RESONANCES AND LOWER RESOLVENT BOUNDS

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ABSTRACT. We show how the presence of resonances close to the real axis implies exponential lower bounds on the norm of the cut-off resolvent on the real axis.

In this note we establish exponential lower bounds on the scattering resolvent on the real line. We show that these lower bounds can be understood in terms of resonances close to the real axis.

To fix the concepts, consider a *semiclassical Schrödinger operator* on $\mathbb{R}^n$:

$$P(h) = -h^2 \Delta + V(x), \quad x \in \mathbb{R}^n, \quad V \in C_c^\infty(\mathbb{R}^n; \mathbb{R}), \tag{1.1}$$

$$\operatorname{supp} V \subset B(0, R_0); \quad V(0) = V_0 > 0, \ V'(0) = 0, \ V''(0) > 0;$$
$$x \cdot V'(x) \le 0 \quad \text{on } \{V \le V_0\}, \quad x \cdot V'(x) < 0 \quad \text{on } \{V = V_0\} \setminus \{0\}. \tag{1.2}$$

Take $R > R_0$ and define the cutoffs

$$\chi = \mathbb{1}_{B(0,R_0)}, \quad \psi = \mathbb{1}_{B(0,R+1) \setminus B(0,R-1)}. \tag{1.3}$$

Theorem 3 in §4 shows that for any $R > R_0$ there exists a constant $c > 0$ independent of h and $E_0(h) = V_0 + \mathcal{O}(h)$ such that

$$\|\chi(P(h) - E_0(h) \pm i0)^{-1}\chi\|_{L^2 \to L^2} \ge \exp(c/h), \tag{1.4}$$

$$\|\psi(P(h) - E_0(h) \pm i0)^{-1}\chi\|_{L^2 \to L^2} \ge \exp(c/h). \tag{1.5}$$

A very general exponential *upper* bound corresponding to (1.4) was first proved by Burq [Bu], with generalizations by Vodev [Vo], and more recently by Datchev [Da]. The lower bound is immediate from much easier arguments involving quasimodes. The "non-trapping" upper bound (for R large enough)

$$\|\psi(P(h) - E_0(h) \pm i0)^{-1}\psi\|_{L^2 \to L^2} \le \frac{C_0}{h}, \tag{1.6}$$

was again given by Burq [Bu] (with a $\log 1/h$ loss) and Vodev [Vo] – see [Da] for a neat new proof.

It is (1.5) which seems to be the novel aspect. It shows that having a one sided cutoff to the exterior of the interaction region *cannot* prevent exponential blow up of the resolvent.

The method also applies to the case of Riemannian manifolds, (M, g), considered recently by Rodnianski–Tao [RoTa] – see Fig. 1. In that case the support of V is
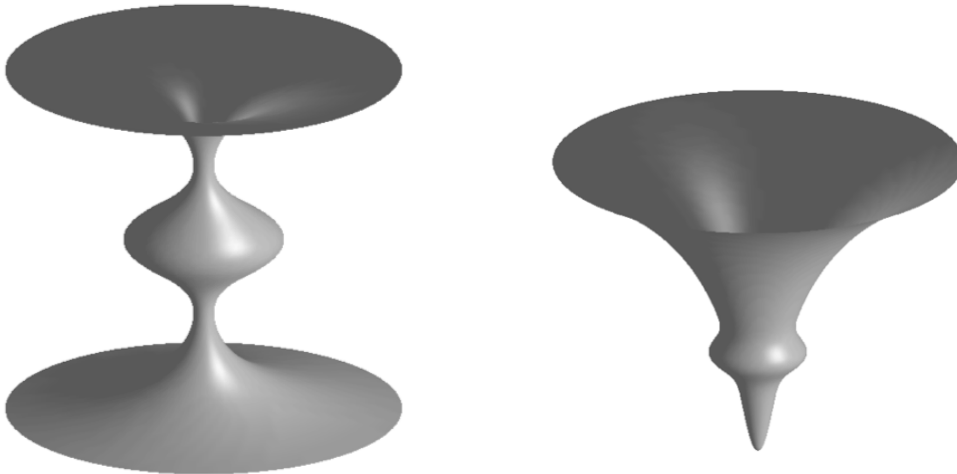
FIGURE 1. Examples of manifolds for which the estimate (1.7) holds: on the left a surface of revolution with two Euclidean ends to which Theorem 4 applies directly; on the right a surface with one end to which a modification of the same method applies (see [Ga]). The same examples work in any dimension.

replaced in (1.3) by the set where the metric is different from the Euclidean metric, and we obtain a sequence of $\lambda_k \to \infty$ such that

$$\|\psi(\Delta_g - \lambda_k \pm i0)^{-1}\chi\|_{L^2(M) \to L^2(M)} \geq e^{c\sqrt{\lambda_k}}. \tag{1.7}$$

See Theorem 4 in §4 for details.

