SCATTERING RESONANCES AS VISCOSITY LIMITS

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Abstract. Using the method of complex scaling we show that scattering resonances of $-\Delta + V$, $V \in L^\infty_{\rm c}(\mathbb{R}^n)$, are limits of eigenvalues of $-\Delta + V - i\varepsilon x^2$ as $\varepsilon \to 0+$. That justifies a method proposed in computational chemistry and reflects a general principle for resonances in other settings.

1. Introduction and statement of results

In this note we show that scattering resonances can be defined as viscosity limits, that is limits of eigenvalues of Hamiltonians suitably regularized as infinity. The detailed proofs are presented in the simplest case of the Schrödinger operator with a compactly supported potential and rely only on standard techniques.

We consider

$$P := -\Delta + V, \quad V \in L^\infty_{\rm comp}(\mathbb{R}^n),$$

where $L^\infty_{\rm comp}$ denotes the spaces of bounded functions vanishing outside of some compact set. (Similarly the subscript $L^\bullet_{\rm loc}$ denotes function in the space $L^\bullet$ on compact sets.) The scattering resonances are defined as the poles of the meromorphic continuation of resolvent:

$$R_V(z) := (-\Delta + V - z)^{-1} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n), \quad \operatorname{Im} z > 0,$$

from the upper half-plane through the continuous spectrum. More precisely,

$$R_V(z) : L^2_{\rm comp}(\mathbb{R}^n) \to L^2_{\rm loc}(\mathbb{R}^n), \tag{1.1}$$

continues meromorphically to the double cover of $\mathbb{C}$ when n is odd and to the logarithmic cover of $\mathbb{C}$ when n is even. The poles of this continuation coincide with the poles of the scattering matrix for the potential V. Their multiplicity (except at the threshold $z = 0$) are given by

$$m(z) := \operatorname{rank} \oint_z R_V(\zeta) d\zeta, \tag{1.2}$$

where the integration is over a small circle around z – see [DyZ2, Chapter 3].

Equivalently, we can consider Green's function, that is the integral kernel of $R_V(z)$,

$$R_V(z)f(x) = \int_{\mathbb{R}^n} G(z, x, y) f(y) dy, \tag{1.3}$$

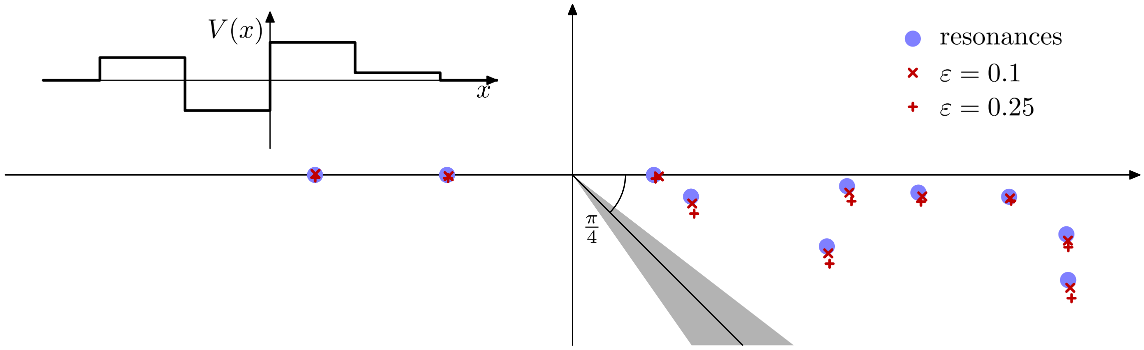

FIGURE 1. An illustration of the results of Theorem 1 in the case of a specific potential shown on the left. Resonances are computed using `squarepot.m` [BiZ]. The eigenvalues of P_ε, $\varepsilon = 1/4$ and $\varepsilon = 1/10$ are computing by discretizing the operator using using the first 151 eigenfunctions of the harmonic oscillator $D_x^2 + x^2$.

and look at the poles of the continuation of $z \mapsto G(z, x, y)$ for x and y fixed. Another way, based on the method of complex scaling, will be reviewed in §2.

We now consider a *regularized* operator,

$$P_\varepsilon := -\Delta + V - i\varepsilon x^2, \quad \varepsilon > 0. \tag{1.4}$$

It is easy to see (with details reviewed in §4) that P_ε is an unbounded operator on $L^2(\mathbb{R}^n)$ with a discrete spectrum. We have

Theorem 1. *Suppose that* $\{z_j(\varepsilon)\}_{j=1}^\infty$ *are the eigenvalues of* P_ε. *Then, uniformly on compact subsets of* $\{z : -\pi/4 < \arg z < 7\pi/4\}$,

$$z_j(\varepsilon) \to z_j, \quad \varepsilon \to 0+,$$

where z_j *are the resonances of* P.

Remarks. 1. A more precise statement involving continuity of spectral projections is given in §5. The term viscosity is motivated by the viscosity definition of Pollicott–Ruelle resonances given in Dyatlov–Zworski [DyZ1] – see Example 3 below.

2. When $\varepsilon < 0$ the spectrum of P_ε is given by complex conjugates of the spectrum of $P_{-\varepsilon}$. Hence we have

$$z_j(\varepsilon) \to \bar{z}_j, \quad \varepsilon \to 0-, \tag{1.5}$$

uniformly on compact subsets of $\{z : -7\pi/4 < \arg z < \pi/4\}$.

3. The term $-i\varepsilon x^2$ is an example of a *complex absorbing potential* and other potentials can also be used – see the discussion below. The proof here requires some analyticity properties of the complex absorbing potential.

4. The restriction to $\arg z > -\pi/4$ when using $-i\varepsilon x^2$ is due to the fact that for $V \equiv 0$ the spectrum of $-\Delta - i\varepsilon x^2$ is given by $\varepsilon^{\frac{1}{2}} e^{-\pi i/4}(2|\ell| + n)$, $\ell \in \mathbb{N}^n$ which is a rescaled spectrum of the Davies harmonic oscillator – see §3. One can expand the range using $\varepsilon e^{-i\alpha} x^2$, $0 < \alpha < \pi$ in which case we recover resonances with $\arg z > -\alpha/2$.

5. The proof applies with simple modifications to compactly supported *black box* perturbations on $\mathbb{R}^n$ introduced in [SjZ] – see [DyZ2, Chapter 4] and [Sj3]. In that case we need to replace $-i\varepsilon x^2$ by $-i\varepsilon(1 - \chi(x))x^2$ where $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ is equal to 1 on a sufficiently large set – see Example 2 below.

The computational method based on calculating eigenvalues of P_ε was introduced in physical chemistry – see Riss–Meyer [RiMe] and Seideman–Miller [SeMi] for the original approach and Jagau et al [JZBRK] for some recent developments and references. However no rigorous mathematical treatments seem to be available and some new interesting open questions can be posed – see Example 4 below.

Fixed complex absorbing potentials have already been used in mathematical literature on scattering resonances. Stefanov [St] showed that semiclassical resonances close to the real axis can be well approximated using eigenvalues of the Hamiltonian modified by a complex absorbing potential. Nonnenmacher–Zworski [NZ1],[NZ2] used fixed complex absorbing potentials to study resonance problems employing gluing techniques of Datchev–Vasy [DV1],[DV2]. Yet another application was given by Vasy in [V] where microlocal complex absorbing potentials were used to obtain Fredholm properties and meromorphic continuation of the resolvents (see also [DyZ2, Chapter 5]).

We conclude this section with some examples to which Theorem 1 does *not* apply directly but which fit in the same framework.

Example 1. As explained in [Sj2, (c.31)–(c.33)] the theory of Helffer–Sjöstrand [HeSj] applies to the case of potentials which are homogeneous of degree m and satisfy the condition $V(x) = 0,\ x \neq 0 \Longrightarrow \nabla V(x) \neq 0$. That means that resonances of $P = -\Delta + V$ can be defined in $\{z \in \mathbb{C}, \arg z > -\theta_0\}$ for some $\theta_0 > 0$. It is interesting to ask if the viscosity limit gives a global definition in that case.

