EQUATIONS FOR $\overline{M}_{0,n}$

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$\overline{M}_{0,n}$ represents the functor $\mathcal{F}\colon$ **Schemes** $\to$ **Sets** where $\mathcal{F}(T)$ is the set of the proper flat maps $S \to T$ with n sections $T \to S$ such that the geometric fibers are reduced connected curves of genus 0 with nodal singularities and without automorphisms fixing marked points and nodes.

It is a smooth variety.

What are the equations of $\overline{M}_{0,n}$?

For example, $\overline{M}_{0,4} = \mathbb{P}^1$, and $\overline{M}_{0,5}$ is a Del Pezzo surface of degree 5.

Here are two applications:

(1) Mumford: Describe the ample cone of $\overline{M}_{g,n}$.

(2) Fulton conjecture: D is nef $\iff$ $D.C \geq 0$, where C is a 1-stratum of $\overline{M}_{g,n}$. (Note: $\overline{M}_{g,n}$ is stratified by the number of nodes.)

Gibney-Keel-Morrison, Keel-McKernan: D is nef on $\overline{M}_{g,n}$ $\iff$ $D.C \geq 0$ and $D|_F$ is nef, where F is the set of curves consisting of a $\mathbb{P}^1$ with n points on it and g nodal curves attached (at other points); this F is an image of $\overline{M}_{0,g+n}/S_g$.

1) Approach via degenerations.

2) To describe effective divisors of $\overline{M}_{0,n}$.

Conjecture 0.1 (Keel-Hu)**.** The total coordinate ring $\bigoplus_{\text{effective } L} H^0(\overline{M}_{0,n}, L)$ of $\overline{M}_{0,n}$ is finitely generated.

Let $B := \overline{M}_{0,n} - M_{0,n}$ (a divisor with simple normal crossings). The components of B are in bijection with partitions $\{1, \dots, n\} = I \coprod I^c$. Given I, I^c, consider the image of $\overline{M}_{0,|I|+1} \times \overline{M}_{0,|I^c|+1}$ obtained by attaching the two curves at one of the marked points on each.

Theorem 0.2. *The log canonical line bundle $K = K_{\overline{M}_{0,n}} + B$ is very ample.*

Fix a flag of subsets

$$\{1, \dots, n\} \supset S_{n-1} \supset S_{n-2} \supset \cdots \supset S_3$$

with $|S_i| = i$. There is a canonical identification $H^0(\overline{M}_{0,n}, K) = \mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \cdots \otimes \mathbb{C}^{n-2} = V$. Then $\overline{M}_{0,n} \subseteq \mathbb{P}(V)$ factors through the Segre embedding

$$\overline{M}_{0,n} \subseteq \mathbb{P}^1 \times \mathbb{P}^2 \times \cdots \times \mathbb{P}^{n-3} \subseteq \mathbb{P}(V).$$

The homogeneous ideal of $\overline{M}_{0,n}$ is a sum of homogeneous ideals of all $\sigma(\mathbb{P}^1 \times \cdots \times \mathbb{P}^{n-3})$ for all $\sigma \in S_n$. Here S_n acts on $\overline{M}_{0,n}$ and $H^0(\overline{M}_{0,n}, K)$, and

$$\overline{M}_{0,n} = \bigcap_{\sigma \in S_n} \sigma(\mathbb{P}^1 \times \mathbb{P}^{n-3}).$$

In particular, $\overline{M}_{0,n}$ is cut out by explicit quadrics of rank 4.

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What is the action of S_n on $H^0(\overline{M}_{0,n}, K) = H_{n-3}(M_{0,n}, \mathbb{C})$? This is called the Whitehouse module: it was introduced by Kontsevich as a building block of a Lie operad.

Let L be a free Lie algebra in generators $x_1, \dots, x_n$. Then S_n acts on L. Add a formal invariant scalar product on L: consider bilinear symbols $\{a, b\}$ where $a, b \in L$ modulo the relations $\{[a, b], c\} = -\{b, [a, c]\}$. Let W be the symbols that are linear in each of $x_1, \dots, x_n$. Then W has a basis of elements of the form

$$\left\{x_n, [[[x_{n-1}, x_{i_1}], x_{i_2}], \dots, x_{i_{n-2}}]\right\}:$$

there are $(n-2)!$ such basis vectors. Then $W \simeq H^0(\overline{M}_{0,n}, K) = H_{n-3}(M_{0,n}, \mathbb{C})$.

