Semiclassical Scattering and Applications

Stanford Maths Colloquium

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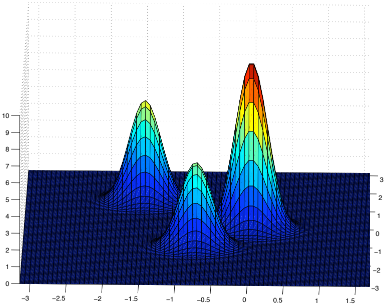
Motivation



We consider

$$ih\partial_t u = -h^2 \Delta u + V(x)u$$

$$u(x,0) = \exp\left(\frac{i}{h}\langle x, \xi_0\rangle - \frac{1}{2h}\langle x - x_0, \xi_0\rangle^2\right) .$$

$$ih\partial_t u = -h^2 \Delta u + V(x)u$$

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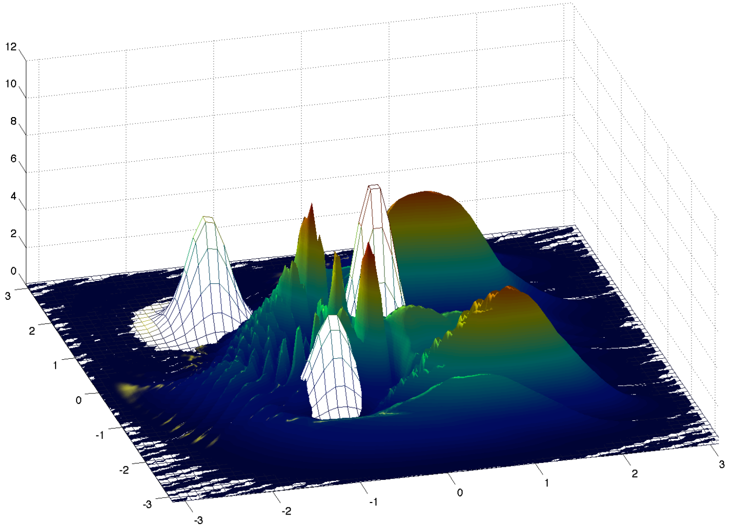

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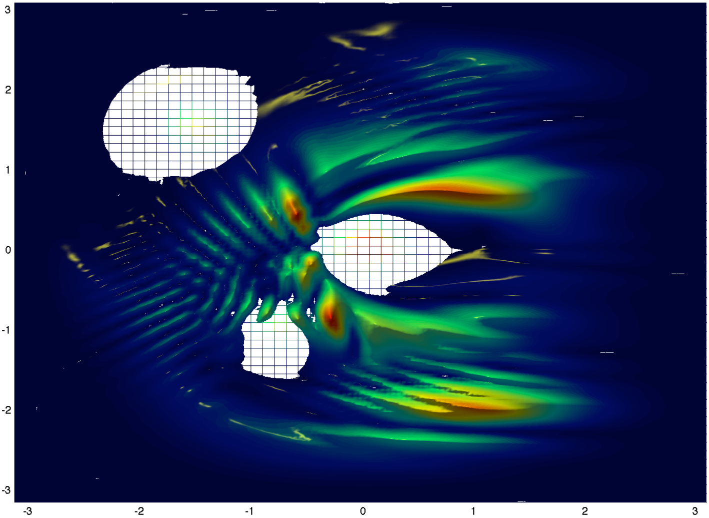
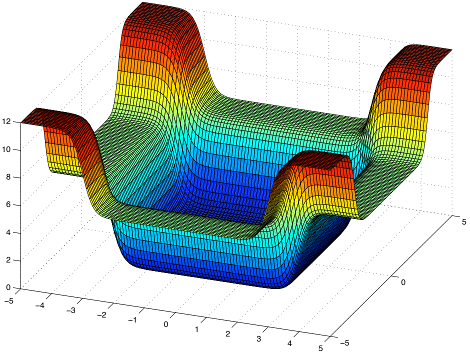


