# FRACTAL WEYL LAW FOR OPEN QUANTUM CHAOTIC MAPS

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1. INTRODUCTION

In this paper we study semiclassical quantizations of Poincaré maps arising in scattering problems with hyperbolic classical flows which have topologically one dimensional trapped sets. The main application is the proof of a fractal Weyl upper bound for the number of resonances/scattering poles in small domains.

The reduction of open scattering problems with hyperbolic classical flows to quantizations of open maps has been recently described by the authors in [28]. In this introduction we show how the main result of the current paper applies to the case of scattering by several convex obstacles, and explain ideas of the proof in that particular setting.
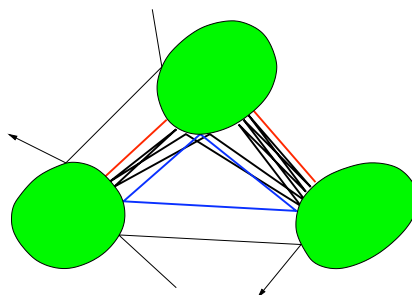
Let $\mathcal{O} = \bigcup_{j=1}^{J} \mathcal{O}_j$ where $\mathcal{O}_j \Subset \mathbb{R}^n$ are open, strictly convex, have smooth boundaries, and satisfy the *Ikawa condition*:

$$(1.1) \qquad \overline{\mathcal{O}}_k \cap \operatorname{convex\ hull}(\overline{\mathcal{O}}_j \cup \overline{\mathcal{O}}_\ell) = \emptyset, \quad j \neq k \neq \ell.$$

The classical flow on $(\mathbb{R}^n \setminus \mathcal{O}) \times \mathbb{S}^{n-1}$ (with the second factor responsible for the direction) is defined by free motion away from the obstacles, and normal reflection on the obstacles – see the figure above and also §6.3 for a precise definition. An important dynamical object is the *trapped set*, K, consisting of $(x, \xi) \in (\mathbb{R}^n \setminus \mathcal{O}) \times \mathbb{S}^{n-1}$ which do not escape to infinity under forward or backward flow.

The high frequency waves on $\mathbb{R}^n \setminus \mathcal{O}$ are given as solutions of the Helmholtz equation with Dirichlet boundary conditions:

$$(-\Delta - \lambda^2)u = 0, \quad u \in H^2(\mathbb{R}^n \setminus \mathcal{O}) \cap H_0^1(\mathbb{R}^n \setminus \mathcal{O}), \quad \lambda \in \mathbb{R}.$$

The scattering resonances are defined as poles of the meromorphic continuation of

$$R(\lambda) = (-\Delta - \lambda^2)^{-1} \; : \; L^2_{\mathrm{comp}}(\mathbb{R}^n \setminus \mathcal{O}) \longrightarrow L^2_{\mathrm{loc}}(\mathbb{R}^n \setminus \mathcal{O})$$

to the complex plane for n odd and to the logarithmic plane when n is even.

The multiplicity of a (non-zero) resonance λ is given by

(1.2) $m_R(\lambda) = \mathrm{rank} \oint_\gamma R(\zeta)d\zeta\,, \;\; \gamma : t \mapsto \lambda + \epsilon e^{2\pi it}\,, \; 0 \le t < 1\,, \;\; 0 < \epsilon \ll 1\,.$

Our general result for hyperbolic quantum monodromy operators (see Theorem 4 in §5.4) leads to the following result for scattering by several convex obstacles.

Theorem 1. *Let* $\mathcal{O} = \bigcup_{j=1}^J \mathcal{O}_j$ *be a union of strictly convex smooth obstacles satisfying* (1.1). *Then for any fixed* $\alpha > 0$,

(1.3) $\displaystyle\sum_{\substack{-\alpha < \mathrm{Im}\,\lambda \\ r \le |\lambda| \le r+1}} m_R(\lambda) = \mathcal{O}(r^{\mu+0})\,, \quad r \longrightarrow \infty\,,$

where $2\mu + 1$ *is the box dimension of the* trapped set.