The reason behind these estimates is the presence of resonances close to real axis. The resolvent

$$R_h(z) = (P(h) - z)^{-1} : L^2(\mathbb{R}^n) \to H^2(\mathbb{R}^n), \quad z \notin [0, +\infty),$$

has a meromoprhic continuation to the Riemann surface of $\sqrt{z}$ for n odd, and to the Riemann surface of $\log z$ for n even, as a family of operators $L^2_{\mathrm{comp}}(\mathbb{R}^n) \to H^2_{\mathrm{loc}}(\mathbb{R}^n)$, see for example [DyZw] and references given there. *Resonances*, defined as the poles of this continuation, replace discrete spectral data for problems on non-compact domains. For the situations considered here, in particular for the case (1.1), Rellich's theorem (see [DyZw]) shows that there are no resonances on the positive real axis, that is, the operators $(P(h) - E \pm i0)^{-1}$ are well-defined for $E > 0$.

Theorem 2 in §3 gives general lower bounds based on existence of resonances with certain properties. It is then applied in Theorems 3 and 4 in §4 to obtain examples, in particular of Riemannian manifolds with Euclidean ends.

The simple proofs here are based on previous work on scattering resonances, in particular those by Bony–Michel [BoMi], Gérard–Martinez [GéMa], Helffer–Sjöstrand
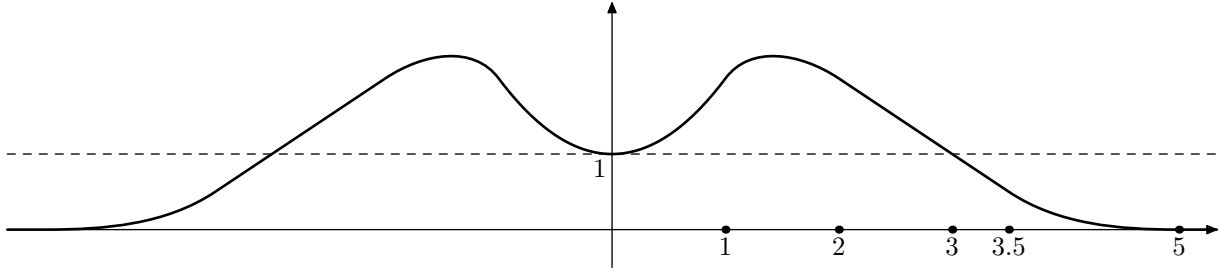
FIGURE 2. The potential V used in Theorem 1.

[HeSj2], Tang-Zworski [TaZw] and Nakamura–Stefanov–Zworski [NaStZw]. To make the basic idea accessible, we present in §2 an elementary and self-contained one dimensional example which captures the basic reason for (1.5); the argument of §2 does not directly use resonances though it could be used to show their existence.

2. An explicit example

The following one dimensional example shows the reasons for (1.5) in an explicit setting:

Theorem 1. *Let $V \in C_c^\infty(\mathbb{R})$ be a nonnegative potential satisfying the following conditions (see Figure 2):*

$$V(x) = V(-x), \quad V(x) = x^2 + 1 \ \text{for } x \in [-1,1];$$
$$V(x) = 4 - x \text{ for } x \in [2, 3.5]; \quad V(x) < 1 \text{ for } x > 3; \ \text{supp } V \subset [-5, 5]. \tag{2.1}$$

Put $R_0 = 4$, fix $R > 5$, and define χ, ψ by (1.3). Then there exists $c > 0$ and families $E_0(h) = 1 + \mathcal{O}(h)$, $u(h), f(h) \in C_c^\infty(\mathbb{R})$ such that

$$(P(h) - E_0(h))u = f, \quad f = \psi f;$$
$$\|\chi u\|_{L^2(\mathbb{R})} = 1, \quad \|f\|_{L^2(\mathbb{R})} \le e^{-c/h}. \tag{2.2}$$

Note that (2.2) implies (henceforth suppressing the dependence on h)

$$\|\chi R_h(E_0 \pm i0)\psi\|_{L^2 \to L^2} \ge e^{c/h}, \tag{2.3}$$

$$\|\psi R_h(E_0 \pm i0)\chi\|_{L^2 \to L^2} \ge e^{c/h}. \tag{2.4}$$

Indeed, since $u \in C_c^\infty$ and $f = \psi f$, we have $\chi u = \chi R_h(E_0 \pm i0)\psi f$; this shows (2.3). The bound (2.4) follows since $R_h(E_0 \pm i0)^* = R_h(E_0 \mp i0)$.

The key component of the proof of Theorem 1 is the existence of *quasimodes* for the operator $P - E_0$, namely functions that satisfy $(P - E_0)v = \mathcal{O}(e^{-c/h})$:

Lemma 2.1. *There exist h-dependent families $E_0 = E_0(h) \in \mathbb{R}$, and $v = v(r; h) \in C^\infty([-R, R])$, such that $E_0 = 1 + \mathcal{O}(h)$ and for some $C, c > 0$,*

$$(P - E_0)v = 0, \quad \|v\|_{L^2(-3.5, 3.5)} \geq C^{-1}, \quad \|v\|_{H_h^1(\{R-1 < |x| < R\})} \leq C e^{-c/h}.$$

Here H_h^1 denotes the semiclassical Sobolev space where in the standard definition D_x is replaced by hD_x – see for instance [Zw1, §7.1, §8.3].

