From classical to quantum and back

Purdue University Colloquium

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UC Berkeley

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Wave front set: location and "directions" of singularities

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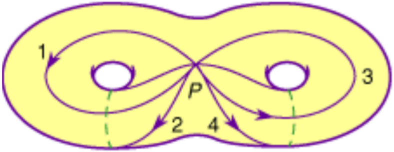

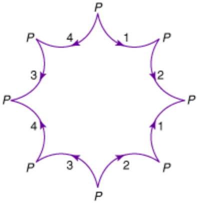

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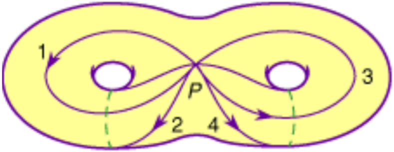

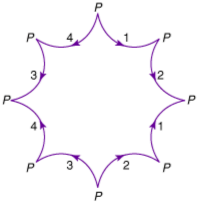

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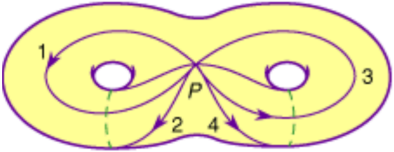

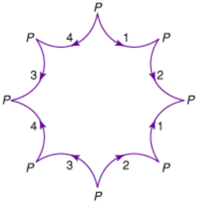

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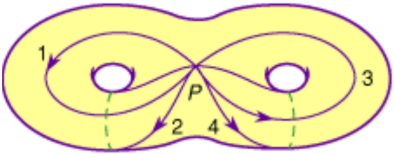

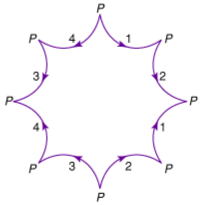

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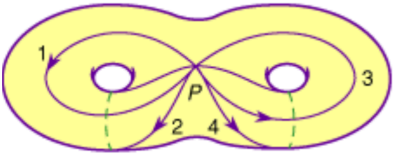

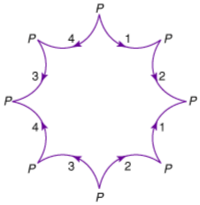

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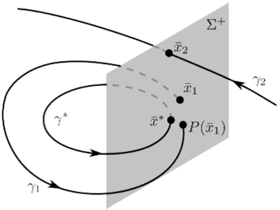

$f_{\mathrm{Sel}} \in \mathcal{C}^\infty((0,\infty))$ is an explicit function

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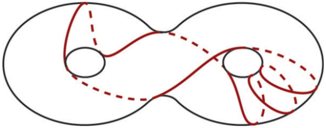

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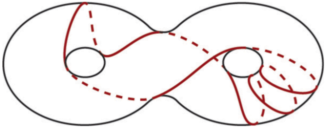

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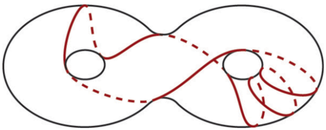

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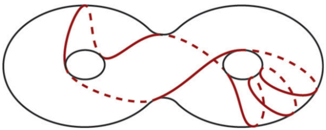

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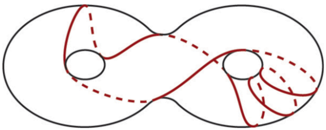

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Key fact: $\mathrm{WF}(K_{\varphi_{-t}^{*}}) \cap N^{*}(\mathbb{R}_t \times \{x = y\}) = \emptyset$

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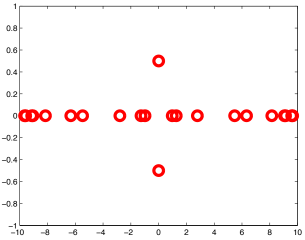
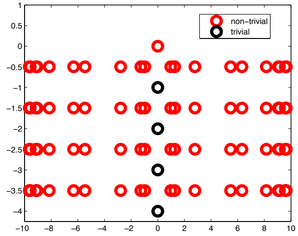


λ_j



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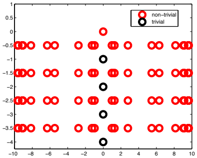
So, in the quantum case $(-\Delta_M)$ the spectral side stays simple, the dynamical side complicates. In the classical case (S^*M, φ_{-t}) the dynamical side stays simple, the spectral side complicates.