Theorem 0.3. *Let $R := \bigoplus_{\ell \geq 0} H^0(\overline{M}_{0,n}, \ell K)$. Let R be a* Koszul quadratic algebra*; i.e., the trivial R-module $k = \mathbb{C}$ has a resolution of the form*

$$\cdots \to R[-2]^{a_2} \to R[-1]^{a_1} \to R \to k.$$

Theorem 0.4. *Consider $\overline{M}_{0,n} \subseteq \mathbb{P}^1 \times \mathbb{P}^2 \times \cdots \times \mathbb{P}^{n-3}$. Then this embedding satisfies the analogue of Green-Lazarsfeld's property N_p for any p; i.e., if*

$$B = \bigoplus_{\ell \geq 0} H^0(\mathbb{P}^1 \times \cdots \times \mathbb{P}^{n-3}, \mathcal{O}(\ell) \otimes \cdots \otimes \mathcal{O}(\ell)),$$

then we have a resolution of the form

$$\cdots \to B[-4]^{a_4} \to B[-3]^{a_3} \to B[-2]^{a_2} \to B \to R \to 0.$$

Remark 0.5. An irreducible $X \subseteq \mathbb{P}^r$ satisfies N_p for every p if and only if X is of minimal degree: $\deg X = \operatorname{codim} X + 1$. This holds if and only if X is a rational normal curve, a rational normal scroll, a quadratic hypersurfaces, or a cone over the Veronese surface in $\mathbb{P}^5$.

We have $\pi\colon \overline{M}_{0,n} \to \overline{M}_{0,n-1}$ (the map that drops a point, and stabilizes). This is the universal family over $\overline{M}_{0,n-1}$. Here $K = \pi^* K' \otimes \omega_\pi(\Sigma)$ where $\Sigma \in \operatorname{Div} \overline{M}_{0,n}$ is the sum of the $n-1$ sections in the universal family over $\overline{M}_{0,n-1}$, and ω_{pi} is the dualizing sheaf. Let $\Psi = \omega_\pi(\Sigma)$. Then

$$H^0(\overline{M}_{0,n}, K) = H^0(\overline{M}_{0,n-1}, K') \otimes H^0(\overline{M}_{0,n}, \Psi).$$

Here Ψ is nef and big, and hence gives

$$\Psi\colon \overline{M}_{0,n} \to \mathbb{P}^{n-3}.$$

The image of $M_{0,n}$ is the complement of the hyperplane arrangement.

For example, $\overline{M}_{0,5}$ is the blowup of $\mathbb{P}^2$ in four points. We have

$$\Psi(\Sigma) = \{p_1, \dots, p_{n-1}\} \subseteq \mathbb{P}^{n-3}.$$

And $\overline{M}_{0,n}$ is an iterated blowup of $\mathbb{P}^{n-3}$ in these points, and the proper transforms of connecting them are lines.

Define a *Veronese curve* to be $\Psi(C) \subseteq \mathbb{P}^{n-3}$ where C is a fiber of π. Hence one can embed $\overline{M}_{0,n-1}$ in the Hilbert scheme corresponding to rational normal curves.

Also $H^0(\overline{M}_{0,n}, K) = \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^{n-2}$.

Basically, what we were doing was to find an embedding $\overline{M}_{0,n} \subseteq \overline{M}_{0,n-1} \times \mathbb{P}^{n-3}$.

Notation: L is a globally generated line bundle, define V_L by the exact sequence

$$0 \to V_L \to H^0(X, L) \otimes \mathcal{O}_X \to L \to 0.$$

Theorem 0.6. *$R^1\pi_*(\bigwedge^k V_\Psi)$ is a vector bundle on $\overline{M}_{0,n}$, called a* syzygy bundle*. Its fiber at a point corresponding to a curve C is $\mathrm{Tor}^k_A(\mathbb{C}, B)$, where A is the homogeneous coordinate ring of $\mathbb{P}^{n-3}$ and B is the homogeneous coordinate ring of C.*

Here

$$\mathcal{M}_1 = \{\text{quadrics vanishing on } C\}$$
$$\mathcal{M}_2 = \{\text{syzygies of these quadrics}\},$$

and $\mathcal{O}_{\overline{M}_{0,n}}$ has a resolution of the form

$$\cdots \to \mathcal{M}^2 \otimes \mathcal{O}(-3) \to \mathcal{M}^1 \otimes \mathcal{O}(-2) \to \mathcal{O}_{\overline{M}_{0,n-1}\times\mathbb{P}^{n-3}} \to \mathcal{O}_{\overline{M}_{0,n}} \to 0.$$

Remark 0.7. Any Veronese curve satisfies N_p for any p.

If $X \to \mathbb{P}^r$, and $X \hookrightarrow X \to \mathbb{P}^r$. There is a resolution for $\mathcal{O}_X$ inside $\mathcal{O}_X \otimes \mathcal{O}_{\mathbb{P}^r}$.

Next step: resolution of coordinate ring of $\overline{M}_{0,n}$ over the coordinate ring of $\overline{M}_{0,n-1} \times \mathbb{P}^{n-3}$.

By induction, $\overline{M}_{0,n-1} \times \mathbb{P}^{n-3}$ is Koszul (Segre products of Koszul rings is Koszul).

Polischuk's trick: If $A \to B$ is a homomorphism, and A is Koszul, and B is 1-linear over A, then B is Koszul.