To derive Theorem 1 from Lemma 2.1, we take $\chi_0 \in C_c^\infty(-R, R)$ such that $\chi_0 = 1$ on $[-(R-1), R-1]$ and put

$$u := \alpha \chi_0 v, \quad f = (P - E_0)u = \alpha[P, \chi_0]v,$$

here the constant $\alpha = \alpha(h)$ is chosen so that $\|\chi u\|_{L^2} = 1$ and we have $|\alpha| \leq C$. We furthermore see that $\operatorname{supp} f \subset \{R - 1 < |x| < R\}$ and $\|f\|_{L^2} \leq e^{-c/h}$ (the constant C can be absorbed into the exponential by replacing c by a smaller constant and taking h small enough).

The rest of this section contains the proof of Lemma 2.1. We take $\widetilde{R}(h) \geq R$, $\widetilde{R}(h) = R + \mathcal{O}(h)$, to be chosen at the end of this section in (2.15), and let v be an eigenfunction of P on $[-\widetilde{R}(h), \widetilde{R}(h)]$ with Dirichlet boundary conditions with eigenvalue E_0 close to the ground state $1 + h$ of the quantum harmonic oscillator $-h^2\partial_x^2 + x^2 + 1$. The existence of such eigenvalue is given by the following

Lemma 2.2. *For h small enough and given $\widetilde{R}(h) \in [R, R + 1]$, there exists $E_0 \in \mathbb{R}$ and $v \in C^\infty([-\widetilde{R}(h), \widetilde{R}(h)])$ such that*

$$(P - E_0)v = 0, \quad v(\widetilde{R}(h)) = v(-\widetilde{R}(h)) = 0;$$

$$\|v\|_{L^2(-\widetilde{R}(h), \widetilde{R}(h))} = 1, \quad \|v\|_{H_h^1(-\widetilde{R}(h), \widetilde{R}(h))} \leq C, \quad E_0 = 1 + h + \mathcal{O}(e^{-\frac{1}{10h}}).$$

Proof. Define

$$v_1(x) := h^{-1/4} e^{-\frac{x^2}{2h}}, \quad x \in [-1, 1].$$

Note that, since $P = -h^2\partial_x^2 + x^2 + 1$ on $[-1, 1]$, we have

$$(P - (1 + h))v_1 = 0, \quad x \in [-1, 1];$$

$$\|v_1\|_{L^2(-1/2, 1/2)} \geq C^{-1}, \quad \|v_1\|_{H_h^1(\{1/2 < |x| < 1\})} \leq C e^{-\frac{1}{10h}}.$$

Now, take $\widetilde{\chi} \in C_c^\infty(-1, 1)$ such that $\widetilde{\chi} = 1$ on $[-1/2, 1/2]$. Then

$$\|\widetilde{\chi} v_1\|_{L^2} \geq C^{-1}, \quad \|(P - (1 + h))\widetilde{\chi} v_1\|_{L^2} \leq C e^{-\frac{1}{10h}},$$

and $\widetilde{\chi} v_1$ satisfies the Dirichlet boundary conditions at $\pm\widetilde{R}(h)$ (since it vanishes there). Now, $P - (1 + h)$ is self-adjoint on $L^2([-\widetilde{R}(h), \widetilde{R}(h)])$ when Dirichlet boundary conditions are imposed. Since the norm of its inverse is at least $C^{-1} e^{\frac{1}{10h}}$, we see that this

operator has an eigenvalue which is $\mathcal{O}(e^{-\frac{1}{10h}})$; we denote the corresponding eigenvalue of P by E_0 and the corresponding L^2 normalized eigenfunction by v. Finally, to establish a bound on the H_h^1 norm of v it suffices to multiply the equation $(P - E_0)v = 0$ by $\overline{v}$ and integrate by parts. $\qquad\square$

Lemma 2.1 follows once we establish the following exponential bound on v:

$$\|v\|_{H_h^1(\{3.5 \le |x| \le \widetilde{R}(h)\})} \le Ce^{-c/h}. \tag{2.5}$$

We will show (2.5) for positive x; the case of negative x is handled similarly (since V is even, v can be taken to be even as well). The main idea is the following: if v is not exponentially large in $1/h$ near, say, $x = 2$ relative to its size on $[3.5, \widetilde{R}(h)]$, then one expects v to give an approximate Dirichlet eigenfunction to the operator P on $[2, \widetilde{R}(h)]$ with eigenvalue E_0. However, then E_0 has to satisfy a quantization condition determined by the behavior of V on $[2, \widetilde{R}(h)]$; since $E_0 = 1 + h + o(h)$, one can choose $\widetilde{R}(h)$ to ensure that the quantization condition is not satisfied and thus obtain a contradiction.

For $f_1, f_2 \in C^\infty(\mathbb{R})$ we define the (semiclassical) Wronskian by

$$W(f_1, f_2) = f_1 \cdot h\partial_x f_2 - f_2 \cdot h\partial_x f_1,$$

and note that

$$h\partial_x W(f_1, f_2) = f_2 \cdot (P - E_0)f_1 - f_1 \cdot (P - E_0)f_2.$$

The interval $[2, R]$ can be split into three regions where the behavior of v is different, based on the sign of $V(x) - 1$: the "elliptic" or classically forbidden region $[2, 3)$, where v will grow exponentially in h as x decreases, the neighborhood of the turning point $x = 3$, and the "hyperbolic" region $(3, R)$, where the equation $(P(h) - E_0)v = 0$ has two solutions which are bounded as $h \to 0$.