Some cheating: added complex absorbing potential $-iA(x)$:

$$ih\partial_t u = -h^2\Delta u + V(x)u - iA(x)u(x)$$



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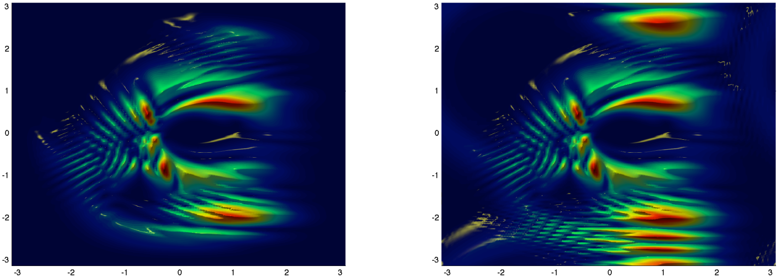


No qualitative change in the interaction region.

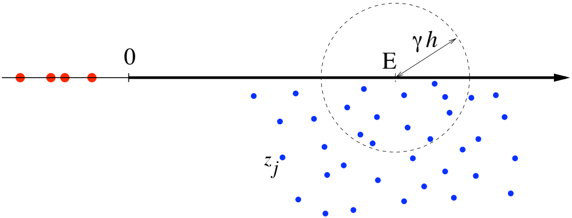
To do this correctly should use the method of complex scaling.

Aguilar-Combes 1971, Balslev-Combes 1971, Simon 1973, Hunziker 1986, Helffer-Sjöstrand 1985, Sjöstrand-Z 1991...

In numerical analysis known as perfectly matched layers (PML)

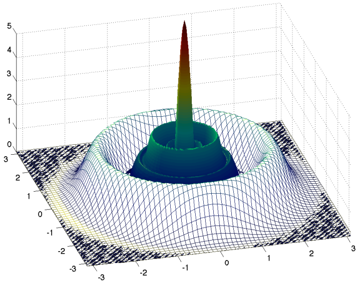
Berenger 1994...

The motivation in mathematical physics came from the study of quantum resonances:



The resolvent $(P(h) - z)^{-1}$ may be continued meromorphically from $\{\text{Im}\, z > 0\}$ to $\{\text{Im}\, z < 0\}$. Its poles $\{z_j(h)\}$ are the **resonances** of $P(h)$.

A strongly trapped example:

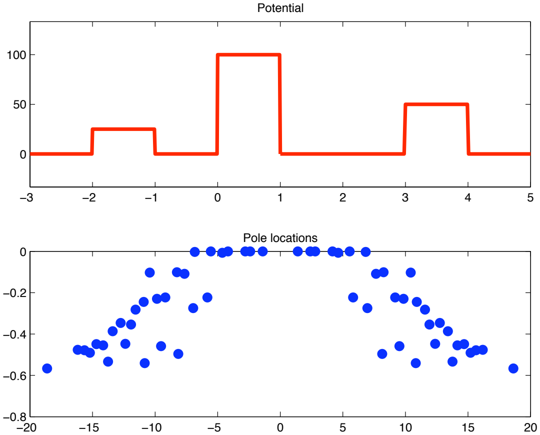


Nakamura-Stefanov-Z 2003

In the example the classical system is completely integrable and

$$u(x,0) = \exp\left(\frac{i}{h}\langle x\rangle - \frac{1}{2h}|x|^2\right) .$$

Quantum Resonances describe these waves resonating in interaction regions.



Computed using `squarepot.m`

`http://www.cims.nyu.edu/∼dbindel/resonant1d/`

Here is how they sound:

```
time = linspace(0,500,5000);
sound(real(exp(-i*z*time)))
```

$$P(h) = -h^2\Delta + V(x) \qquad\qquad p(x,\xi) = \xi^2 + V(x)$$

Classical Flow/Newton's equations

$$\dot{x} = 2\xi\,, \quad \dot{\xi} = -dV(x)\,, \quad \Phi^t(x(0),\xi(0)) = (x(t),\xi(t))\,.$$

How do the properties of the flow affect solutions of PDEs?

That is, how does the classical/quantum correspondence manifests itself in PDEs? Here are some of the different scenarios:

Non trapping $\longleftrightarrow$ *Trapping*

Elliptic trapping $\longleftrightarrow$ *Hyperbolic trapping*

Euclidean infinity $\longleftrightarrow$ *Hyperbolic infinity*

Note that hyperbolic can be meant in three different ways: dynamical, geometric, and (yet to come) analytical (PDE).