That is easily seen in the case of quadratic potentials. In fact, suppose that

$$V(x) = \lambda_1^2 x_1^2 + \cdots + \lambda_r^2 x_r^2 - \mu_1^2 x_{r+1}^2 - \cdots - \mu_{n-r}^2 x_n^2, \quad \lambda_j, \mu_\ell > 0.$$

As recalled in §3 the eigenvalues of P_ε, $\varepsilon > 0$, are given by

$$\sum_{j=1}^{r} (\lambda_j^2 - i\varepsilon)^{\frac{1}{2}} (2k_j + 1) - i \sum_{j=1}^{n-r} (\mu_j^2 - i\varepsilon)^{\frac{1}{2}} (2k_{j+r} + 1), \quad k \in \mathbb{N}_0^n,$$

where the branch of the square root is chosen to be positive on $\mathbb{R}_+$. As $\varepsilon \to 0+$ we obtain the globally defined set of resonances:

$$\sum_{j=1}^{r} \lambda_j (2k_j + 1) - i \sum_{j=1}^{n-r} \mu_j (2k_{j+r} + 1), \quad k \in \mathbb{N}_0^n.$$

Example 2. This example fits in the framework of black box scattering with one dimensional infinity. Consider the modular surface $M = SL_2(\mathbb{Z}) \backslash \mathbb{H}^2$ and $\Delta_M \leq 0$ the Laplacian on M. We then put $P = -\Delta_M - \frac{1}{4}$ where $\frac{1}{4}$ guarantees that the the continuous spectrum of P is given $[0, \infty)$. This is a *black box* Hamiltonian on $\mathcal{H}_0 \oplus L^2([0,\infty))$ in the sense of [SjZ] – see [DyZ2, §4.1]. Traditionally, the resonances of the quotient M are defined as poles of the meromorphic continuation of $(-\Delta_M - s(1-s))^{-1}$ from $\operatorname{Re} s > \frac{1}{2}$ to $\mathbb{C}$ and are given by the embedded eigenvalues with $\operatorname{Re} s = \frac{1}{2}$ and by the non-trivial zeros of $\zeta(2s)$ where ζ is the Riemann zeta function. The resonances of P are then given by $s(1-s)$.

If we choose the fundamental domain of $SL_2(\mathbb{Z})$ to be $\{x + iy : |x| \leq 1, x^2 + y^2 \geq 1\}$ then the Laplacian in the cusp $y > 1$ is $y^{-2}(\partial_x^2 + \partial_y^2)$. The Hamiltonian on $L^2([0,\infty)_r)$ is given by $-\partial_r^2$, $r = \log y$ – see [DyZ2, §4.1, Example 3]. In the language of Theorem 1 (see Remark 5) and in (x, y) coordinates

$$P_\varepsilon = -\Delta_M - \tfrac{1}{4} - i\varepsilon(1 - \chi(y))(\log y)^2,$$

where $\chi \in \mathcal{C}_c^\infty([0,\infty))$, $\chi(y) \equiv 1$ for $y < \frac{3}{2}$ and $\chi(y) \equiv 1$ for $y > 2$. The operator P_ε has discrete spectrum for $\varepsilon > 0$ and the eigenvalues converge to the resonances of P uniformly on compact subsets of $\operatorname{Im} z > -\pi/4$. Equivalently if we define Σ_ε

$$s(\varepsilon) \in \Sigma_\varepsilon \Longleftrightarrow s(\varepsilon)(1 - s(\varepsilon)) \in \sigma(P_\varepsilon)$$

The limit points of Σ_ε, $\varepsilon \to 0+$, in $\operatorname{Re} s < \frac{1}{2}$, $|s| > C$ are given by the nontrivial zeros of $\zeta(2s)$.

Example 3. Suppose that X is a compact manifold and V is a vector field on X generating an Anosov flow, $\varphi^t = \exp tV$. That means that the tangent space to X has a continuous decomposition $T_x X = E_0(x) \oplus E_s(x) \oplus E_u(x)$ which is invariant, $d\varphi_t(x) E_\bullet(x) = E_\bullet(\varphi_t(x))$, $E_0(x) = \mathbb{R}V(x)$, and for some C and $\theta > 0$ fixed

$$\begin{aligned} |d\varphi_t(x)v|_{\varphi_t(x)} &\leq Ce^{-\theta|t|}|v|_x, \quad v \in E_u(x), \quad t < 0, \\ |d\varphi_t(x)v|_{\varphi_t(x)} &\leq Ce^{-\theta|t|}|v|_x, \quad v \in E_s(x), \quad t > 0. \end{aligned} \tag{1.6}$$

where $|\bullet|_y$ is given by a smooth Riemannian metric on X. A class of examples is given by $X = T^1 M$ where M is a negatively curved Riemannian manifold and φ^t is the geodesic flow in its unit tangent bundle X.

If $\Delta_g \leq 0$ is the Laplacian for some metric on X then – see [DyZ1] – the limit set of the spectrum of

$$P_\varepsilon = V/i + i\varepsilon\Delta_g, \quad \varepsilon \to 0+$$

is a discrete set given by the *Pollicott–Ruelle* resonances – see [DyZ1] for the definition and references. Adding the Laplacian corresponds to taking a *viscosity* regularization and that explains our terminology. Another interpretation is given in terms of Brownian motion: the pullback by the flow flow $x(t) := \varphi_t(x(0))$, is given by $e^{itP_0}f(x) = f(x(t))$, $\dot{x}(t) = -V_{x(t)}$, $x(0) = x$. For $\varepsilon > 0$ the evolution equation is replaced by the Langevin equation:

$$e^{-itP_\varepsilon}f(x) = \mathbb{E}\left[f(x(t))\right], \quad \dot{x}(t) = -V_{x(t)} + \sqrt{2\varepsilon}\dot{B}(t), \quad x(0) = x,$$

where $B(t)$ is the Brownian motion corresponding to the metric g on X. Hence considering P_ε corresponds to a stochastic perturbation of the deterministic flow. In the case of scattering resonances the same interpretation can be proposed on the Fourier transform side.

The assumption that the flow satisfies (1.6) is crucial as otherwise the limit set is typically not discrete. The simplest example is given by $X = \mathbb{S}^1 \times \mathbb{S}^1$, $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$, and $V = \partial_{x_1} + \alpha\partial_{x_2}$, $\alpha \notin \mathbb{Q}$, $\Delta_g = \partial_{x_1}^2 + \partial_{x_2}^2$. In that case the limit set of the spectrum of P_ε is the lower half plane. Other limit sets are possible, for instance in the case of the geodesic flow on $\mathbb{S}^2$, $X = T^1\mathbb{S}^2 \simeq SO(3)$. The spectrum of P_0 is given by $\mathbb{Z}$ (with infinite multiplicities) and if we take Δ_g to be the Casimir operator then the limit set of the spectrum of P_ε as $\varepsilon \to 0+$ is $\mathbb{Z} - i[0, \infty)$. For yet another example see [DyZ1, §1].

Example 4. We expect that viscosity definition of resonances remains valid, in a small angle near the real axis, for all *dilation analytic* potentials – see [HeSj] and references given there and §2 below for a review of complex scaling. It would be interesting to find a Schrödinger operator P for which the limit set of the spectrum of P_ε, $\varepsilon \to 0$ is not discrete. Candidates are given by potentials which are *not* dilation analytic, for instance,

$$-\partial_x^2 + \frac{\sin x}{x}, \quad x \in \mathbb{R}.$$

Notation. We use the following notation: $f = \mathcal{O}_\ell(g)_H$ means that $\|f\|_H \leq C_\ell g$ where the norm (or any seminorm) is in the space H, and the constant C_ℓ depends on ℓ. When either ℓ or H are absent then the constant is universal or the estimate is scalar, respectively. When $G = \mathcal{O}_\ell(g) : H_1 \to H_2$ then the operator $G : H_1 \to H_2$ has its norm bounded by $C_\ell g$. Also when no confusion is likely to result, we denote the operator $f \mapsto gf$ where g is a function by g.