We would expect that **?** should be given by eigenvalues of $P = -iV$



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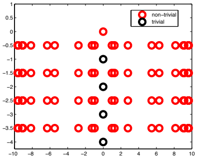

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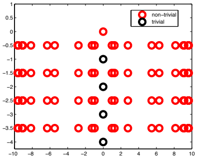

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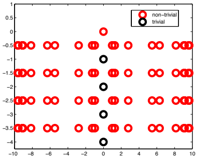

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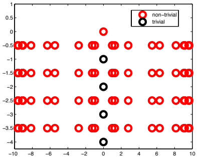

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Pollicott-Ruelle Resonances

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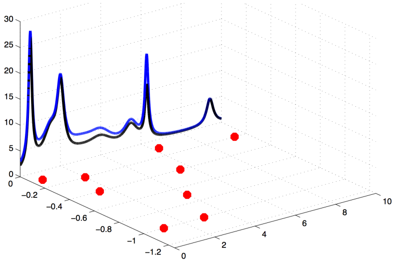

A “real” life example

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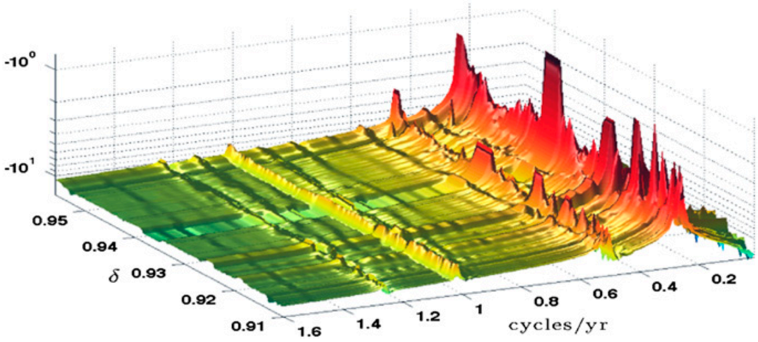

Rough parameter dependence in climate models and the role of Ruelle–Pollicott resonances,
Chekroun–Neelin–Kondrashov–McWilliams–Ghil, 2014

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$\nu_1 > 0$ for contact flows: Dolgopyat '98, Liverani '04, Tsujii '12, Nonnenmacher–Z '15.

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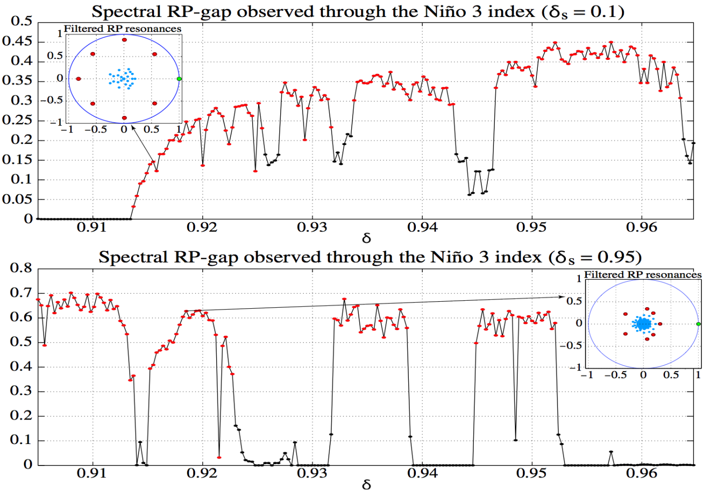

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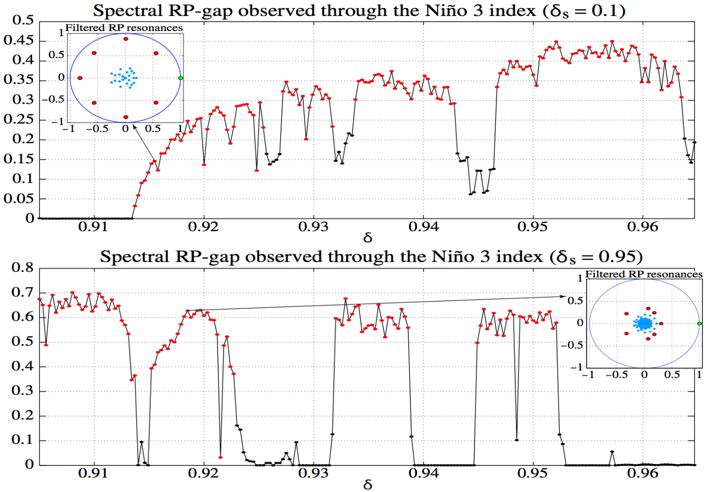

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in the sense of distribution on $(0, \infty)$ *and where the Fourier transform of* F_A *has an analytic extension to* $\operatorname{Im}\lambda < A$ *and for some* M *independent of* A*,*

$$|\widehat{F}_A(\lambda)| = \mathcal{O}_\epsilon((1 + |\lambda|)^M), \quad \operatorname{Im}\lambda < A - \epsilon, \ \textit{for any } \epsilon > 0.$$

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$$\zeta_R(\lambda) := \prod_{\gamma_\#} (1 - e^{i\lambda T_\gamma^\#})^{-1}, \quad \operatorname{Im} \lambda \gg 1,$$

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The noncompact case (the full Smale conjecture) recently completed by Dyatlov–Guillarmou.