Resolvent estimates

Consider the classical flow on $\xi^2 + V(x) = E$:

$$\dot{x} = 2\xi \,, \quad \dot{\xi} = -dV(x) \,, \quad \Phi^t(x(0), \xi(0)) = (x(t), \xi(t)) \,.$$

No trapping (i.e. all trajectories escape):

$$\chi(P(h) - z - i0)^{-1}\chi = O(1/h) \ : \ L^2 \longrightarrow L^2 \,, \quad \chi \in C_c^\infty \,,$$

for $|z - E| \ll 1$.

... Gérard-Martinez 1988, Vasy-Z 2000, Cardoso-Vodev 2002, ...

(The cut-off χ can be replaced by a suitable weight.)

A quick explanation of the estimate:

Consider the simplest non-trapping Hamiltonian:

$$p = \xi_1$$

Then $P(h) = hD_{x_1}$.

Suppose χ vanishes for $|x| > R$.

The fundamental theorem of calculus shows that

$$\chi(P(h) - z - i0)^{-1}\chi = O(1/h) \ : \ L^2 \longrightarrow L^2 \,.$$

Note that

$$(hD_{x_1} - i0)^{-1} f(x) = \frac{i}{h} \int_{-\infty}^{x_1} f(t, x_2, \cdots, x_n) dt \,.$$

Real difficulties at infinity...

Resolvent estimates

Consider the classical flow on $\xi^2 + V(x) = E$:

$$\dot{x} = 2\xi\,, \quad \dot{\xi} = -dV(x)\,, \quad \Phi^t(x(0), \xi(0)) = (x(t), \xi(t))\,.$$

Trapping (i.e. some trajectories of energy E never escape):

$$\|\chi(P(h) - z - i0)^{-1}\chi\| \geq \log(1/h)/h\,, \quad |z - E| \ll 1\,.$$

Bony-Burq-Ramond 2010

A simple but striking observation...

All this applies to very general operators not just $P(h) = -h^2\Delta + V(x)$, including operators on manifolds.

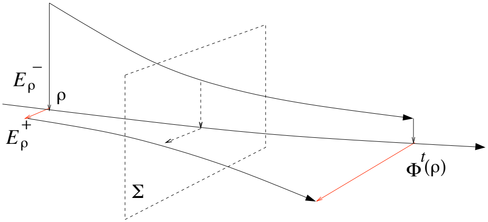
Suppose the classical flow on $\xi^2 + V(x) = E$:

$$\dot{x} = 2\xi \,, \quad \dot{\xi} = -dV(x) \,, \quad \Phi^t(x(0), \xi(0)) = (x(t), \xi(t))$$

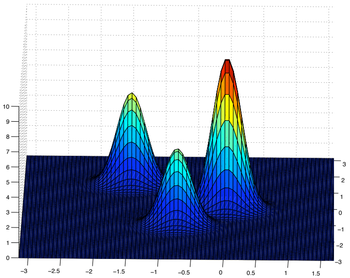
is uniformly hyperbolic:

$$\Gamma_E^{\pm} = \{\rho : \Phi^t(\rho) \not\to \infty \,, \ t \to \mp\infty\} \,, \quad K_E = \Gamma_E^+ \cap \Gamma_E^- \,,$$

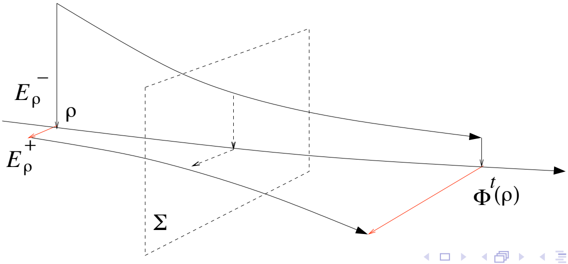
and for $\rho \in K_E$

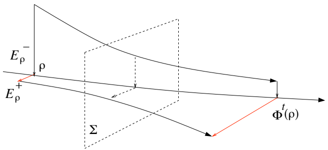


$$p(x,\xi) = \xi^2 + V(x)\,, \quad P(h) = -h^2\Delta + V(x)\,,$$



$E > 0$ below the lowest peak





We define the topological pressure associated to the unstable Jacobian:

$$J_t^+(\rho) = \det\left(d\Phi^t_{|E_\rho^+}\right)$$

$$\mathcal{P}_E(s) = \lim_{T\to\infty} \frac{1}{T} \log \sum_{T_\gamma < T} J^+(\gamma)^{-s},$$

where γ are closed orbits with period T_γ.