We start with the hyperbolic region, considering the phase function

$$\Phi(x) := \int_{4-E_0}^x \sqrt{E_0 - V(y)}\, dy.$$

Note that Φ is well-defined on $x \in [4 - E_0, R+1]$, since $\sqrt{E_0 - V(y)} = \sqrt{y - (4 - E_0)}$ for $y \in [4 - E_0, 3.5]$; in fact, we have

$$\Phi(x) = \frac{2}{3}(x - (4 - E_0))^{3/2} \quad \text{for } x \in [4 - E_0, 3.5]. \tag{2.6}$$

Define now the following *WKB solutions*:

$$v_\pm(x) := (E_0 - V(x))^{-1/4} e^{\pm \frac{i\Phi(x)}{h}}, \quad x \in [3.5, R+1],$$

then we have uniformly in $x \in [3.5, R+1]$,

$$(P - E_0)v_\pm(x) = \mathcal{O}(h^2), \quad W(v_+, v_-)(x) = -2i + \mathcal{O}(h). \tag{2.7}$$

Denote
$$\mathbf{v}(x) = (\mathbf{v}_1(x), \mathbf{v}_2(x)) := (W(v, v_+)(x), W(v, v_-)(x)),$$

then

$$v(x) = \frac{\mathbf{v}_2(x) \cdot v_+(x) - \mathbf{v}_1(x) \cdot v_-(x)}{W(v_+, v_-)(x)}, \tag{2.8}$$

$$h\partial_x v(x) = \frac{\mathbf{v}_2(x) \cdot h\partial_x v_+(x) - \mathbf{v}_1(x) \cdot h\partial_x v_-(x)}{W(v_+, v_-)(x)}. \tag{2.9}$$

Since $(P - E_0)v = 0$, we have

$$h\partial_x W(v, v_\pm) = -v \cdot (P - E_0)v_\pm .$$

From (2.7) and (2.8) we see that for $x \in [3.5, R+1]$,

$$|\partial_x \mathbf{v}(x)| \le Ch|\mathbf{v}(x)|.$$

Therefore,

$$\mathbf{v}(x) = \mathbf{v}(3.5)(1 + \mathcal{O}(h)), \quad x \in [3.5, \widetilde{R}(h)]. \tag{2.10}$$

This and (2.8), (2.9) show that

$$\|v\|_{H_h^1(3.5, \widetilde{R}(h))} \le C|\mathbf{v}(3.5)|. \tag{2.11}$$

The final component of the proof is the following solution in the region $[2, 3.5]$ which describes the transformation from the hyperbolic to the elliptic region via the turning point, and is exponentially decaying in the elliptic region. Since $V(x) = 4 - x$ in this region, the solution is given by an Airy function, and its properties are as follows:

Lemma 2.3. *There exists a solution $w(x)$ to the equation $(P - E_0)w = 0$ for $x \in [2, 3.5]$ such that $\|w\|_{H_h^1(2, 2.5)} \le Ce^{-c/h}$ for some constants $C, c > 0$ and*

$$\begin{pmatrix} w(x) \\ h\partial_x w(x) \end{pmatrix} = e^{\frac{i\pi}{4}} \begin{pmatrix} v_+(x) \\ h\partial_x v_+(x) \end{pmatrix} - e^{-\frac{i\pi}{4}} \begin{pmatrix} v_-(x) \\ h\partial_x v_-(x) \end{pmatrix} + O(h), \quad x \in [3.25, 3.5]. \tag{2.12}$$

Proof. The solution w is given by

$$w(x) = 2i\sqrt{\pi}h^{-1/6} \operatorname{Ai}(h^{-2/3}(4 - E_0 - x)),$$

and its properties follow from the following asymptotic formulæ for the Airy function Ai as $y \to +\infty$:

$$\operatorname{Ai}(y) = \frac{y^{-1/4}}{2\sqrt{\pi}} \exp\left(-\frac{2}{3}y^{3/2}\right)(1 + \mathcal{O}(y^{-3/2})),$$

$$\operatorname{Ai}(-y) = \frac{y^{-1/4}}{\sqrt{\pi}}\left(\sin\left(\frac{2}{3}y^{3/2} + \frac{\pi}{4}\right) + \mathcal{O}(y^{-3/2})\right),$$

and similar formulæ for its derivatives, see for example [HöI–II, (7.6.20), (7.6.21)]. $\square$

We are now ready to finish the proof of (2.5). Since $(P - E_0)v = (P - E_0)w = 0$ on $[2, 3.5]$, the Wronskian $W(v, w)$ is constant on this interval. Using the estimates $\|v\|_{H^1_h(2,2.5)} \leq C$, $\|w\|_{H^1_h(2,2.5)} \leq Ce^{-c/h}$, we see that

$$|W(v, w)| \leq Ce^{-c/h}.$$

Now, computing the same Wronskian at $x = 3.5$ and using (2.12), we get

$$W(v, w) = e^{\frac{i\pi}{4}} \mathbf{v}_1(3.5) - e^{\frac{-i\pi}{4}} \mathbf{v}_2(3.5) + \mathcal{O}(h)|\mathbf{v}(3.5)|,$$

It remains to prove that, for a certain choice of $\widetilde{R}(h)$ independent of E_0, we have

$$|e^{\frac{i\pi}{4}} \mathbf{v}_1(3.5) - e^{\frac{-i\pi}{4}} \mathbf{v}_2(3.5)| \geq C^{-1}|\mathbf{v}(3.5)|. \tag{2.13}$$

Indeed, in this case $|\mathbf{v}(3.5)| \leq Ce^{-c/h}$, which together with (2.11) gives (2.5).