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2. Review of complex scaling

The complex scaling method changes the original Hamiltonian $P = P_0$ to a non-self-adjoint Hamiltonian $P_{0,\theta}$ such that $P_{0,\theta} - z : H^2 \to L^2$ is a Fredholm operator when $\arg z > -2\theta$. It was introduced by Aguilar–Combes [AgCo], Balslev–Combes [BaCo] and Simon [Si]. For a review of practical applications of this method in computational chemistry see Reinhardt [Rei]. As the method of *perfectly matched layers* (PML) it has reappeared in numerical analysis – see Berenger [Be]. The presentation here follows the geometric approach of Sjöstrand–Zworski [SjZ]. Eventually the proof that the viscosity eigenvalues converge to scattering resonances is a straightforward application of the methods of [SjZ] (see also [Sj3, §7.2] for a more detailed presentation and [DyZ2, §4.5] for an approach to complex scaling based on the continuation of the Green function $G(z, x, y)$ in (1.3) in variables x and y).

Suppose that $\Omega \subset \mathbb{C}^n$ is an open subset and that

$$P(z, D_z) = \sum_{|\alpha| \le m} a_\alpha(z) D_z^\alpha, \quad D_{z_j} := \tfrac{1}{i}\partial_{z_j}, \quad D_z^\alpha = D_{z_1}^{\alpha_1} \cdots D_{z_n}^{\alpha_n}, \tag{2.1}$$

is a differential operator with holomorphic coefficients. For instance we can have $P(z, D_z) = \sum_{j=1}^n D_{z_j}^2 - i\varepsilon z_j^2$.

Suppose that $\Omega \subset \mathbb{C}^n$ is an open subset and that $\Gamma \subset \Omega$ is a *maximal totally real* submanifold. That means that Γ is a smooth real submanifold of dimension n such that

$$\forall\, x \in \Gamma, \quad T_x\Gamma \cap iT_x\Gamma = \{0\}. \tag{2.2}$$

Here we identify $T_x\Gamma$ with a real subspace of $\mathbb{C}^n$. The condition (2.2) means that there exists a *complex linear* change of variables $A : \mathbb{C}^n \to \mathbb{C}^n$ such that $A(T_x\Gamma) = \mathbb{R}^n \subset \mathbb{C}^n$. Locally, Γ can be represented using real coordinates:

$$\mathbb{R}^n \supset U \ni x \mapsto f(x) = (f_1(x), \cdots, f_n(x)) \in \Gamma \subset \Omega \subset \mathbb{C}^n. \tag{2.3}$$

Composing the matrix $\partial_x f(x) := (\partial_{x_j} f_k(x))_{1 \le k,j \le n}$ with A we obtain an invertible matrix $\mathbb{R}^n \to \mathbb{R}^n$. That means that

$$\det \left(\frac{\partial f_k(x)}{\partial x_j} \right)_{1 \le k,j \le n} \neq 0. \tag{2.4}$$

Conversely, if (2.4) holds, then $\partial f(x)$ is an injective complex linear matrix and for any sets $U, V \subset \mathbb{C}^n$, $\partial f(x)(U) \cap \partial f(x)(V) = \partial f(x)(U \cap V)$. Hence,

$$\begin{aligned} T_x\Gamma \cap iT_x\Gamma &= \partial f(x)(\mathbb{R}^n) \cap i\partial f(x)(\mathbb{R}^n) = \partial f(x)(\mathbb{R}^n) \cap \partial f(x)(i\mathbb{R}^n) \\ &= \partial f(x)(\mathbb{R}^n \cap i\mathbb{R}^n) = \{0\}, \end{aligned}$$

and (2.4) implies (2.2). The volume form on Γ is obtained by pushing forward the standard volume form on $\mathbb{R}^n$ by f. That of course depends on the choice of f (in what follows the uniformity will be guaranteed by (2.8) below).

Example. As a simple illustration consider $n = 2$ and $f(x_1, x_2) = (x_1 + ix_2, 0) \in \mathbb{C}^2$. Then

$$\partial_x f(x) = \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix}, \quad T_x f(\mathbb{R}^2) = \mathbb{C} \oplus \{0\} \subset \mathbb{C}^2.$$

The tangent space is not totally real and condition (2.4) is violated. To introduce the next topic we also note we cannot restrict operators, P, with holomorphic coefficients to $f(\mathbb{R}^2)$ in a way that for holomorphic functions, u, $(Pu)|_{f(\mathbb{R}^2)} = (P_{f(\mathbb{R}^n)})(u|_{f(\mathbb{R}^n)})$. As an example consider $P = \partial_{z_2}$ and $u = z_2$.

The point of introducing totally real submanifolds Γ is the fact that an operator, P, with holomorphic coefficients can be restricted to an operator with complex smooth coefficients on Γ, P_Γ, in such a way that for u holomorphic near Γ, $Pu|_\Gamma = P_\Gamma(u|_\Gamma)$.

The differential operator $P(z, D_z)$ given in (2.1) defines a unique P_Γ a differential operator on Γ as follows. Using (2.3) we can identify a small neighbourhood of any $z_0 \in \Gamma$ with $U \subset \mathbb{R}^n$. Then $u \in \mathcal{C}^\infty(\Gamma \cap f(U))$ can be identified with $u \circ f \in \mathcal{C}^\infty(U)$. We then have

$$(P_\Gamma u) \circ f(x) = \sum_{|\alpha| \le m} (a_\alpha \circ f)(x)(({}^t\partial_x f(x)^{-1} D_x)^\alpha (u \circ f)(x). \tag{2.5}$$

It is easy to see that this definition is independent of the choice of f and that the condition (2.4) is crucial.

The key fact is the standard result about continuation of solutions to $P_\Gamma u$. The proof based on [Le], [Ma] and [Sj1] can be found in [SjZ, Lemma 3.1] and (in more detail) [Sj3, Lemma 7.2]. With the notation above we have the following:

Lemma 1. *Suppose that $W \subset \mathbb{R}^n$ is open and that $F : [0, 1] \times W \ni (s, x) \mapsto F(s, x) \in \mathbb{C}^n$, is a smooth proper map satisfying for all $s \in [0, 1]$*

$$\det \partial_x F(s, x) \neq 0, \quad \textit{and } x \mapsto F(s, x) \textit{ is injective.}$$

In addition assume that there exists a compact set $K \subset W$ such that

$$x \in W \setminus K \implies F(0, x) = F(s, x), \quad 0 \le s \le 1,$$

and that $F([0,1]\times W)\subset\Omega$ with $P(z,D_z)$ a differential operator with holomorphic coefficients in Ω.

Now assume that for $\Gamma_s := F(\{s\}\times W)$, P_{Γ_s} is an elliptic differential operator in the sense that

$$\Big|\sum_{|\alpha|=m} a_\alpha(z)\zeta^\alpha\Big| \geq C|\zeta|^m, \quad (z,\zeta)\in T^*\Gamma_s.$$

If $u_0\in\mathcal{C}^\infty(\Gamma_0)$ and $P_{\Gamma_0}u_0$ extends to a holomorphic function on Ω, then for every $s\in[0,1]$ there exists a holomorphic function, U_s defined near Γ_s such that, for some ε,

$$U_0|_{\Gamma_0}=u_0, \quad |s-s'|<\varepsilon \implies U_s=U_{s'} \text{ on the intersection of their domains.}$$

In other words, the function u_0 defined on Γ_0 extends to a possibly multivalued function U in a neighbourhood of $f([0,1]\times W)$.