Theorem (Nonnemacher-Z 2009)

Assume the topological pressure $\mathcal{P}_E(1/2) < 0$. Then for $|z - E| \ll 1$,

$$\chi(P(h) - z - i0)^{-1}\chi = O(\log(1/h)/h) \;:\; L^2 \longrightarrow L^2 .$$

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In dimension $d = 2$, the gap condition $\mathcal{P}_E(1/2) < 0$ is equivalent to

$$\dim K_E < 2$$

i.e. the trapped set is "filamentary", $1 \leq \dim K_E \leq 3$, in general.

Recall that Bony-Burq-Ramond 2010 taught us that

$$K_E \neq \emptyset \quad \Longrightarrow \quad \|\chi(P(h) - z - i0)^{-1}\chi\| \geq \log(1/h)/h \,.$$

Applications

- Resolvent estimates on manifolds with more complicated infinities Datchev 2008, Datchev-Vasy 2010.
- Local smoothing for the Schrödinger equation with a log loss of regularity Datchev 2008; follows the classical approach of Kato, brought to this setting by Burq.
- Exponential decay of energy for wave equations on manifolds Christianson 2009
- Exponential decay of damped waves $\partial_t^2 - a(x)\partial_t - \Delta_g$ under a pressure condition on $a(x)$ Schenck 2010
- Strichartz estimates with no loss despite trapping Burq-Guillarmou-Hassell 2010
- Unique quantum ergodicity for (certain) manifolds of infinite volume Guillarmou-Naud 2011

(each item has its own major challenges...)

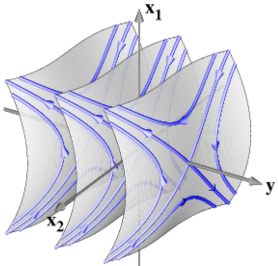
Another setting for hyperbolic systems:

Normally hyperbolic trapped sets

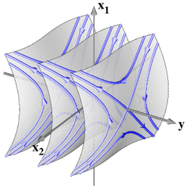
K_E , $\Gamma^{\pm}_E$ are smooth and the hyperbolicity condition holds on K_E,

$$E^{\pm}_{\rho} = T_{\rho}\Gamma^{\pm}_E , \ \rho \in K_E , \quad \dim K_E + 2 = \dim p^{-1}(E) = 2d - 1 .$$

This dynamical structure is stable under perturbations
Hirsh-Pugh-Shub 1977 (smoothness means some finite, but arbitrary large, regularity).



Resolvent estimates



Theorem (Wunsch-Z 2010)

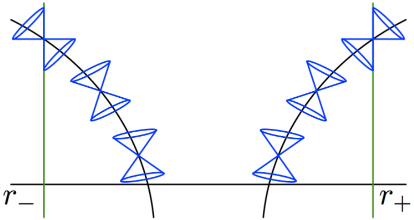
Assume normal hyperbolicity. Then, for $|z - E| \ll 1$,

$$\chi(P(h) - z - i0)^{-1}\chi = O(\log(1/h)/h) \ : \ L^2 \longrightarrow L^2 \, .$$

When $d = 2$ this is the case of a single closed hyperbolic orbit: $\dim K_E = 2d - 3 = 1$. Then the theorem follows from the work of Christianson 2008 who covered the case of arbitrary closed (weakly) hyperbolic orbit in any dimension.

Applications

Normally hyperbolic trapped sets occur in the geometry of the Schwarzschild, Kerr, and Kerr-DeSitter black holes.



Two radial timelike geodesics, with light cones shown; $r = r_\pm$ are the event horizons

But for that we need to consider infinities which are different from the Euclidean case.