If the trapped set is of pure dimension (see §5.4) then the bound is $\mathcal{O}(r^\mu)$. *In the case of* $n = 2$ *the trapped set is always of pure dimension, which is its Hausdorff dimension.*

We should stress that even a weaker bound,

$$\sum_{\substack{-\alpha < \mathrm{Im}\,\lambda \\ 1 \le |\lambda| \le r}} m_R(\lambda) = \mathcal{O}(r^{\mu+1+0})\,,$$

corresponding the standard Weyl estimate $\mathcal{O}(r^n)$ for frequencies of a bounded domain, was not known previously. Despite various positive indications which will be described below no lower bound is known in this setting.

The study of counting of scattering resonances was initiated in physics by Regge [34] and in mathematics by Melrose [24] who proved a global bound in odd dimensions:

$$\sum_{|\lambda| \le r} m_R(\lambda) = \mathcal{O}(r^n)\,.$$

This bound is optimal for the sphere and for obstacles with certain elliptic trapped trajectories but the existence of a general lower bound remains open – see [44],[45] and references given there. For even dimensions an analogous bound was established by Vodev [52]

The fractal bound (1.3) for obstacles was predicted by the second author in [37] where fractal upper bounds for the number of resonances were established for a wide class of semiclassical operators with analytic coefficients (such as $-h^2\Delta + V$ with V equal to a restriction of a suitable holomorphic function in a complex neighbourhood of $\mathbb{R}^n$); promising numerics were obtained in [21] for the case of a three bump potential, and in [35] for the Hénon–Heiles Hamiltonian. The bound of the type (1.3) was first proved for resonances
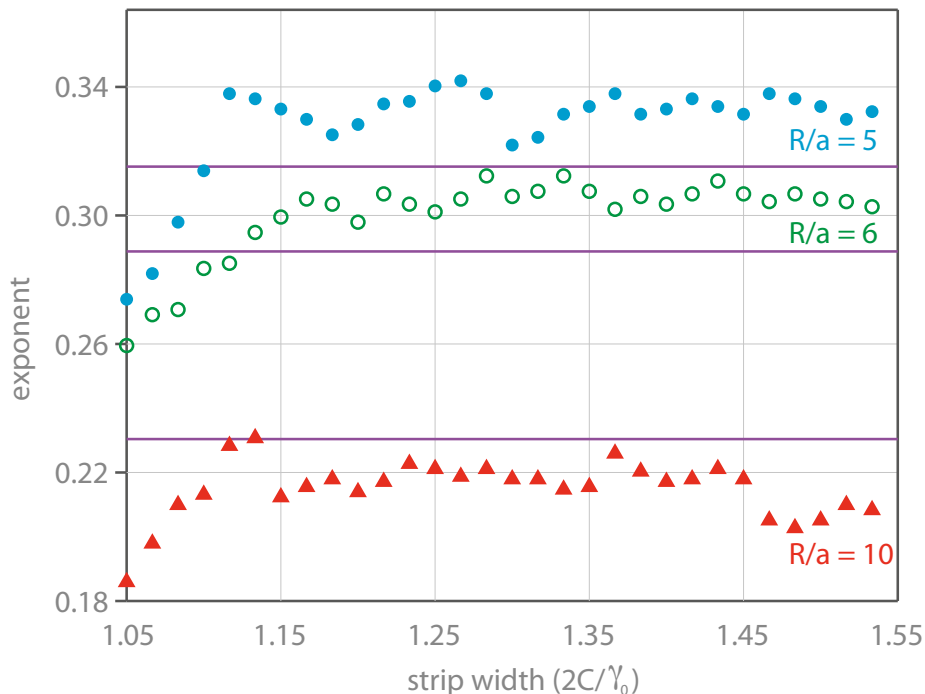
FIGURE 1. This figure, taken from [22], shows resonances computed using the semiclassical zeta function for scattering by three symmetrically placed discs of radii a and distances R between centers. The horizontal axis represents $2C/\gamma_0$ where γ_0 is a classical rate of decay (the imaginary parts of resonances tend to cluster at $\operatorname{Im}\lambda \sim -\gamma_0/2$ and that motivates this rescaling — see [22]) and the vertical axis corresponds to the best fit for the slope of $\log N(C,r)/\log r - 1$, that is the exponent in the fractal Weyl law (1.4). The three lines correspond to the three values of the Hausdorff dimension μ.