The lemma will be applied to a family of deformations of $\mathbb{R}^n$ in $\mathbb{C}^n$. Our goal is to restrict the operator $P_\varepsilon = -\Delta - i\varepsilon x^2 + V$, $\varepsilon\geq 0$, to the corresponding totally real submanifolds. For that the deformation has to avoid the support of V and we choose r_0 such that $\operatorname{supp} V\subset B(0,r_0)$. We then construct

$$[0,\pi)\times[0,\infty)\ni(\theta,t)\longmapsto g_\theta(t)\in\mathbb{C} \tag{2.6}$$

which is $\mathcal{C}^\infty$, is injective on $[0,\infty)$ for every fixed θ and satisfies

$$g_\theta(t)=t \text{ for } 0\leq t\leq r_0, \tag{2.7}$$

$$0\leq \arg g_\theta(t)\leq\theta, \quad \partial_t g_\theta\neq 0, \tag{2.8}$$

$$\arg g_\theta(t)\leq \arg\partial_t g_\theta(t)\leq \arg g_\theta(t)+\varepsilon_0, \tag{2.9}$$

$$g_\theta(t)=e^{i\theta}t \text{ for } t\geq T_0 \text{ where } T_0 \text{ depends only on } \varepsilon_0 \text{ and } r_0. \tag{2.10}$$

We now define the totally real submanifolds, Γ_θ, as images of $\mathbb{R}^n$ under the maps

$$f_\theta:\mathbb{R}^n\to\mathbb{C}^n, \quad f_\theta(x):=g_\theta(|x|)x/|x|, \quad \Gamma_\theta:=f_\theta(\mathbb{R}^n). \tag{2.11}$$

For $\varepsilon\geq 0$ and $0\leq\theta<\pi$ we put

$$\begin{gathered} -\Delta_\theta := (\Delta_z)|_{\Gamma_\theta}, \qquad x_\theta := z|_{\Gamma_\theta}, \\ Q_{\varepsilon,\theta} := -\Delta_\theta - i\varepsilon x_\theta^2, \qquad P_{\varepsilon,\theta} := Q_{\varepsilon,\theta}+V. \end{gathered} \tag{2.12}$$

Parametrizing Γ_θ by $(t,\omega)\in[0,\infty)\times\mathbb{S}^{n-1}$, $(t,\omega)\mapsto g_\theta(t)\omega$, we have

$$-\Delta_\theta = \big(g_\theta'(t)^{-1}D_t\big)^2 - i(n-1)g_\theta(t)^{-1}g_\theta'(t)^{-1}D_t + g_\theta(t)^{-2}D_\omega^2, \tag{2.13}$$

where $D_t=\partial_t/i$ and $D_\omega^2=-\Delta_{\mathbb{S}^{n-1}}$. The symbol is given by

$$\sigma(-\Delta_\theta)=g_\theta'(t)^{-1}\tau^2+g_\theta(t)^{-2}w^2, \quad (t,\omega;\tau,w)\in T^*([0,\infty)\times\mathbb{S}^{n-1}).$$

The basic result based on ellipticity at infinity is

$$-2\theta+\delta<\arg z<2\pi-2\theta-\delta, \ |z|\geq\delta \implies (-\Delta_\theta-z)^{-1}=\mathcal{O}_\varepsilon(1): L^2(\Gamma_\theta)\to H^2(\Gamma_\theta).$$

This follows from [SjZ, Lemmas 3.2–3.5] applied with $P = -\Delta$. As will be reviewed in §4 this shows that $P_{0,\theta} - z : H^2 \to L^2$ is a Fredholm operator in this range of values of z and that the eigenvalues are independent of θ.

The crucial property is

Lemma 2. *Let $R_0(z) = (-\Delta - z)^{-1} : L^2 \to H^2$, $\operatorname{Im} z > 0$, be the free resolvent and let $R_0(z)$ also denote its analytic continuation across $[0,\infty)$ as an operator $L^2_{\rm comp} \to H^2_{\rm loc}$.*

Suppose that $\chi \in \mathcal{C}^\infty_{\rm c}(B(0,r_0))$ so that χ is defined on Γ_θ. Then for $-2\theta < \arg z < 2\pi - 2\theta$, $\theta < \pi$,

$$\chi R_0(z)\chi = \chi(-\Delta_\theta - z)^{-1}\chi. \tag{2.14}$$

Proof. We recall the main features of the proof which is implicit in [SjZ, §3]. It is sufficient to establish the identity (2.14) for $0 < \arg z < 2\pi - 2\theta$ as it then follows by analytic continuation. It is also enough to show that in this range of z and $0 \le \theta_1 < \theta_2 \le \theta$, $|\theta_1 - \theta_2| \ll 1$,

$$\chi(-\Delta_{\theta_1} - z)^{-1}\chi = \chi(-\Delta_{\theta_2} - z)^{-1}\chi. \tag{2.15}$$

For that we show that for $f \in L^2(B(0,r_0)) \subset L^2(\Gamma_{\theta_j})$ there exists U holomorphic in a neighbourhood $\Omega_{\theta_1,\theta_2}$ of

$$\bigcup_{\theta_1 \le \theta \le \theta_2} (\Gamma_\theta \setminus B(0,r_0)) \subset \mathbb{C}^n$$

such that

$$U|_{\Gamma_{\theta_j}}(x) = [(-\Delta_{\theta_j} - z)^{-1}\chi f](x) \ \text{ for } x \in \Gamma_{\theta_j} \setminus B(0,r_0). \tag{2.16}$$

The unique continuation property for second order elliptic operators then shows that

$$\chi(\Delta_{\theta_1} - z)^{-1}\chi f = \chi(\Delta_{\theta_2} - z)^{-1}\chi f,$$

proving (2.14).

To show the existence of U such that (2.16) holds we use Lemma 1 applied to a modified family of deformations. The key is to show that a holomorphic extension, U, of the solution to $(-\Delta_{\theta_1} - z)u_1 = \chi f$, $u_1 \in L^2(\Gamma_{\theta_1})$, restricts to $u_2 \in L^2(\Gamma_{\theta_2})$ (the equation $(-\Delta_{\theta_2} - z)u_2 = \chi f$ is automatically satisfied). That means that $u_2 = (-\Delta_{\theta_2} - z)^{-1}(\chi f)$ proving (2.16).

The modified family of contours is obtained as follows. Fix $T \gg 1$ and choose $\chi \in \mathcal{C}^\infty_{\rm c}((2,5);[0,1])$ equal to 1 near $[3,4]$. Then define

$$g_{\theta_1,\theta_2,T}(t) := g_{\theta_1}(t) + \chi(t/T)(g_{\theta_2}(t) - g_{\theta_1}(t)),$$
$$\Gamma_{\theta_1,\theta_2,T} := \{g_{\theta_1,\theta_2,T}(t)\omega \,:\, t \in [0,\infty),\ \omega \in \mathbb{S}^{n-1}\} \subset \mathbb{C}^n.$$

We can apply Lemma 1 to the family of totally real submanifolds interpolating between Γ_{θ_1} and $\Gamma_{\theta_1,\theta_2,T}$: $[0,1] \ni s \longmapsto \Gamma_{\theta_1,\theta_1+s(\theta_1,\theta_2),T}$. That implies that there exists a

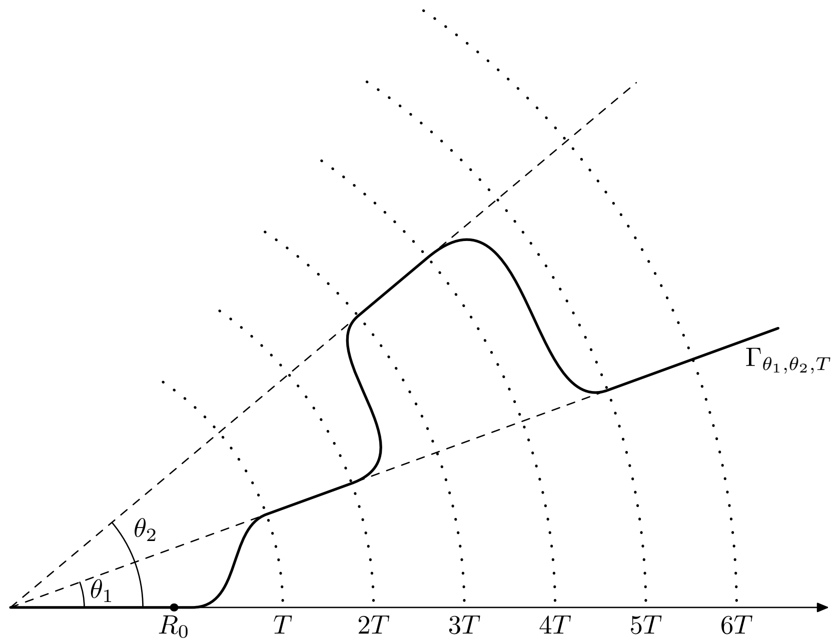

FIGURE 2. The deformed totally real submanifold $\Gamma_{\theta_1,\theta_2,T}$ interpolating between Γ_{θ_1} and Γ_{θ_2}.

