$, draw a degree 6 curve with nodes at four specified points.

Let $\mathcal{M}_g$ be the moduli space of genus g curves. Does there exist a dominant rational map $\mathbb{P}^n \to \mathcal{M}_g$? Severi(?) thought yes. But Eisenbud, Harris, Mumford showed that for $g \geq 24$, the answer is no. Later, Farkas proved a negative answer for $g \geq 22$.

Let X be a smooth projective variety. Associated to X we have the canonical bundle K_X. Its tensor powers $K_X^{\otimes n}$ may determine a rational map $\phi_{|K_X^{\otimes n}|} \colon X \dashrightarrow \mathbb{P}^n$. Define the *Kodaira dimension* $\kappa(X)$ as $\max_n \dim \phi_{|K_X^n|}(X)$. We have

$$\kappa(X) \in \{0, 1, \ldots, \dim X, -1\}.$$

If there exists a dominant rational map $\mathbb{P}^N \dashrightarrow X$, then $\kappa(X) = -1$. Say that X is of *general type* if $\kappa(X) = \dim X$.

Harris-Mumford-Eisenbud: $\mathcal{M}_g$ is of general type for $g \geq 24$.

Farkas: Same for $g \geq 22$.

Let $\overline{\mathcal{M}}_g$ be the moduli space of stable curves.

Let C be a curve. Let $\Lambda_C := H^0(C, \omega_C)$. This gives a bundle of rank g over $\overline{\mathcal{M}}_g$. Let $\lambda = c_1(\Lambda)$. Let $\delta = [\Delta]$, where $\Delta = \overline{\mathcal{M}}_g - \mathcal{M}_g$. Then

$$K_{\overline{\mathcal{M}}_g} \sim 13\lambda - 2\delta.$$

Look in the 2-dimensional subspace of $\mathrm{NS}(\overline{\mathcal{M}}_g) \otimes \mathbb{R}$ spanned by λ and δ. Plot $a\lambda - b\delta$ at $(b, a) \in \mathbb{R}^2$. So K is at $(2, 13)$. The ample cone (intersected with the 2-dimensional subspace) is the upper wedge between the lines of slope 11 and ∞. The effective cone is the wedge between a ray in the first quadrant and the negative x-axis. If $(2, 13)$ is in the effective cone, then $\overline{\mathcal{M}}_g$ is not dominated by $\mathbb{P}^n$.

Define the *slope* of $\mathcal{M}_g$ by

$$s_g = \inf\left\{\frac{a}{b} : a, b \geq 0, a\lambda - b\delta \text{ effective}\right\}.$$

If $s_g < 6.5$, then $\mathcal{M}_g$ is of general type. If $s_g > 6.5$, then $\kappa(\mathcal{M}_g) = -1$.

1. Brill-Noether theory

Let $\mathrm{Pic}^d C$ be the line bundles of degree d on C. Let $W^r_d(C)$ be the subset of $L \in \mathrm{Pic}^d C$ such that $h^0(L) \geq r + 1$. Let $G^r_d(C)$ be the subset of pairs (L, V) where L is a bundle, and $V \subseteq H^0(L)$ is of dimension $r+1$. We have $G^r_d(C) \to W^r_d(C) \subseteq \mathrm{Pic}^d C$. An element of $G^r_d(C)$ is called a g^r_d.

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Theorem 1.1. *For a general curve C, $G^r_d(C)$ is a smooth variety of dimension*

$$\rho = g - (r+1)(g-d+r).$$

Example: Choose (g, r, d) such that $\rho = -1$. Let D^r_d be the subset of $\mathcal{M}_g$ consisting of C such that C has a g^r_d. Then one can show that D^r_d is a divisor in $\mathcal{M}_g$, and

$$D^r_d \sim (g+3)\lambda - \left(\frac{g+1}{6}\right)\delta_0 - \sum_{i=1}^{\lfloor g/2 \rfloor} i(g-i)\delta_i.$$

The slope of $D^d_r + \sum_{i=1}^{\lfloor g/2 \rfloor} i(g-i)\delta_i$ equals

$$\frac{g+3}{(g+1)/6} = 6 + \frac{12}{g+1}.$$

This already shows that $\mathcal{M}_g$ is of general type for $g \geq 24$.