associated to hyperbolic quotients by Schottky groups without parabolic elements (that is for the zeros of the Selberg zeta functions) in [13], and for a general class of semiclassical problems in [43]. Theorem 4 below provides a new proof of the result in [43] in the case of topologically one dimensional trapped sets. The new proof is simpler by avoiding the complicated second microlocalization procedure of [43, §5]. The reduction to Poincaré sections obtained using the Schrödinger propagator [28] replaces that step. The only rigorous fractal lower bound was obtained in a special toy model of open quantum map in [29]. For some classes of hyperbolic surfaces lower bounds involving the dimension were obtained in [18].

In the case of three discs in the plane, results of numerical experiments based on semi-classical zeta function calculations [10] were presented in [22]. They suggest that a global

version of the Weyl law might be valid:

$$(1.4) \qquad N(\alpha, r) \stackrel{\text{def}}{=} \sum_{\text{Im}\,\lambda > -\alpha,\; |\lambda| \le r} m_R(\lambda) \sim C(\alpha) r^{\mu+1}\,, \quad r \longrightarrow \infty\,,$$

see Fig. 1. A similar study for the scattering by four hard spheres centered on a tetrahedron in three dimensions was recently conducted in [7], and lead to a reasonable agreement with the above fractal Weyl law, at least for α large enough. We stress however that the method of calculation based on the zeta function, although widely accepted in the physics literature, does not have a rigorous justification and may well be inaccurate. Experimental validity of the fractal Weyl laws has been investigated in the setting of microwave cavities [20] — see Fig. 2. The theoretical model is precisely the one for which Theorem 1 holds. The fractal Weyl law has been considered (and numerically checked) for various open chaotic quantum maps like the open kicked rotator [36] and the open baker's map [29, 32]. Theorems 2 and 3 below lead to a rigorous fractal Weyl upper bound in this setting of open quantum maps with a hyperbolic trapped set. Fractal Weyl laws have also been proposed in other types of chaotic scattering systems, like dielectric cavities [53], as well as for resonances associated with classical dynamical systems [47, 3, 8].

Since the posting of this paper two important mathematical contributions have appeared: Datchev-Dyatlov [4] generalized estimates of [13] to arbitrary asymptotically hyperbolic manifolds, and Naud [27] showed that for hyperbolic surfaces considered in [13] fractal upper bounds can be improved in the strip $\text{Re}\,s > \delta/2$. That is consistent with predictions about the density of decaying rates made in [22] (see in particular Fig. 2 (c) there and Fig. 1 above).

The proof of Theorem 1 uses Theorem 4 below, which holds for general *hyperbolic quantum monodromy operators* defined in §2. Here we will sketch how these operators appear in the framework of scattering by several convex bodies. In the case of two obstacles they were already used in the precise study of resonances conducted by Ch. Gérard [11]. The detailed analysis will be presented in §6.