Using (2.10), we rewrite (2.13) as follows:

$$|e^{\frac{i\pi}{4}} \mathbf{v}_1(\widetilde{R}(h)) - e^{\frac{-i\pi}{4}} \mathbf{v}_2(\widetilde{R}(h))| \geq C^{-1}|\mathbf{v}(\widetilde{R}(h))|. \tag{2.14}$$

Since v satisfies the Dirichlet boundary condition at $\widetilde{R}(h)$, we have

$$W(v, v_\pm)(\widetilde{R}(h)) = -E_0^{-1/4} e^{\pm \frac{i\Phi(\widetilde{R}(h))}{h}} \cdot h\partial_x v(\widetilde{R}(h)),$$

so that $|\mathbf{v}_1(\widetilde{R}(h))|, |\mathbf{v}_2(\widetilde{R}(h))| \geq C^{-1}|\mathbf{v}(\widetilde{R}(h))|$ and

$$\frac{e^{-\frac{i\pi}{4}} \mathbf{v}_2(\widetilde{R}(h))}{e^{\frac{i\pi}{4}} \mathbf{v}_1(\widetilde{R}(h))} = \exp\left(-\frac{i}{h}\big(2\Phi(\widetilde{R}(h)) + \pi h/2\big) \right).$$

To prove (2.14), we choose $\widetilde{R}(h) = R + \mathcal{O}(h)$, $\widetilde{R}(h) \geq R$, so that

$$\min_{j \in \mathbb{Z}} \left| \Phi(\widetilde{R}(h)) + (j + 1/4)\pi h \right| \geq \frac{\pi h}{4}; \tag{2.15}$$

this can be done independently of E_0, since $E_0 = 1 + h + \mathcal{O}(e^{-\frac{1}{10h}})$ and thus

$$\Phi(\widetilde{R}(h)) = \int_{3-h}^{5} \sqrt{1 + h - V(y)}\, dy + \sqrt{1+h}(\widetilde{R}(h) - 5) + \mathcal{O}(e^{-\frac{1}{10h}}).$$

3. A GENERAL ARGUMENT

Suppose that $P(h)$ is an operator satisfying the general assumptions of [NaStZw], that is a black box self-adjoint operator, close to the Laplacian and having analytic coefficients near infinity and with a barrier at energy V_0. (A barrier separates the interaction region from infinity – see (3.3) and Fig. 3). We assume that $\mathbb{R}^n \setminus B(0, R_0)$ is contained in the "outside" of the black box and the trapped set at energy V_0. We also assume the Hilbert space on which the operator acts, $\mathcal{H} = \mathcal{H}_{R_0} \oplus L^2(\mathbb{R}^n \setminus B(0, R_0))$, is equipped with an involution $u \mapsto \bar{u}$ equal to complex conjugation on $L^2(\mathbb{R}^n \setminus B(0, R_0))$ and satisfying $\overline{zu} = \bar{z}\bar{u}$, $z \in \mathbb{C}$. The abstract reality assumption on $P(h)$ reads $\overline{P(h)u} = P(h)\bar{u}$.
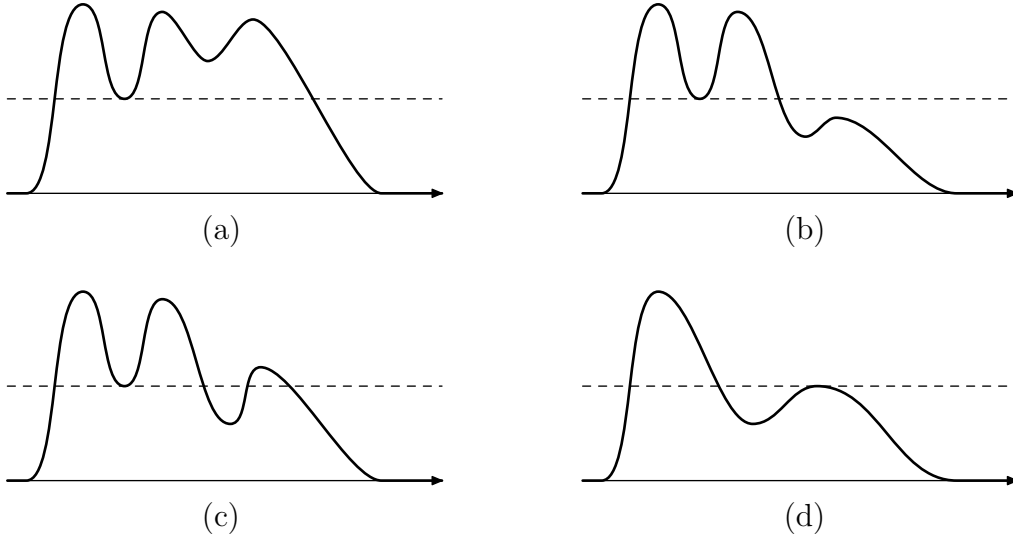
FIGURE 3. Examples of one-dimensional potentials satisfying: (a) condition (1.2) and hence (3.3) and (4.1); (b) conditions (3.3) and (4.1), but not (1.2); (c) condition (3.3), but not (4.1); (d) neither (3.3) nor (4.1). The dashed line corresponds to V_0. In particular, examples (a) and (b) satisfy the assumptions of Theorems 2 and 3.

