LINEARLY PRESENTED IDEALS AND RATIONAL MAPS OF PROJECTIVE SPACE

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1. Varieties of minimal degree

Let $\mathbb{C}$ be an algebraically closed field (of arbitrary characteristic!) Let $\mathbb{P}^r = \mathbb{P}^r_{\mathbb{C}}$ with coordinates $x_0, \ldots, x_r$. Let $S = \mathbb{C}[x_0, \ldots, x_r]$.

Varieties are irreducible and reduced. They correspond to homogeneous prime ideals in S. More generally one has algebraic sets, which correspond to intersections of prime ideals, and schemes which correspond to arbitrary ideals.

The degree $\deg V$ of a variety $V \subset \mathbb{P}^r$ is the number of points in which V meets a general plane of complementary dimension.

$\operatorname{Span} V$ is the minimal linear space containing V.

Elementary proposition: $\deg V \geq \dim \operatorname{Span} V - \dim V + 1$.

Proof: Induction. Intersect with hyperplane, or perform a linear projection.

Definition 1.1. A variety is of *minimal degree* if equality holds.

Conics are rational curves. For instance, there is the image of

$$\mathbb{P}^1 \to \mathbb{P}^2$$
$$(s,t) \mapsto (s^2, st, t^2).$$

Similarly, the image of

$$\mathbb{P}^1 \to \mathbb{P}^3$$
$$(s,t) \mapsto (s^3, s^2t, st^2, t^3)$$

is of minimal degree. This is called the *twisted cubic curve.* (The degree is the number of zeros of a generic cubic form in s, t, hence 3.)

Similarly, the *rational normal curve $S(n)$ of degree n in $\mathbb{P}^n$* is of minimal degree.

The *Veronese surface* is the image of

$$\mathbb{P}^2 \to \mathbb{P}^5$$
$$(s,t,u) \mapsto (s^2, \ldots, t^2).$$

Its degree is the number of intersection points of two conics, which equals 4. Hence it is of minimal degree.

The image of

$$\mathbb{P}^2 \to \mathbb{P}^9$$
$$(s,t,u) \mapsto (s^3, \ldots, t^3)$$

has degree 9, but $\dim \operatorname{Span} V - \dim V + 1 = 8$.

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2. Rational normal scrolls

In $\mathbb{P}^6$ we have a disjoint $\mathbb{P}^2$ and $\mathbb{P}^3$ (think of the direct sum of vector spaces of dimension 3 and 4). Map $\mathbb{P}^1$ into the $\mathbb{P}^2$ and $\mathbb{P}^3$ as conic and twisted cubic, and take the union of the line joining each pair. This is a surface $S(2,3)$ of minimal degree.

More generally, one can define $S(d_1, \dots, d_k)$ for any nonnegative integers by taking a union of $\mathbb{P}^{k-1}$'s. All these have minimal degree.

3. Classification

Theorem 3.1 (Del Pezzo 1886, Bertini 1907)**.** *A variety of minimal degree is either a quadric hypersurface or a cone over the Veronese surface or a rational normal scroll.*

Does the inequality

$$\deg V \geq \dim \operatorname{Span} V - \dim V + 1.$$

hold for algebraic sets?

No, not even for equidimensional algebraic sets. Consider the union of two skew lines in $\mathbb{P}^3$: we get

$$2 \geq 3 - 1 + 1,$$

which is wrong.

Fix: One should check intersections with arbitrary linear spaces. We said $X^s \subseteq \mathbb{P}^r$ is minimal means that $X \cap \mathbb{P}^{r-s}$ is linearly independent for a general $\mathbb{P}^{r-s}$. We say $X^s \subseteq \mathbb{P}^r$ is small if $X \cap \mathbb{P}^n$ is linearly independent for a plane of any dimension, whenever the intersection is finite. (One should take the intersection scheme-theoretically and define linearly independent appropriately.)

4. Castelnuovo-Mumford regularity

Let M be a graded finitely generated module over S. Choose a resolution

$$0 \leftarrow M \leftarrow F_0 \leftarrow F_1 \leftarrow \cdots$$

where $F_j = \bigoplus Se_{ji}$ with $\deg e_{ji} = d_{ji}$. Define the *regularity* as

$$\operatorname{reg} M := \max\{d_{ji} - j\}.$$

For $X \subseteq \mathbb{P}^r$ corresponding to a graded S-ideal $I = I(X)$, define $\operatorname{reg} X := \operatorname{reg} I$.

Theorem 4.1. *For a variety X in $\mathbb{P}^r$,*

$$X \text{ has minimal degree} \iff \operatorname{reg} X \leq 2.$$

Regularity 1 means that X is a plane.

Regularity 2: If X is not contained in a hyperplane, then $I(X)$ generated by quadrics, linear relations.

For example, the ideal of the twisted cubic is generated by the minors of

$$\begin{pmatrix} s^3 & s^2t & st^2 \\ s^2t & st^2 & t^3 \end{pmatrix}.$$

Now we have

$$I \leftarrow S^3(-2) \leftarrow \cdots$$

Eisenbud, Green, Hulek, Popescu show that small varieties are the same as 2-regular varieties.

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