To connect this setting with the general semiclassical point of view, we write

$$z = \frac{i}{h}(h^2\lambda^2 - 1)\,, \quad h \sim |\,\text{Re}\,\lambda|^{-1}\,, \quad z \sim 0\,,$$

and consider the problem

$$P(z)u = 0\,, \quad P(z) \stackrel{\text{def}}{=} \left(\frac{i}{h}(-h^2\Delta - 1) - z\right)u(x)\,, \quad u\!\restriction_{\partial\mathcal{O}} = 0\,, \quad u \text{ outgoing.}$$

The precise meaning of "outgoing" will be recalled in §6. This rescaling means that investigating resonances in $\{r \le |\lambda| \le r+1,\; \text{Im}\,\lambda > -\alpha\}$, corresponds to investigating the poles of the meromorphic continuation of $P(z)^{-1}$ in a fixed size neighbourhood of 0 (in these notations the resonances are situated on the half-plane $\text{Re}\,z > 0$).
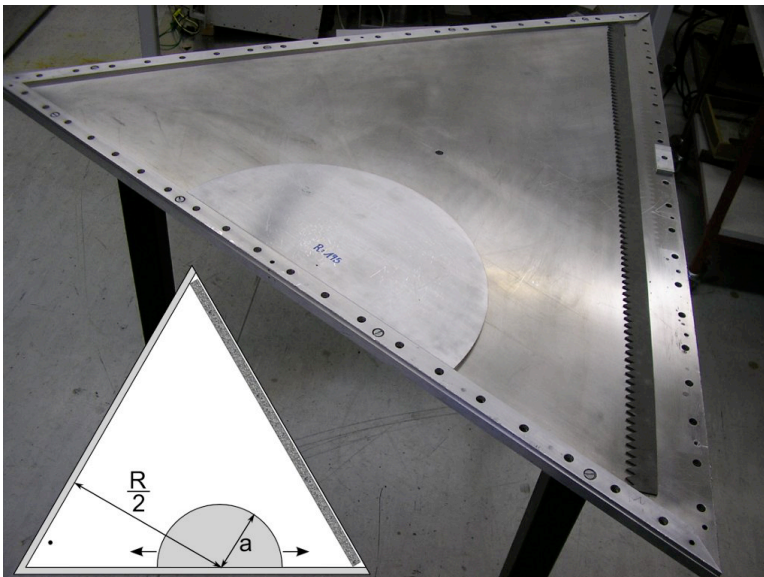
FIGURE 2. The experimental set-up of the Marburg quantum chaos group [20] the five disc, symmetry reduced, system. The hard walls correspond to the Dirichlet boundary condition, that is to odd solutions (by reflection) of the full problem. The absorbing barrier, which produces negligible reflection at the considered range of frequencies, models escape to infinity.

The study of this meromorphic continuation can be reduced to the boundary through the following well-known construction. To each obstacle $\mathcal{O}_j$ we associate a Poisson operator $H_j(z) : \mathcal{C}^\infty(\partial\mathcal{O}_j) \to \mathcal{C}^\infty(\mathbb{R}^n \setminus \mathcal{O}_j)$ defined by

$$P(z)H_j(z)v(x) = 0\,, \quad x \in \mathbb{R}^n \setminus \mathcal{O}_j\,, \quad H_j(z)v\!\restriction_{\partial\mathcal{O}_j} = v\,, \quad H_j(z)v \text{ outgoing.}$$

Besides, let $\gamma_j : \mathcal{C}^\infty(\mathbb{R}^n) \to \mathcal{C}^\infty(\partial\mathcal{O}_j)$ be the restriction operators, $\gamma_j u \overset{\text{def}}{=} u \restriction_{\partial\mathcal{O}_j}$. As described in detail in §6.1 the study of the resolvent $P(z)^{-1}$ can be reduced to the study of

$$(I - \mathcal{M}(z, h))^{-1} : \bigoplus_{j=1}^J \mathcal{C}^\infty(\partial\mathcal{O}_j) \longrightarrow \bigoplus_{j=1}^J \mathcal{C}^\infty(\partial\mathcal{O}_j)\,,$$

where

(1.5)
$$(\mathcal{M}(z, h))_{ij} = \begin{cases} -\gamma_i H_j(z) & i \neq j\,, \\ 0 & i = j\,. \end{cases}$$

The structure of the operators $\gamma_i H_j(z)$ is quite complicated due to diffractive phenomena. In the semiclassical/large frequency regime and for complex values of z, the operators $H_j(z)$ have been analysed by Gérard [11, Appendix] ($\text{Im}\, z > -C$) and by Stefanov-Vodev [46, Appendix] ($\text{Im}\, z > -C\log(1/h)$). We refer to these papers and to [16, Chapter 24] and

[25] for more information about propagation of singularities for boundary value problems and for more references.