Why?

Gravitational wave detectors: GEO 600, LIGO, MiniGRAIL, VIRGO, . . .

Quasi-normal modes (QNMs) are the frequencies of the gravitational waves emitted by a black hole.

In principle we would like to "listen" to them as we did with resonances.

Why?

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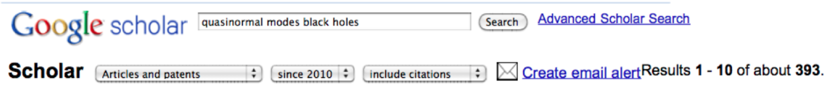


Most of it, alas, by string theorists...

There are many works by physicists on quasi-normal modes; however, there have been only a handful of attempts to put these works on a mathematical foundation: Bachelot 1991, Bachelot–Motet-Bachelot 1993, Sá Barreto–Z 1997, Bony–Häfner 2007, Melrose–Sá Barreto–Vasy 2008,

Decay of the wave equation for black hole metrics has been much studied recently: Bony–Häfner 2007, 2010, Dafermos–Rodnianski 2007, 2008, 2009,2010, Donninger–Schlag–Soffer 2009, Finster–Kamran–Smoller–Yau 2009, Marzuola–Metcalfe–Tataru–Tohaneanu 2008, Tataru 2009, Tataru–Tohaneanu 2008. . .

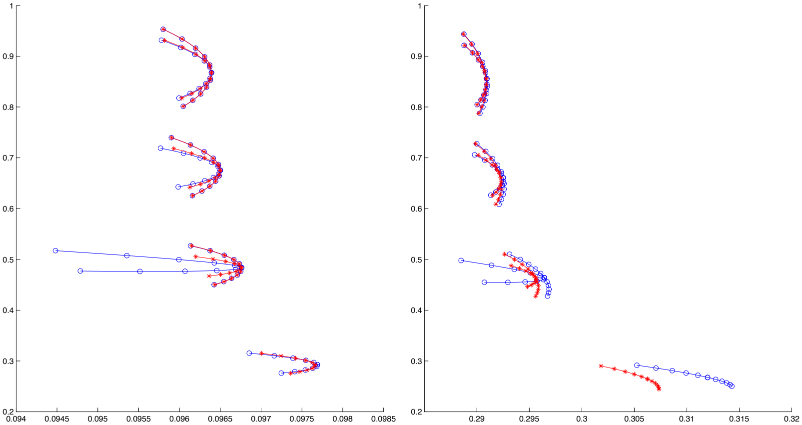
Applications of normally hyperbolic estimates

Dyatlov 2010: exponential decay of waves for the Kerr-DeSitter metric

Vasy 2011: exponential decay for general "Kerr-DeSitter" metrics without any symmetry assumptions (fully uses the general result about normally hyperbolic trapped sets).

Dyatlov 2011: asymptotic behaviour of quasinormal modes for Kerr (rotating) black holes

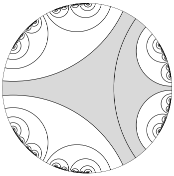
Comparison of Dyatlov's semiclassical results with exact numerical results of Berti-Cardoso-Starinets 2009 for low energy [sic!] QNM:

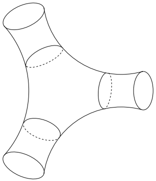


The real and imaginary axis interchanged; semiclassical and exact; the different points correspond to increase in rotation of the black hole: Zeeman effect.

These quasinormal modes are poles of the meromorphic continuation of the resolvent $(P - z)^{-1}$.

Different kinds of infinities:





Mazzeo-Melrose 1987: modeled on $\Gamma \backslash \mathbb{H}^n$, geometrically finite without parabolic elements (mixed ranked cusps).

Guillarmou 2004 – a more precise statement...

Mazzeo-Guilllarmou 2010: modeled on $\Gamma \backslash \mathbb{H}^n$, geometrically finite with parabolic elements.

Many contributions in the meantime: Froese-Hislop-Perry 1991, Guillope-Z 1995, Bunke-Olbrich 1999, ...

But none of these results are constructive enough to give estimates on the resolvent, even if there is no trapping...