An example to keep in mind is given by the operator

$$P(h) = -h^2 \Delta_g + V : H^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n),$$

where the measure on L^2 is obtained from the Riemannian metric and $\mathcal{H} := L^2(\mathbb{R}^n)$. The potential $V(x)$ and the metric coefficients $g^{ij}(x)$ are smooth, extend analytically to $\Omega := \{z \in \mathbb{C}^n : |z| \geq R_1, \ |\operatorname{Im} z| \leq \delta|z|\}$, and

$$g^{ij}(z) - \delta^{ij} \to 0, \quad V(z) \to 0, \quad |z| \to \infty, z \in \Omega. \tag{3.1}$$

The trapped set, K_E, at energy $E > 0$ is then defined by

$$K_E := \{(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n : p(x, \xi) = E, \ e^{tH_p}(x, \xi) \not\to \infty, \ t \to \pm\infty\}, \tag{3.2}$$

where $H_p := \sum_{j=1}^n \partial_{\xi_j} p \cdot \partial_{x_j} - \partial_{x_j} p \cdot \partial_{\xi_j}$ and $p(x, \xi) := \sum_{i,j=1}^n g^{ij}(x)\xi_j\xi_i + V(x)$.

The assumption on the interaction region in this case means that $\pi(K_{V_0}) \subset B(0, R_0)$. The barrier assumption means that

$$p^{-1}(V_0) = K_{V_0} \cup \Sigma_{V_0}, \quad \Sigma_{V_0} \cap K_{V_0} = \emptyset, \quad \Sigma_{V_0} \text{ is closed.} \tag{3.3}$$

The more general black box setting allows obstacle problems and other geometric situations. However, the barrier assumption cannot be satisfied for connected manifolds without having a nontrivial potential V.

We denote by $\operatorname{Res}(P(h))$ the set of resonances of $P(h)$ (in an h-independent neighborhood of $\{\operatorname{Re} z > 0, \operatorname{Im} z > 0\}$).

Theorem 2. *Let $P(h)$ satisfy the general assumptions above. Suppose that $z_0 = z_0(h) \in \mathrm{Res}(P(h))$, $\mathrm{Re}\, z_0 = V_0 + \mathcal{O}(h)$, z_0 is simple, and*

$$|\mathrm{Im}\, z_0| = \mathcal{O}(h^\infty), \quad d(z_0(h), \mathrm{Res}(P(h)) \setminus \{z_0(h)\}) > h^N, \tag{3.4}$$

for some N. Suppose that χ and ψ are given by (1.3) *with $R > R_0$. Then there exist $C_0 > 0$ and h_0 such that for $0 < h < h_0$,*

$$\|\chi(P(h) - \mathrm{Re}\, z_0 - i0)^{-1}\chi\|_{\mathcal{H}\to\mathcal{H}} \geq \frac{1}{C_0|\mathrm{Im}\, z_0|}, \tag{3.5}$$

and

$$\|\psi(P(h) - \mathrm{Re}\, z_0 - i0)^{-1}\chi\|_{\mathcal{H}\to\mathcal{H}} \geq \frac{1}{C_0\sqrt{|\mathrm{Im}\, z_0|h}}. \tag{3.6}$$

Remark. As stated in the introduction, it is (3.6) that seems to be the novel aspect. The presence of the square root is (morally) consistent with the results of [BuZw, Lemma A.2] and of [DaVa, Theorem 2].

Proof. We only present the proof of (3.6). Using the involution $u \mapsto \bar{u}$ we define $(u \otimes v)f := u\langle f, \bar{v}\rangle$, where we use the inner product on the black box Hilbert space. Since z_0 is simple, we have

$$\psi(P(h) - z)^{-1}\chi = \frac{\psi u \otimes \chi u}{z - z_0} + \psi R_{z_0}(z, h)\chi,$$

where u is the corresponding normalized resonant state and $R_{z_0}(z, h)$ is holomorphic in

$$[\mathrm{Re}\, z_0 - h^N, \mathrm{Re}\, z_0 + h^N] + i(-h^N, \infty).$$

From [NaStZw, (5.1)] and [BoMi, (1.12)] (or [NoZw, (8.18)]) we see that

$$\|\chi u\|_{\mathcal{H}} = 1 + \mathcal{O}(h^\infty), \quad u = \chi u + \mathcal{O}(h^\infty)_{L^2_{\mathrm{loc}}}. \tag{3.7}$$

Using the maximum principle as in [TaZw, Lemma 2] and the estimates on the resolvent in [TaZw, Lemma 1] we see that

$$\|\chi R_{z_0}(\mathrm{Re}\, z_0, h)\psi\|_{\mathcal{H}\to\mathcal{H}} = \mathcal{O}(h^{-M}), \tag{3.8}$$

for some M. Hence to obtain (3.6) we need to estimate

$$\frac{1}{|\mathrm{Im}\, z_0|}\|\psi u \otimes \chi u\|_{\mathcal{H}\to\mathcal{H}} = \frac{\|\psi u\|_{\mathcal{H}}\|\chi u\|_{\mathcal{H}}}{|\mathrm{Im}\, z_0|}, \tag{3.9}$$

from below.