holomorphic function U^T defined in a neighbourhood of the union of these submanifolds and such that $u_1 = U^T|_{\Gamma_{\theta_1}}$. Changing T we obtain a family of functions agreeing on the intersections of their domains and that gives U defined in the neighbourhood $\Omega_{\theta_1,\theta_2}$. To see that $U|_{\Gamma_{\theta_2}} \in L^2(\Gamma_{\theta_2})$ it suffices to show that

$$\|U^T|_{\Gamma_{\theta_1,\theta_2,T}}\|_{L^2(\Gamma_{\theta_1,\theta_2,T})} \leq C_0 \|u_1\|_{L^2(\Gamma_{\theta_1}\cap\{T\leq|z|\leq 6T\},)}, \tag{2.17}$$

where C_0 is independent of T. (We apply (2.17) with $T = 2^j$ and sum over j.)

To see (2.17)

$$\Omega_1(T) = \{z \in \mathbb{C}^n : 2T \leq |z| \leq 5T\} \cap \Gamma_{\theta_1,\theta_2,T} \supset \Gamma_{\theta_1,\theta_2,T} \setminus \Gamma_{\theta_1},$$
$$\Omega_2(T) = \{z \in \mathbb{C}^n : T \leq |z| \leq 6T\} \cap \Gamma_{\theta_1,\theta_2,T}, \quad \Omega_2(T) \setminus \Omega_1(T) \subset e^{i\theta_1}\mathbb{R}^n.$$

We claim that for T large and $u \in \mathcal{C}^\infty(\Gamma_{\theta_1,\theta_2,T})$,

$$\|u\|_{L^2(\Omega_1(T))} \leq C\|(-\Delta_{\Gamma_{\theta_1,\theta_2,T}} - z)u\|_{L^2(\Omega_2(T))} + C\|u\|_{L^2(\Omega_2(T)\setminus\Omega_1(T))}. \tag{2.18}$$

For $|\theta_2 - \theta_1| \ll 1$, this estimate is a perturbation of a standard semiclassical elliptic estimate: treating $h := 1/T$ as a semiclassical parameter, uniform ellipticity of $-e^{-2i\theta}h^2\Delta - z$ shows that for $v \in \mathcal{C}^\infty(\mathbb{R}^n)$,

$$\|v\|_{L^2(\{2\leq|x|\leq5\})} \leq C\|(-e^{-2i\theta}h^2\Delta - z)v\|_{L^2(\{1\leq|x|\leq6\})} + C\|v\|_{L^2(\{1\leq|x|\leq2\}\cup\{5\leq|x|\leq6\}}.$$

(This can be seen applying the inverse from [Z, Theorem 4.29] to χv where $\chi \in \mathcal{C}^\infty_{\rm c}((1,6))$ is equal to 1 on $[2,5]$.) The properties of $\Omega_j(T)$ then imply (2.17) completing the argument. ☐

3. The Davies harmonic oscillator

The operator $H_{\varepsilon,\gamma} := -\Delta + e^{-i\gamma}\varepsilon x^2$, $\varepsilon > 0$, $0 \le \gamma < \pi$, was used by Davies [Da] to illustrate properties of non-normal differential operators. We recall the following basic result:

Lemma 3. *The operator $H_{\varepsilon,\gamma}$ is an unbounded operator on L^2 with the discrete spectrum given by*

$$\sigma(H_{\varepsilon,\gamma}) = e^{-i\gamma/2}\sqrt{\varepsilon}(n + 2|\mathbb{N}_0^n|), \quad |\alpha| = \alpha_1 + \cdots + \alpha_n. \tag{3.1}$$

If $\Omega \Subset \{z : -\gamma < \arg z < 0\} \setminus e^{-i\gamma/2}[0,\infty)$, then for some constant $C_1 = C_1(\Omega)$,

$$\frac{1}{C_1} e^{\varepsilon^{-\frac12}/C_1} \le \|(H_{\varepsilon,\gamma} - z)^{-1}\|_{L^2 \to L^2} \le C_1 e^{C_1 \varepsilon^{-\frac12}} \quad , z \in \Omega. \tag{3.2}$$

In addition for any $\delta > 0$ there exists a constant C_2 such that, uniformly in $\varepsilon > 0$,

$$\|(H_{\varepsilon,\gamma} - z)^{-1}\|_{L^2\to L^2} \le C_2/|z|, \quad \delta < \arg z < 2\pi - \gamma - \delta, \quad |z| > \delta. \tag{3.3}$$

Proof. By rescaling $y = \sqrt{\varepsilon}x$ this operator in unitarily equivalent to $-\varepsilon\Delta_y + e^{-i\gamma}y^2$, that is a semiclassical, $h = \sqrt{\varepsilon}$, quadratic operator. For the analysis of the spectrum and upper bounds on the resolvent for general quadratic operators see Hitrik–Sjöstrand–Viola [HSV] and references given there – in particular we obtain (3.1) and the upper bound in (3.2). The lower bound in (3.2) follows from general arguments for operators with analytic coefficients – see [DeSZ, §3] and the bound (3.3) from (semiclassical) ellipticity of $-h^2\Delta_y + e^{-i\gamma}y^2 - z$ for $\delta < \arg z < 2\pi - \gamma - \delta$, $|z| > \delta$. ☐

We now consider the special case of $H_{\varepsilon,\pi/2} = Q_{\varepsilon,0}$ and of its deformation $Q_{\varepsilon,\theta}$ – see (2.12). The facts we need are given in the next two lemmas. The first is the analogue of Lemma 2:

Lemma 4. *In the notation of Lemma 2, $0 \le \theta \le \pi/8$, $\varepsilon > 0$, and $-2\theta < \arg z < 3\pi/2 + 2\theta$ we have*

$$\chi(Q_{\varepsilon,0} - z)^{-1}\chi = \chi(Q_{\varepsilon,\theta} - z)^{-1}\chi. \tag{3.4}$$

In particular, for $0 \le \theta \le \pi/8$, the spectrum is independent of θ and given by $\sqrt{\varepsilon}e^{-i\pi/4}(n + 2|\mathbb{N}_0^n|)$.