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We define one of the building blocks of the zeta function:

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$$\mathrm{WF}'(A) = \{(x, \xi, y, \eta) : (x, y, \xi, -\eta) \in \mathrm{WF}(K_A)\},$$
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Microlocal analysis (semiclassical version)

- Phase space: $(x, \xi) \in T^*X$
- Semiclassical parameter: $h \to 0$, the effective wavelength
- Classical observables: $a(x, \xi) \in C^\infty(T^*X)$
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Basic examples

- $a(x, \xi) = x_j \quad \Longrightarrow \quad \mathrm{Op}_h(a) = x_j$ multiplication operator
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Classical-quantum correspondence

- $[\mathrm{Op}_h(a), \mathrm{Op}_h(b)] = \frac{h}{i}\,\mathrm{Op}_h(\{a, b\}) + \mathcal{O}(h^2)$
- $\{a, b\} = \partial_\xi a \cdot \partial_x b - \partial_x a \cdot \partial_\xi b = H_a b, \quad e^{tH_a}$ Hamiltonian flow
- Example: $[\mathrm{Op}_h(\xi_k), \mathrm{Op}_h(x_j)] = \frac{h}{i}\delta_{jk}$

Standard semiclassical estimates

General question

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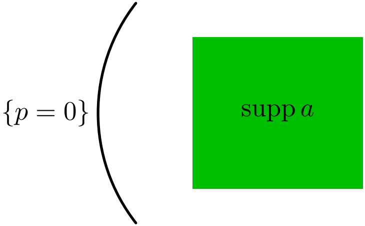

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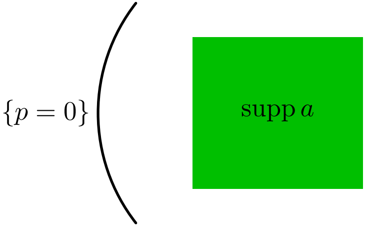
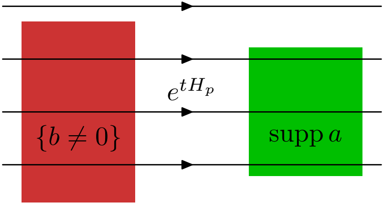


Propagation of singularities



The microlocal picture of the Anosov case

$P - z := (hV/i - z) = \mathrm{Op}_h(p)$,
$p(x,\xi) = \langle \xi, V_x \rangle$
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$$T^*X = E_0^* \oplus E_s^* \oplus E_u^*,$$
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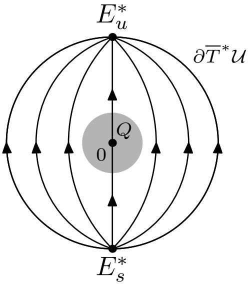

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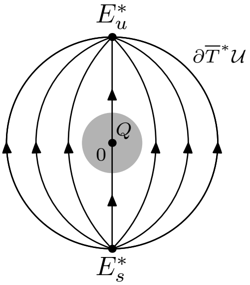

Key Fact

$P - z : \mathcal{H}^r \to \mathcal{H}^r$ continues meromorphically to $z \in \mathbb{C}$.

By Fredholm theory, enough if for $Q = \mathrm{Op}_h(q)$, $q \in C_0^\infty(T^*X)$

$$\|u\|_{\mathcal{H}^r} \le Ch^{-1}\|Pu\|_{\mathcal{H}^r} + C\|Qu\|_{\mathcal{H}^r}$$

where $\mathcal{H}^r$ is an anisotropic Sobolev space

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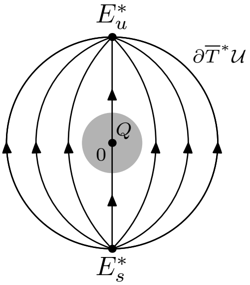

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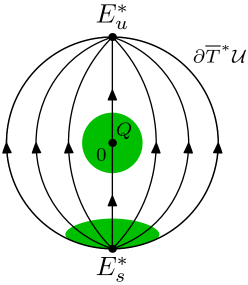

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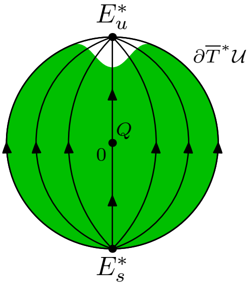

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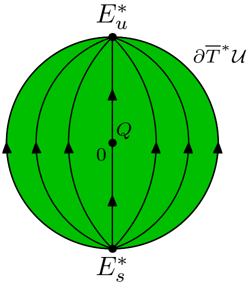

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Further applications to purely geometric inverse problems Guillarmou '14, Guillarmou–Salo–Uhlmann '15.