Harris-Morrison slope conjecture: $s_g \geq 6 + \frac{12}{g+1}$. This would imply that $\kappa(\overline{\mathcal{M}}_g) = -1$ for $g \leq 22$.

Farkas-Popa: $\mathcal{K} \subseteq \mathcal{M}_{10}$ is a divisor consisting of curves C such that C lies on a $K3$ surface. We have

$$\mathcal{K} \sim 7\lambda - \delta_0 - 5\delta_1 - \cdots,$$

and the slope of $\mathcal{K}$ is $7 < 6 + \frac{12}{11}$, so this violates the slope conjecture.

Also, $\mathcal{K}$ is the set of C such that there exists an embedding $C \hookrightarrow \mathbb{P}^4$ of degree 12 such that C lies on a quadric. For $(g, d, r) = (10, 12, 4)$, we have $\rho = 0$, so the general curve has a g^4_{12}, but it does not lie on a quadric. The dimensions of

$$H^0(\mathcal{O}_{\mathbb{P}^4}(2)) \to H^0(\mathcal{O}_C(2))$$

are $\binom{4+2}{2} = 15$ and $24 + 1 - 10 = 15$, so the condition that this have a kernel should define a divisor in $\mathcal{M}_g$.

Let E be the set of C of genus 21 admitting an embedding in $\mathbb{P}^6$ of degree 24 such that C lies on a quadric. So $(g, d, r) = (21, 24, 6)$, and $\rho = 21 - 7 \cdot 3 = 0$ and $H^0(\mathcal{O}_{\mathbb{P}^6}(2)) \to H^0(\mathcal{O}_C(2))$ have dimensions $\binom{8}{2} = 28$ and $48 + 1 - 21 = 28$.

Theorem 1.2. *$E \subset \mathcal{M}_{21}$ is a divisor.*

We have a space $\mathcal{G}^r_d(\mathcal{M}_g) \xrightarrow{\eta} \mathcal{M}_g$ parameterizing (C, L, V) where $C \in \mathcal{M}_g$, $L \in \operatorname{Pic}^d C$, and $V \subseteq H^0(L)$ is a subspace of dimension $r+1$. The divisor E is naturally the image (under η) of a divisor $\tilde{E}$ on $\mathcal{G}^r_d(\mathcal{M}_g)$: namely, $\tilde{E}$ is the set of (C, L, V) such that $\operatorname{Sym}^2 V \to H^0(L^{\otimes 2})$ has a nonzero kernel.

We can compute the class of $\tilde{E}$ using the Porteous formula. What is

$$\eta_* \colon A^{*+\rho}(\mathcal{G}^r_d(\tilde{\mathcal{M}}_g)) \to A^*(\tilde{\mathcal{M}}_g),$$

where $\tilde{\mathcal{M}}_g$ is some partial compactification of $\mathcal{M}_g$.

Theorem 1.3. *If $\rho = 0$, we can compute*

$$\eta_* \colon A^1(\mathcal{G}^r_d(\tilde{\mathcal{M}}_g)) \to A^1(\tilde{\mathcal{M}}_g).$$

Corollary 1.4. *E is proportional to*

$$\eta_* \tilde{E} = \eta_*(2\alpha - \beta - 8\gamma + \lambda),$$

which is proportional to

$$\frac{2459}{95}\lambda - \frac{377}{95}\delta_0 - \cdots ,$$

and the slope of E is

$$\frac{2459}{377} \approx 6.52 < 6 + \frac{12}{22} \approx 6.55.$$

We have $\tilde{E} \subseteq \mathcal{G}^6_{24}(\mathcal{M}_{21}) \to \mathcal{M}_{21}$. We want to show that $\tilde{E} \neq \mathcal{G}^6_{24}(\mathcal{M}_{21})$. If L has $h^0 = 6$ and $d = 24$, then $K \otimes L^\vee$ has $h^0 = 3$ and $d = 16$. So $\mathcal{G}^6_{24}(\mathcal{M}_{21})$ is dominated by $V_{16,21}$, which is irreducible, so $\mathcal{G}^6_{24}(\mathcal{M}_{21})$ is irreducible too. Now it suffices to find a single point in $\mathcal{G}^6_{24}(\mathcal{M}_{21}) - \tilde{E}$. Find such a curve in the surface S in $\mathbb{P}^6$ given by the condition that a 6×3 matrix of linear forms have rank 2.