Proof. We follow the argument in the proof of Lemma 2 and use the notation introduced there. Hence it is enough to prove that $0 \le \theta_1 < \theta_2 \le \pi/8$ and $|\theta_1 - \theta_2|$ small it is enough to show that

$$\chi(Q_{\varepsilon,\theta_1} - z)^{-1}\chi = \chi(Q_{\varepsilon,\theta_2} - z)^{-1}\chi.$$

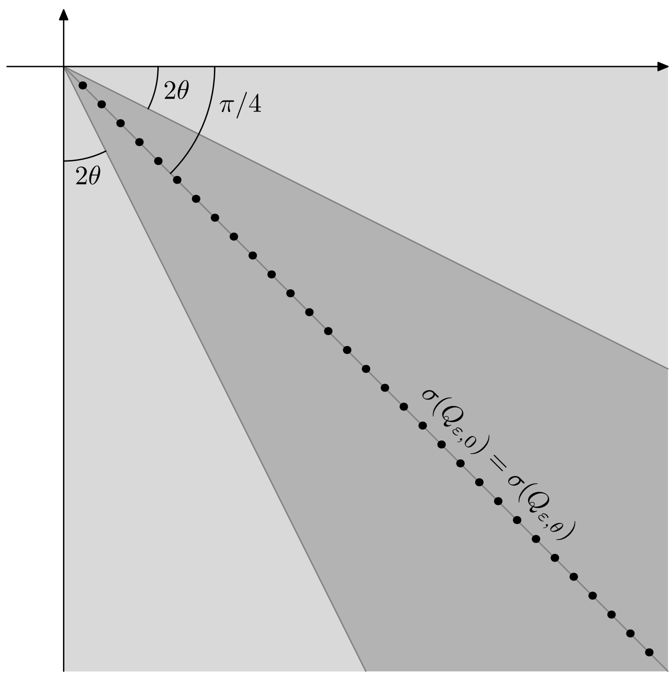

FIGURE 3. A visualization of the spectrum of $Q_{\varepsilon,0} = -\Delta - i\varepsilon x^2$ which is equal to the spectrum of the deformed operator $Q_{\varepsilon,\theta}$. The lightly shaded region is the numerical range of $Q_{\varepsilon,0}$ and the darker shaded region, the numerical range of $-e^{-2i\theta}\Delta - ie^{2i\theta}\varepsilon x^2$. The estimates for the resolvents of $Q_{\varepsilon,\theta}$ improve outside of that region.

We only need to establish this for $z \in e^{i(-2\theta_1+\pi/2)}(1,\infty)$ as then the result follows by analytic continuation. The only difference is an estimate which replaces (2.18): for $\tau > 1$,

$$\begin{gathered}\|u\|_{L^2(\Omega_1(T))} \leq C\|(Q_{\Gamma_{\theta_1,\theta_2,T}} - ie^{-2\theta_1}\tau)u\|_{L^2(\Omega_2(T))} + C\|u\|_{L^2(\Omega_2(T)\setminus\Omega_1(T))}, \\ Q_{\theta_1,\theta_2,T} := -\Delta_{\Gamma_{\theta_1,\theta_2,T}} - i\varepsilon(x|_{\Gamma_{\theta_1,\theta_2,T}})^2\end{gathered} \tag{3.5}$$

uniformly for $T \gg 1$. To see this we first note that for $\varepsilon > 0$, $Q_{\theta_1,\theta_2,T} - z$, $z \in \mathbb{C}$, is a Fredholm operator (since it is elliptic and near infinity it is equal to $e^{-2i\theta}H_{\varepsilon,\pi/2-4\theta}$).

To obtain an estimate we notice that for $t > T$

$$g'_{\theta_1,\theta_2,T}(t) = \chi(t/T)e^{i\theta_2} + (1-\chi(t/T))e^{i\theta_2} + (t/T)\chi'(t/T)(e^{i\theta_2} - e^{i\theta_2}),$$

so that from (2.8) and (2.10),

$$\theta_1 - C|\theta_2 - \theta_1| \leq \arg g'_{\theta_1,\theta_2,T}(t) \leq \theta_2.$$

Also, $\theta_1 \leq \arg g_{\theta_1,\theta_2,T}(t) \leq \theta_2$. Hence,

$$\mathrm{Re}\langle (e^{2i\theta_1}Q_{\theta_1,\theta_2,T} - i\tau)u, u\rangle \geq \|Du\|^2/C$$

where we used the fact that for for $0 \leq \theta \leq \pi/8$, $\mathrm{Re}(-ie^{4\theta}) \geq 0$. The imaginary part is then estimated as follows,

$$- \operatorname{Im}\langle (e^{2i\theta_1} Q_{\theta_1,\theta_2,T} - i\tau)u, u\rangle \geq \tau \|u\|_{L^2(\Gamma_{\theta_1,\theta_2,T})} - \mathcal{O}(|\theta_2 - \theta_1|)\|Du\|^2.$$

We conclude that when $|\theta_2 - \theta_1|$ is small enough

$$\|(Q_{\Gamma_{\theta_1,\theta_2,T}} - ie^{-2i\theta_1}\tau)u\| \geq (\|u\| + \|Du\|)/C,$$

This and the Fredholm property imply that

$$(Q_{\theta_1,\theta_2,T} - ie^{-2i\theta_1}\tau)^{-1} = \mathcal{O}(1) : L^2(\Gamma_{\theta_1,\theta_2,T}) \to H^1(\Gamma_{\theta_1,\theta_2,T}).$$

that is the operator is invertible with bounds independent of T. From this (3.5) follows by a standard localization argument: we choose $\chi_T \in \mathcal{C}^\infty(\Omega_2(T), [0,1])$, such that $\chi_T = 1$ on $\Omega_1(T)$ with derivative bounds independent of T. We then apply the inverse above to $(Q_{\theta_1,\theta_2,T} - ie^{-2i\theta_1}\tau)\chi_T u$ with the commutator terms estimated by $\|u\|_{L^2(\Omega_2(T)\setminus\Omega_1(T))}$. ☐

The next lemma shows how complex scaling dramatically improves the exponential bound (3.2):

Lemma 5. *Suppose that $0 \leq \theta \leq \pi/8$ and that $\Omega \Subset \{z : -2\theta < \arg z < 3\pi/2 + 2\theta\}$. Then there exists $C = C(\Omega)$ (in particular independent of $\varepsilon > 0$) such that*

$$\|(Q_{\varepsilon,\theta} - z)^{-1}\|_{L^2 \to L^2} \leq C, \quad z \in \Omega.$$

Proof. Let $\chi_j \in \mathcal{C}^\infty_{\mathrm{c}}([0,\infty))$ be equal to 1 on $[0, r_0]$ and satisfy $\chi_j = 1$ on $\operatorname{supp}\chi_{j+1}$, $j = 0, 1$. Parametrizing Γ_θ by $F_\theta : [0,\infty)_t \times \mathbb{S}^{n-1} \to \Gamma_\theta$, $F_\theta(t,\omega) = g_\theta(t)\omega$ (with g_θ given in (2.6)) we define functions $\chi^h_j \in \mathcal{C}^\infty_{\mathrm{c}}(\Gamma_\theta)$ as

$$\chi^h_j \circ F_\theta(t,\omega) := \chi_j(th), \quad 0 < h \leq 1.$$

In view of (2.10) and (2.13) we see that for h small enough

$$\begin{aligned} Q_{\varepsilon,\theta}(1 - \chi^h_1) &= (-e^{-2i\theta}\Delta_x - i\varepsilon e^{2i\theta}x^2)(1 - \chi^h_1) \\ &= e^{-2i\theta} H_{\varepsilon,\gamma}(1 - \chi^h_1), \quad \gamma := \pi/2 - 4\theta, \quad x = t\omega. \end{aligned}$$

In view of (3.3) we have

$$(1 - \chi^h_2)e^{2i\theta}(H_{\varepsilon,\gamma} - e^{2i\theta}z)^{-1}(1 - \chi^h_2) = \mathcal{O}_\delta(1) : L^2(\Gamma_\theta) \to H^2(\Gamma_\theta), \tag{3.6}$$

for

$$-\delta < 2\theta + \arg z < 2\pi - \gamma - \delta = 3\pi/2 + 4\theta - \delta, \quad |z| > \delta,$$

and in particular for $z \in \Omega$. We stress that the bounds are independent of ε.