To estimate $\|\psi u\|_{\mathcal{H}}$ from below we write

$$0 = \mathrm{Im}\langle(P(h) - z_0)u, \mathbb{1}_{B(0,R+t)}u\rangle_{\mathcal{H}} = \mathrm{Im}\langle P(h)u, \mathbb{1}_{B(0,R+t)}u\rangle_{\mathcal{H}} - \mathrm{Im}\, z_0\|\mathbb{1}_{B(0,R+t)}u\|^2_{\mathcal{H}}.$$

Since $P(h)$ is self-adjoint on $\mathcal{H}$ and acts as a symmetric second order operator on $C_c^\infty(\mathbb{R}^n \setminus B(0, R_0))$, we obtain

$$\operatorname{Im} z_0 \| \mathbb{1}_{B(0,R+t)} u \|_{\mathcal{H}}^2 = -\operatorname{Im} \int_{\partial B(0,R+t)} \bar{u} N(x, hD) u \, dS(x),$$

where $N(x, hD_x)$ is a first order semiclassical differential operator. For example, if $P(h) = -h^2 \Delta + V$, then $N(x, hD_x) = i(x/|x|) \cdot hD_x$.

Since from (3.7), $\| \mathbb{1}_{B(0,R+t)} u \|_{\mathcal{H}}^2 = 1 + \mathcal{O}_t(h^\infty)$, we see that

$$\frac{1}{3} |\operatorname{Im} z_0| \leq h \int_{\frac{1}{3}}^{\frac{2}{3}} \int_{\partial B(0,R+t)} |u||N(x, hD_x) u| \, dS(x) dt$$

$$= h \int_{B(0,R+\frac{2}{3}) \setminus B(0,R+\frac{1}{3})} |u||N(x, hD_x) u| \, dx$$

$$\leq C'h \int_{B(0,R+1) \setminus B(0,R)} |u|^2 dx \leq C'h \|\psi u\|_{\mathcal{H}}^2,$$

where we used the equation $(P - z_0)u = 0$ and elliptic estimates to control the first order term term $N(x, hD_x)u$. This shows that

$$\|\psi u\|_{\mathcal{H}} \geq \sqrt{|\operatorname{Im} z_0|/Ch},$$

which combined with (3.8),(3.9) and (3.7) completes the proof of (3.6).   □

4. A METRIC EXAMPLE

We start by using Theorem 2 to obtain a generalization of Theorem 1 to higher dimensions and to more general potentials:

Theorem 3. *Consider a Schrödinger operator $P(h) = -h^2 \Delta + V$ on $L^2(\mathbb{R}^n)$ where V satisfies (3.1) and (3.3). Suppose also that*

$$K_{V_0} = \{(x_0, 0)\}, \quad V'(x_0) = 0, \quad V''(x_0) > 0. \tag{4.1}$$

Then there exists z_0 satisfying the assumptions of Theorem 2 with

$$d(z_0, \operatorname{Res}(P(h)) \setminus \{z_0\}) > h/C, \quad |\operatorname{Im} z_0| < e^{-c_0/h}. \tag{4.2}$$

In particular in the notation of (3.6),

$$\|\psi(P(h) - \operatorname{Re} z_0 \pm i0)^{-1}\chi\|_{L^2 \to L^2} \geq \exp \frac{c}{h}, \tag{4.3}$$

for $0 < h < h_0$ and some $c > 0$.

Proof. The existence of z_0 follows from Corollary in [NaStZw, §5]. The reference operator $P^\sharp(h)$ there can be chosen as $P^\sharp(h) = -h^2 \Delta + V^\sharp(x)$, where $V^\sharp(x) = V(x)$ in a small neighbourhood of x_0 where x_0 is the only critical point and $V^\sharp(x) > V(x_0) + \varepsilon$, $\varepsilon > 0$, outside of that neighbourhood. Since the eigenvalue of $P^\sharp(h)$ corresponding to

the minimum $V_0 = V(x_0)$ is separated from other eigenvalues by h/C (see for instance [HeSj1] and references given there) the same corollary shows the separation from other resonances. □

Remarks. 1. The condition (1.2) implies that (3.3) and (4.1) hold (since $H_p(x \cdot \xi) > 0$ on $\{p = V_0\}$ except at $x = \xi = 0$), but the converse is not true – see Figure 3.

2. When V is analytic and satisfies certain "well-in-the-island" hypotheses, Theorem 3 follows from the work of Helffer–Sjöstrand [HeSj2] and under these stronger assumptions Theorem 2 can then be proved in the same way using the earlier results of Gérard–Martinez [GéMa] in place of the results of [NaStZw].

3. For $P(h) = -h^2\Delta + V$, and for E's satisfying (4.1) (with $V_0 = E$), a result of Nakamura [Na, Proposition 4.1] and [NaStZw, Corollary, §5] show that

$$\|\chi(P(h) - E - i0)^{-1}\chi\|_{L^2 \to L^2} \leq Ch^{-q}, \quad |E - z_j(h)| \geq h^q,$$

where $q \geq 1$ and $z_j(h)$ are the resonances of $P(h)$. Since the density of $\mathrm{Re}\, z_j(h)$ satisfies a Weyl law, this means that the bound is $\mathcal{O}(h^{-q})$, outside of a set of measure $\mathcal{O}(h^{q-n})$, $q > n$.