The Mazzeo-Melrose result was used to define quasinormal modes for Schwarzschild-de Sitter black holes by Sá Barreto-Z.

Vasy recently realized that the black hole setting, including Kerr-de Sitter (not covered by M-M but described by Dyatlov) is easier to study: reversing the S-Z strategy gives a simple and more effective proof of resolvent continuation for $\Gamma\backslash\mathbb{H}^n$-like spaces (no parabolic elements).

It eliminates a lot of "degenerate" microlocal analysis and replaces it by better known microlocal methods.

Simplest model

$$P = (xD_x)^2 + x^2 D_w^2 \, , \quad -1 < x < 0 \, , \quad w \in \mathbb{S}^1 \, ,$$

The hyperbolic infinity is given by $x = 0$ is (at $x = -1$, Dirichlet boundary condition, say).

We want to continue $(P - z)^{-1}$ from $\mathrm{Im}\, z > 0$ to $\mathbb{C} \setminus (-\infty, 0]$ as an operator on some weighted spaces.

That is normally achieved by obtaining some Fredholm properties.

Now, put $y = -x^2$ and change the C^∞ structure so that y is a new coordinate on our cylinder. Then

$$\widetilde{P}(z) := y^{\mu - \frac{1}{2}} (P - z) y^{-\mu - \frac{1}{2}} = -D_y y D_y + D_w^2 + \sqrt{z} D_y - \sqrt{z} \, .$$

(for some suitable $\mu = \mu(z)$; this defines the weighted spaces)

$$\widetilde{P}(z) = -D_y y D_y + D_w^2 + \sqrt{z} D_y - \sqrt{z}$$

Now forget that $y < 0$ and consider the equation on a larger cylinder. In the black hole setting this corresponds to going through the event horizon – you cannot come back but you do not even notice this happening!

The behaviour changes from elliptic to hyperbolic (the latter in the PDE rather than geometric sense now):

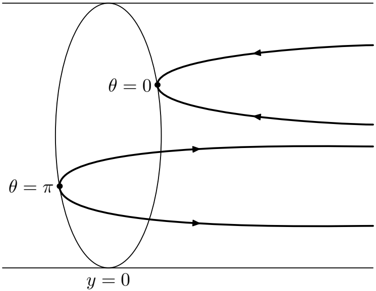
$$\widetilde{p} = -y\eta^2 + \omega^2$$

$$(y, w; \eta, \omega) \in T^*(\mathbb{R}_y \times \mathbb{S}^1_w)$$

$$\widetilde{p} = -y\eta^2 + \omega^2$$

The phase space picture is obtained by using polar coordinates for the phase variables:

$$(\eta, \omega) = \rho^{-1}(\cos\theta, \sin\theta)$$



We see the radial points at $y = 0$ and $\omega = 0$.

One still needs to modify the operator by adding a suitably chosen complex absorbing term:

$$-D_y y D_y + D_w^2 + \sqrt{z} D_y - \sqrt{z} - iA(y, w, D)\,,$$

$$A(y, w, D) = 0\,, \quad y \leq 0$$

Since $y = 0$ is characteristic A supported in $y > 0$ does not affect the solution in $y < 0$.

The classical propagation estimates of Hörmander away from the radial points, and the estimates developed by Melrose for asymptotically Euclidean scattering near the radial points, give the desired Fredholm properties.

Vasy's method combined with the "gluing" arguments of Datchev-Vasy gives also semiclassical estimates based on the hyperbolic (now in the dynamical sense!) analysis of Nonnenmacher-Z and Wunsch-Z:

Theorem (Datchev 2008, Datchev-Vasy 2010, Vasy 2011)

Assume hyperbolicity on the trapped set and the pressure assumption, or the normal hyperbolicity. Then, for asymptotically euclidean, asymptotically hyperbolic, or asymptotically Kerr-de Sitter (with some modifications in the statement), we have

$$\chi(P(h) - z - i0)^{-1}\chi = O(\log(1/h)/h) \; : \; L^2 \longrightarrow L^2 \, .$$

Hence many things developed for Euclidean infinities are now established for more general infinities.

And the black hole examples show that this is not a totally "academic" direction...