Noting that

$$(-\Delta_\theta - z)^{-1} = \mathcal{O}(1) : L^2(\Gamma_\theta) \to H^2(\Gamma_\theta), \quad z \in \Omega, \tag{3.7}$$

(for $0 \le \theta \le \pi/8$, $-2\theta < \arg z < 2\pi - 2\theta$) we now put

$$T^h_{\varepsilon,\theta}(z) := \chi^h_0(-\Delta_\theta - z)^{-1}\chi^h_1 + (1-\chi^h_1)e^{2i\theta}(H_{\varepsilon,\gamma} - e^{2i\theta}z)^{-1}(1-\chi^h_2),$$

so that $(Q_{\varepsilon,\theta} - z)T^h_{\varepsilon,\theta}(z) = I + K^h_{\varepsilon,\theta}(z)$, where

$$\begin{aligned} K^h_{\varepsilon,\theta}(z) := & - i\varepsilon x^2_\theta \chi^h_0(-\Delta_\theta - z)^{-1}\chi^h_1 - [\Delta_\theta, \chi^h_0](-\Delta_\theta - z)^{-1}\chi_1 \\ & + [\Delta_\theta, \chi^h_1]e^{2i\theta}(1-\chi^h_2)(H_{\varepsilon,\gamma} - e^{2i\theta}z)^{-1}(1-\chi^h_2). \end{aligned}$$

Since $[\Delta_\theta, \chi^h_j] = \mathcal{O}(h) : H^1(\Gamma_\theta) \to L^2(\Gamma_\theta)$ and $x^2_\theta\chi^h_1 = \mathcal{O}(h^{-2}) : L^2(\Gamma_\theta) \to L^2(\Gamma_\theta)$, we conclude from (3.6) and (3.7) that for $z \in \Omega$,

$$K^h_{\varepsilon,\theta}(z) = \mathcal{O}(h^{-2}\varepsilon) + \mathcal{O}(h) : L^2(\Gamma_\theta) \to L^2(\Gamma_\theta).$$

Hence by choosing h first, we see that for $\varepsilon < \varepsilon_0(h)$, $I + K^h_{\varepsilon,\theta}(z)$ has a uniformly bounded inverse and $0 \le \varepsilon < \varepsilon_0$

$$(Q_{\varepsilon,h} - z)^{-1} = T^h_{\varepsilon,\theta}(z)(I + K^h_{\varepsilon,\theta}(z))^{-1} = \mathcal{O}(1) : L^2(\Gamma_\theta) \to L^2(\Gamma_\theta), \quad z \in \Omega.$$

In view of Lemma 4 we know that for $z \in \Omega$, $(Q_{\varepsilon,h} - z)^{-1}$ exists for $\varepsilon > \varepsilon_0$ and that gives the bound for all values ε. ☐

4. Meromorphic continuation

In this section we will review the meromorphy of the resolvent $R_V(z)$, see (1.1), in a way connecting it to the resolvent of P_ε given in (1.4), $\varepsilon \ge 0$. For that we define

$$R_\varepsilon(z) = (-\Delta - i\varepsilon x^2 - z)^{-1}, \quad R_{V,\varepsilon}(z) = (-\Delta - i\varepsilon x^2 + V - z)^{-1}, \quad \varepsilon \ge 0. \tag{4.1}$$

For $\varepsilon > 0$, these operators are meromorphic for $z \in \mathbb{C}$ as operators on L^2. For $\varepsilon = 0$, $R_0(z)$ is holomorphic in the sense of (1.1) on the double cover of $\mathbb{C} \setminus \{0\}$ when n is odd and on the logarithmic cover when n is even – see for instance [DyZ2, §3.1]. We are only concerned with continuation to $\arg z \ge -\pi/4$.

Let $\rho \in \mathcal{C}^\infty_{\rm c}(\mathbb{R}^n; [0,1])$ be equal to 1 on a neighbourhood of $\operatorname{supp} V$. We have

Lemma 6. *For* $\varepsilon \ge 0$

$$z \mapsto (I + VR_\varepsilon(z)\rho)^{-1}, \quad -\pi/4 < \arg z < 7\pi/4,$$

is a meromorphic family of operators on $L^2(\mathbb{R}^n)$ *for with poles of finite rank. Then*

$$m_\varepsilon(z) := \frac{1}{2\pi i} \operatorname{tr} \oint_z (I + VR_\varepsilon(w)\rho)^{-1}\partial_w(VR_\varepsilon(w)\rho)dw, \tag{4.2}$$

where the integral is over a positively oriented circle enclosing z *and containing no poles other than possibly* z*, satisfies*

$$m_\varepsilon(z) = \begin{cases} \frac{1}{2\pi i}\oint_z (w - P_\varepsilon)^{-1}dw, & \varepsilon > 0 \\ m(z), & \varepsilon = 0, \end{cases} \tag{4.3}$$

where $m(z)$ is the multiplicity of the resonance z given by (1.2).

Proof. We recall the standard argument (see [DyZ2, §2.2, 3.2] and references given there). For any $\delta > 0$ and uniformly in $\varepsilon \geq 0$,

$$R_\varepsilon(z) = \mathcal{O}_\delta(1/|z|) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n), \quad \delta < \arg z < 3\pi/2 - \delta, \quad |z| > \delta. \tag{4.4}$$

This follows from self-adjointness for $\varepsilon = 0$ and from (3.3) for $\varepsilon > 0$.

For z in (4.4) and $Q_\varepsilon := -\Delta - i\varepsilon x^2$,

$$\begin{aligned}(P_\varepsilon - z) &= (Q_\varepsilon - z)(I + R_\varepsilon(z)V) \\ &= (I + VR_\varepsilon(z)\rho)(I + VR_\varepsilon(z)(1-\rho))(Q_\varepsilon - z).\end{aligned} \tag{4.5}$$

Noting that

$$(I + VR_\varepsilon(z)(1-\rho))^{-1} = I - VR_\varepsilon(z)(1-\rho)$$

we obtain from (4.4) and (4.5) that

$$\begin{gathered}R_{V,\varepsilon}(z) = R_\varepsilon(z)(I + VR_\varepsilon(z)\rho)^{-1}(I - VR_\varepsilon(z)(1-\rho)), \\ \delta < \arg z < 3\pi/2 - \delta, \quad |z| \gg 1,\end{gathered} \tag{4.6}$$

where for large $|z|$, $I + VR_\varepsilon(z)\rho$ is invertible by a Neumann series argument. Since $z \mapsto VR_\varepsilon(z)\rho$ is a holomorphic family of compact operators for $-\pi/4 < \arg z < 3\pi/4$ (see Lemma 3 for the case $\varepsilon > 0$), $z \mapsto (I + VR_\varepsilon(z)\rho)^{-1}$ is a meromorphic family operators in the same range of z. (For $\varepsilon > 0$ the meromorphy is in fact valid for $z \in \mathbb{C}$ – see [DyZ2, §C.4].) The formula (4.6) remains valid for that range of z with boundedness on L^2 for $\varepsilon > 0$. For $\varepsilon = 0$ we note that

$$(I - VR_0(z)(1-\rho)) \, , (I + VR_0(z))^{-1} : L^2_{\rm comp} \to L^2_{\rm comp}, \quad R_0(z) : L^2_{\rm comp} \to L^2_{loc},$$

and we obtain the meromorphic continuation of $R_{V,0}(z) : L^2_{\rm comp} \to L^2_{loc}$. Arguing as in the proof of [DyZ2, Theorem 3.23] we obtain the multiplicity formula (4.3). (This can also seen using complex scaling as reviewed in the proof of Theorem 2 below.) ☐

5. Proof of convergence

The proof of convergence is based on Lemma 6 and on the following lemma in which we use the complex variable techniques of §§2,3.