The example in Theorem 3 can be used directly to obtain examples of resolvent growth for asymptotically conic metrics of the type studied by Rodnianski–Tao [RoTa].

Theorem 4. *Let (M, g) be the following Riemannian manifold:*

$$M = \mathbb{R}_x \times \mathbb{S}_\theta^{n-1}, \quad g = dx^2 + V(x)^{-1}\, d\theta^2, \quad n > 1,$$

where $d\theta^2$ is the round metric on the sphere of radius 1 and $V(x) \in C^\infty(\mathbb{R}; (0, \infty))$ is a function satisfying the assumptions of Theorem 3 and

$$V(x) = \frac{1}{x^2}, \quad |x| \geq R_0.$$

Put

$$\chi(x) = \mathbb{1}_{|x| < R_0}, \quad \psi(x) = \mathbb{1}_{R-1 < |x| < R+1}, \quad R > R_0.$$

Then there exists a sequence $\lambda_k \to \infty$ such that

$$\|\psi(-\Delta_g - \lambda_k \pm i0)^{-1}\chi\|_{L^2(M) \to L^2(M)} \geq \exp(c\sqrt{\lambda_k}), \tag{4.4}$$

for some constant $c > 0$.

Proof. In the (x, θ) coordinates, the Laplacian Δ_g has the form

$$\Delta_g = \partial_x^2 - \frac{(n-1)V'(x)}{2V(x)}\partial_x + V(x)\Delta_S.$$

Here Δ_S is the Laplacian on $\mathbb{S}^{n-1}$. For $k \geq 0$, let $Y_k(\theta)$ be (any) spherical harmonic of order k, i.e. a smooth function on $\mathbb{S}^{n-1}$ such that

$$(-\Delta_S - k(k + n - 2))Y_k = 0, \quad \|Y_k\|_{L^2(\mathbb{S}^{n-1})} = 1,$$

see for example [HöI–II, §17.2] for the spectrum of Δ_S. Then for $u(x) \in C^\infty(\mathbb{R})$ and $\lambda \in \mathbb{R}$, we have

$$-\Delta_g(u(x)Y_k(\theta)) = \Big(-\partial_x^2 + \frac{(n-1)V'(x)}{2V(x)}\partial_x + k(k+n-2)V(x) \Big) u(x)Y_k(\theta).$$

Put $h_k := \big(k(k+n-2)\big)^{-1/2}$ so that

$$h_k^2(-\Delta_g - \lambda)(u(x)Y_k(\theta)) = (P(h_k) - h_k^2\lambda)u(x)Y_k(\theta),$$

where

$$P(h) := -h^2\partial_x^2 + \frac{(n-1)V'(x)}{2V(x)}h^2\partial_x + V(x)\,.$$

Let $R(\lambda) := (-\Delta_g - \lambda)^{-1}$ for $\lambda \notin [0, +\infty)$. It follows that

$$R(\lambda) = \sum_{k\in\mathbb{N}} h_k^2(P(h_k) - h_k^2\lambda)^{-1} \otimes \Pi_k : L^2(M) \to L^2(M),$$

$$L^2(M) \simeq L^2(\mathbb{R}, V(x)^{-\frac{n-1}{2}}dx) \otimes L^2(\mathbb{S}^{n-1}),$$

(4.5)

where $\Pi_k : L^2(\mathbb{S}^{n-1}) \to L^2(\mathbb{S}^{n-1}))$ is the orthogonal projection onto the space of spherical harmonics of order k. The operator $R(\lambda) : C_c^\infty(M) \to C^\infty(M)$ continues meromorphically to $\mathrm{Im}\,\lambda \le 0$, and $(P(h_k) - h_k^2\lambda)^{-1} : C_c^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ continues meromorphically for each k. Hence (4.5) is valid for $\mathrm{Im}\,\lambda \le 0$, with the operator acting on $C_c^\infty(M) \simeq C_c^\infty(\mathbb{R}) \otimes C_c^\infty(\mathbb{S}^{n-1})$.

Hence,

$$\begin{aligned} \|\psi(-\Delta_g - \lambda \pm i0)^{-1}\chi\|_{L^2(M)\to L^2(M)} &= \|\psi R(\lambda \pm i0)\chi\|_{L^2(M)\to L^2(M)} \\ &= \sup_{k\in\mathbb{N}} h_k^2\|\psi(P(h_k) - h_k^2\lambda \pm i0)^{-1}\chi\|_{L_x^2\to L_x^2}, \end{aligned}$$

where $L_x^2 := L^2(\mathbb{R}, V(x)^{-\frac{n-1}{2}}dx)$. We now apply Theorem 3 to $P(h_k)$ and put $\lambda_k = \mathrm{Re}\,z_0(h_k)/h_k^2$. The estimate (4.4) follows from (4.3). Theorem 3 applies to the operator $P(h)$ despite the presence of a first order term, as this term is of order $\mathcal{O}(h)$ in the semiclassical calculus and thus does not affect the classical Hamiltonian flow H_p, and the results of [NaStZw] and Theorem 2 apply to a wide class of semiclassical differential operators including $P(h)$. $\qquad\square$

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