WHERE DO NOWHERE VANISHING 1-FORMS COME FROM?

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Let $\mathbb{C}$ be any algebraically closed field of characteristic zero. Let X be a smooth projective variety of dimension n.

Carrell-Lieberman: If there exists a nowhere vanishing (holomorphic) vector field on X, then (**) all characteristic numbers of X are 0. E.g., $c_1^n = 0$, $c_1^{n-2}c_2 = 0$, $\dots$, $c_n = 0$, where $c_i := c_i(T_X)$.

Example: If $X = A \times V$ is a product of a nonzero abelian variety A and another variety V.

Carrell asked about the dual situation, for differential 1-forms. That is, if (*) there exists a nowhere vanishing differential 1-form on X, does that imply that (**) all characteristic numbers of X are 0.

If (*), then we have an exact sequence

$$0 \to \mathcal{O}_X \to \Omega_X^1 \to \mathcal{E} \to 0$$

with $\mathcal{E}$ locally free, and

$$c_n(T_X) = (-1)^n c_n(\Omega_X^1) = (-1)^n c_n(\mathcal{E}) = 0.$$

The most interesting characteristic number is c_1^n.

(1) Curves: Trivially true: (*) holds only for genus-1 curves.
(2) Abelian varieties of dimension > 0: (*) and (**) both hold.
(3) More generally, if $\alpha\colon X \to A$ is a smooth map to an abelian variety of dimension > 0, and $0 \neq \eta \in H^0(A, \Omega_A)$, then $\alpha^*\eta$ is a nonvanishing 1-form on X, so (*) holds.
(4) Surfaces: (*) implies $c_1^2 = c_2 = 0$.
(5) 3-folds: There is a $\mathbb{P}^1$-bundle over an abelian surface with $c_1^3 \neq 0$. (This is uniruled. It happens often that uniruled varieties give counterexamples to statements that would otherwise be true.)

Kodaira dimension: For curves we have $g = 0$, $g = 1$, $g \geq 2$; these correspond to $\kappa = -1$ (or $-\infty$) or $\kappa = 0$ or $\kappa = 1$. By definition, $\kappa(X)$ is

$$\kappa(X) = \operatorname{trdeg} \bigoplus_{m \geq 0} H^0(X, \omega_X)^{\otimes m} - 1.$$

For hypersurfaces of degree d in $\mathbb{P}^N$, we have

$$\kappa(X) = \begin{cases} -1 & \text{if } d < N+1 \\ 0 & \text{if } d = N+1 \\ \dim X & \text{if } d > N+1 \end{cases}$$

To get intermediate values of κ, one can take products. This is almost the only way, by a theorem of Iitaka.

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Definition 0.1. Say that X is of *general type* if $\kappa(X) = \dim X$.

Carrell's conjecture 1: If (*) and $\kappa(X) \geq 0$, then (**) holds.

Minimal models: Let us call a surface X *almost minimal* if it does not contain a (-1)-curve.

Lemma 0.2. *If $n = 2$ (X is a surface) and (*) holds, then X is almost minimal.*

Proof. We have

$$0 \to \mathcal{O}_X \to \Omega_X \to \omega_X \to 0,$$

since on the right we have a line bundle whose determinant equals the determinant of Ω_X. Suppose $\mathbb{P}^1 \simeq C \subseteq X$. Restrict to C:

$$0 \to \mathcal{O}_C \to \Omega_X|_C \to \omega_X|_C \to 0.$$

The map from $\Omega_X|_C$ to ω_C is a surjection killing $\mathcal{O}_C$, so it induces a surjection $\omega_X|_C \to \omega_C$, which must be an isomorphism. Therefore $N_{C/X} = \mathcal{O}_X$, so $C^2 = 0$. □

Definition 0.3. Call X *minimal* if K_X is nef, i.e., $K_X.C \geq 0$ for all proper curves $C \subseteq X$.

If X is minimal, then X is almost minimal: if $C \subseteq X$ is a (-1)-curve on a surface, then $K_X.C = -1$.

Conversely, almost minimal implies minimal or ruled or $\mathbb{P}^2$.

If X is minimal, then $K_X^n \geq 0$ (follows from the fact that nef divisors are limits of ample divisors). Then X is of general type if and only if $K_X^n > 0$, or equivalently $K_X^n \neq 0$, or equivalently $c_1^n \neq 0$.

Corollary to Green-Lazarsfeld: (*) implies $\chi(\mathcal{O}_X) = 0$.

Example: If $n = 2$, then Riemann-Roch gives $\chi(\mathcal{O}_X) = \frac{1}{12}(c_1^2 + c_2)$. If this is 0, and $c_2 = 0$, then one gets $c_1^2 = 0$.

Example: If $n = 3$ and (*) holds, then $\chi(\mathcal{O}_X) = \frac{1}{24}c_1 c_2 = 0$; and $c_3 = 0$, so we need only check whether $c_1^3 = 0$. (And it need not be, as in our ruled 3-fold counterexample.)

Conjecture 2: (*) implies that X is not of general type.

This conjecture is true for $\dim X = 3$.

Carroll also conjectured that (*) implies that X is minimal, but this is false, because one could take a product of a variety satisfying (*) and a non-minimal variety.

Plan for proving Conjecture 2: First prove

Conjecture: (*) and X general type implies that X is minimal.

Then prove Conjecture 2 for X minimal.

Tie Luo and Qi Zhang 2003: This plan works in dimension 3.

Theorem 0.4 (Hacon,Kovács)**.** *Let X be a smooth projective variety of general type over $\mathbb{C}$. Assume that either X is minimal or that* $\operatorname{Alb} X$ *is simple. Then there does not exist a nowhere vanishing $\eta \in H^0(X, \Omega_X)$.*

Conjecture: (*) implies that there exists a smooth $\alpha\colon X \to A \times V$ with A an abelian variety of dimension > 0 such that every nowhere vanishing 1-form on X is $\alpha^*\theta$ for some nowhere vanishing 1-form θ on $A \times V$,

Does (*) imply that there exists a smooth $X \to A$, where A is an abelian variety of dimension > 0?

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