Lemma 7. *For $\chi \in \mathcal{C}^\infty_{\rm c}(\mathbb{R}^n)$ consider*

$$T^\chi_\varepsilon(z) := \chi(-\Delta - i\varepsilon x^2 - z)^{-1} x^2 (-\Delta - z)^{-1}\chi, \quad 0 < \arg z < 3\pi/2. \tag{5.1}$$

Then T^χ_ε continues to a holomorphic family of operators

$$T^\chi_\varepsilon(z) : L^2 \to L^2, \quad -\pi/4 < \arg z < 7\pi/4.$$

If $\Omega \Subset \{z : -\pi/4 < \arg z < 3\pi/2\}$ *then there exists* $C = C_{\Omega,\chi}$ *(independent of* ε*) such that*

$$\|T_\varepsilon^\chi(z)\|_{L^2\to L^2} \leq C, \quad z \in \Omega, \quad \varepsilon > 0. \tag{5.2}$$

Proof. In the notation of (4.1) we see that for $\delta < \arg z < 3\pi/2 - \delta$, $|z| > \delta$,

$$\chi(R_\varepsilon(z) - R_0(z))\chi = i\varepsilon\chi R_\varepsilon(z)x^2 R_0(z)\chi,$$

where we note that, for in our range of z, $R_0(z)\chi : L^2 \to e^{-c_\delta|x|}L^2$ (by looking, for instance at the explicit formulas for the resolvent, see [DyZ2, §3.1], or by conjugation with exponential weights) and consequently $x^2R_0(z)\chi : L^2 \to L^2$. Hence

$$T_\varepsilon^\chi(z) = -\frac{i}{\varepsilon}(\chi R_\varepsilon(z)\chi - \chi R_0(z)\chi). \tag{5.3}$$

The right hand side is holomorphic for $-\pi/4 < \arg z < 5\pi/4$ which provides holomorphic continuation of $T_\varepsilon^\chi(z)$, $\varepsilon > 0$.

We now use Lemmas 2 and 4. For that we choose r_0 in the definition of Γ_θ large enough so that $\operatorname{supp}\chi \subset B(0, r_0)$ and take $\theta = \pi/8$. Then we have

$$\begin{aligned} T_\varepsilon^\chi(z) &= -\frac{i}{\varepsilon}\left(\chi(Q_{\varepsilon,\theta} - z)^{-1}\chi - \chi(Q_{0,\theta} - z)^{-1}\chi\right) \\ &= \chi(Q_{\varepsilon,\theta} - z)^{-1}x_\theta^2(Q_{0,\theta} - z)^{-1}\chi, \end{aligned} \tag{5.4}$$

where, in the notation of (2.12), $x_\theta := x|_{\Gamma_\theta}$. We now note that for $z \in \Omega$,

$$(Q_{0,\theta} - z)^{-1}\chi : L^2(\Gamma_\theta) \to e^{-c_\Omega|x|}L^2(\Gamma_\theta). \tag{5.5}$$

This can be seen by conjugation by exponential weights or by constructing a parametrix for $Q_{0,\theta}$ as in the proof of Lemma 4 and using the explicit properties of $(-e^{-2i\theta}\Delta - z)^{-1} = e^{2i\theta}R_0(e^{2i\theta}z)$. From this and Lemma 4 we obtain

$$\|(Q_{\varepsilon,\theta} - z)^{-1}x_\theta^2(Q_{0,\theta} - z)^{-1}\chi\|_{L^2\to L^2} \leq C_\Omega, \quad z \in \Omega.$$

Inserting this into (5.4) concludes the proof. ☐

We can now state a stronger version of Theorem 1 formulated using the projections appearing in (4.2):

Theorem 2. *Suppose that* $-\pi/4 < \arg z < 5\pi/4$ *and that* $m(z) = m \geq 0$, *where* $m(z)$ *is the multiplicity of the resonance of* $P := -\Delta + V$ *at* z *– see (1.2).*

Then there exists ε_0 *and* δ *such that for* $0 < \varepsilon \leq \varepsilon_0$, $P_\varepsilon = -\Delta + V - i\varepsilon x^2$ *has* m *eigenvalues in* $D(z, \delta)$:

$$\operatorname{tr}\Pi_\varepsilon = m, \quad \Pi_\varepsilon := \frac{1}{2\pi i}\int_{\partial D(z,\delta)}(\zeta - P_\varepsilon)^{-1}d\zeta, \quad \Pi_\varepsilon^2 = \Pi_\varepsilon, \tag{5.6}$$

and for any $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$,

$$\chi\Pi_\varepsilon\chi \in C^\infty([0, \varepsilon_0], \mathcal{L}(L^2(\mathbb{R}^n), L^2(\mathbb{R}^n))). \tag{5.7}$$

Remarks. 1. Notation $f \in C^1([a,b])$ means that f, and f' continuous in $[a,b]$; here $f'(a), f'(b)$ are the left and right derivatives of f at those points. By induction we then define $C^k([a,b])$ and $C^\infty([a,b])$. In view of (1.5) we cannot expect analytic dependence on ε.

2. For $\chi \equiv 1$ on $\operatorname{supp} V$, $m(z) = \operatorname{rank} \chi \Pi_0 \chi$ and (5.7) shows the convergence of resonant states in the case of simple resonances. As the proof shows a stronger statement is obtained by using the complex scaled operators: for $\theta = \pi/8$,

$$\begin{gathered}\Pi_{\varepsilon,\theta} := \frac{1}{2\pi i} \int_{\partial D(z,\delta)} (\zeta - P_{\varepsilon,\theta})^{-1} d\zeta, \quad P_{\varepsilon,\theta} = -\Delta|_{\Gamma_\theta} - i\varepsilon (x|_{\Gamma_\theta})^2 + V, \\ \Pi_{\varepsilon,\theta} \in C([0,\varepsilon_0); \mathcal{L}^1(L^2(\Gamma_\theta), L^2(\Gamma_\theta))), \quad \Pi_{\varepsilon,\theta}\chi \in \mathcal{C}^\infty([0,\varepsilon_0); \mathcal{L}^1(L^2(\Gamma_\theta), L^2(\Gamma_\theta))),\end{gathered} \tag{5.8}$$

where Γ_θ is the deformation defined in (2.11).

Proof. We first note that (4.6) and Lemma 4 imply that for $-\pi/4 \le -2\theta < \arg z < 2\pi - 2\theta$, $\varepsilon \ge 0$,

$$(P_{\varepsilon,\theta} - z)^{-1} = (Q_{\varepsilon,\theta} - z)^{-1} (I + V R_\varepsilon(z)\rho)^{-1} (I - V(Q_{\varepsilon,\theta} - z)^{-1}(1-\rho)). \tag{5.9}$$

Since $z \mapsto (Q_{\varepsilon,\theta} - z)^{-1}$ is a holomorphic family in our range of z's, the Gohberg-Sigal theory – see [DyZ2, §C.4] – shows that the poles of $(P_{\varepsilon,\theta} - z)^{-1}$ with $\arg z > -2\theta$ are independent of $0 \le \theta \le \pi/8$ and

$$\operatorname{tr} \frac{1}{2\pi i} \oint (P_{\varepsilon,\theta} - \zeta)^{-1} d\zeta = \operatorname{tr} \frac{1}{2\pi i} \oint (P_\varepsilon - \zeta)^{-1} d\zeta, \quad \varepsilon > 0.$$

If in the definition of Γ_θ we take r_0 large enough so that $\operatorname{supp} \chi \subset B(0, r_0)$ then Lemmas 2 and 4 show that $\chi \Pi_{\varepsilon,\theta} \chi = \chi \Pi_\varepsilon \chi$.

Hence it is enough to prove (5.8). For that we note that in the notation of Lemma 7,

$$\begin{aligned}(I + V R_\varepsilon(z)\rho)^{-1} - (I + V R_0(z)\rho)^{-1} &= i\varepsilon (I + V R_\varepsilon(z)\rho)^{-1} T_\varepsilon^\rho(z) (I + V R_0(z)\rho)^{-1} \\ &= \mathcal{O}_z(\varepsilon \|(I + V R_\varepsilon(z)\rho)^{-1}\|_{L^2 \to L^2}) : L^2 \to L^2.\end{aligned}$$

We can now apply the Gohberg–Sigal–Rouché theorem [DyZ2, Theorem C.9] to see that the poles of $(I + V R_0(z)\rho)^{-1}$ and $(I + V R_\varepsilon(z)\rho)^{-1}$ coincide with multiplicities. This and (5.9) prove the first statement in (5.8). The second statement follows from differentiation and estimates similar to (5.5). □

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