Using the propagation of singularities results obtained from the parametrix (see §6.2) the issue of invertibility of $(I - \mathcal{M}(z, h))$ can be microlocalized to a neighbourhood of the trapped set, where the structure of $\mathcal{M}(z, h)$ is described using h-Fourier integral operators — see §3.4 for the definition of these objects.

At the classical level, the reduction of the flow to the boundary of the obstacles proceeds using the standard construction of the billiard map on the reduced phase space

$$B^* \partial \mathcal{O} = \bigsqcup_{j=1}^{J} B^* \partial \mathcal{O}_j \,,$$

where $B^* \partial \mathcal{O}_k$ is the co-ball bundle over the boundary of $\mathcal{O}_k$ defined using the induced Euclidean metric (see Fig. 3 in the two dimensional case). Strictly speaking, the billiard map is not defined on the whole reduced phase space (for instance in Fig. 3 it is not defined on the point ρ'); one can describe it as a symplectic relation $F \subset B^* \partial \mathcal{O} \times B^* \partial \mathcal{O}$, union of the relations $F_{ij} \subset B^* \partial \mathcal{O}_i \times B^* \partial \mathcal{O}_j$, $i \neq j$ encoding the trajectory segments going from $\mathcal{O}_j$ to $\mathcal{O}_i$:

$$(\rho', \rho) \in F_{ij} \subset B^* \partial \mathcal{O}_i \times B^* \partial \mathcal{O}_j$$

$$\Longleftrightarrow$$

(1.6)
$$\exists\, t > 0 \,, \xi \in \mathbb{S}^{n-1} \,,\ x \in \partial \mathcal{O}_j \,,\ x + t\xi \in \partial \mathcal{O}_i \,,\ \langle \nu_j(x), \xi \rangle > 0 \,,$$
$$\langle \nu_i(x + t\xi), \xi \rangle < 0 \,,\ \pi_j(x, \xi) = \rho \,,\ \pi_i(x + t\xi, \xi) = \rho' \,.$$

(here $\pi_k : S^*_{\partial \mathcal{O}_k}(\mathbb{R}^n) \to B^* \partial \mathcal{O}_k$ is the natural orthogonal projection.)

To the relation F we can associate various *trapped* sets:

(1.7)
$$\mathcal{T}_\pm \overset{\text{def}}{=} \bigcap_{k=0}^{\infty} F^{\pm k}(B^* \partial \mathcal{O}) \,, \qquad \mathcal{T} \overset{\text{def}}{=} \mathcal{T}_+ \cap \mathcal{T}_- \,.$$

Notice that $\mathcal{T}$ is directly connected with the trapped set K for the scattering flow, defined in the beginning of this introduction:

$$\mathcal{T} \cap B^* \partial \mathcal{O}_j = \pi_j(K \cap S^*_{\partial \mathcal{O}_j}(\mathbb{R}^n)) \,.$$

The sets $\mathcal{T}_\pm$ and $\mathcal{T}$ for the 2D scattering problem of Fig. 3 are shown in Fig. 4.

The strict convexity of the obstacles entails that the trapped rays are uniformly unstable: in the dynamical systems terminology, $\mathcal{T}$ is an invariant hyperbolic set for the relation F (see §2). The set $\mathcal{T}$ is a closed totally disconnected subset of $B^* \partial \mathcal{O}$. That corresponds to the fact that the set of trapped rays on the energy surface is topologically one dimensional – see 2 and [28, §2.2].

The boundary of $B^* \partial \mathcal{O}_k$, which consists of covectors of length one, corresponds to rays which are tangent to $\partial \mathcal{O}_k$ (the glancing rays), and these produce complicated effects in
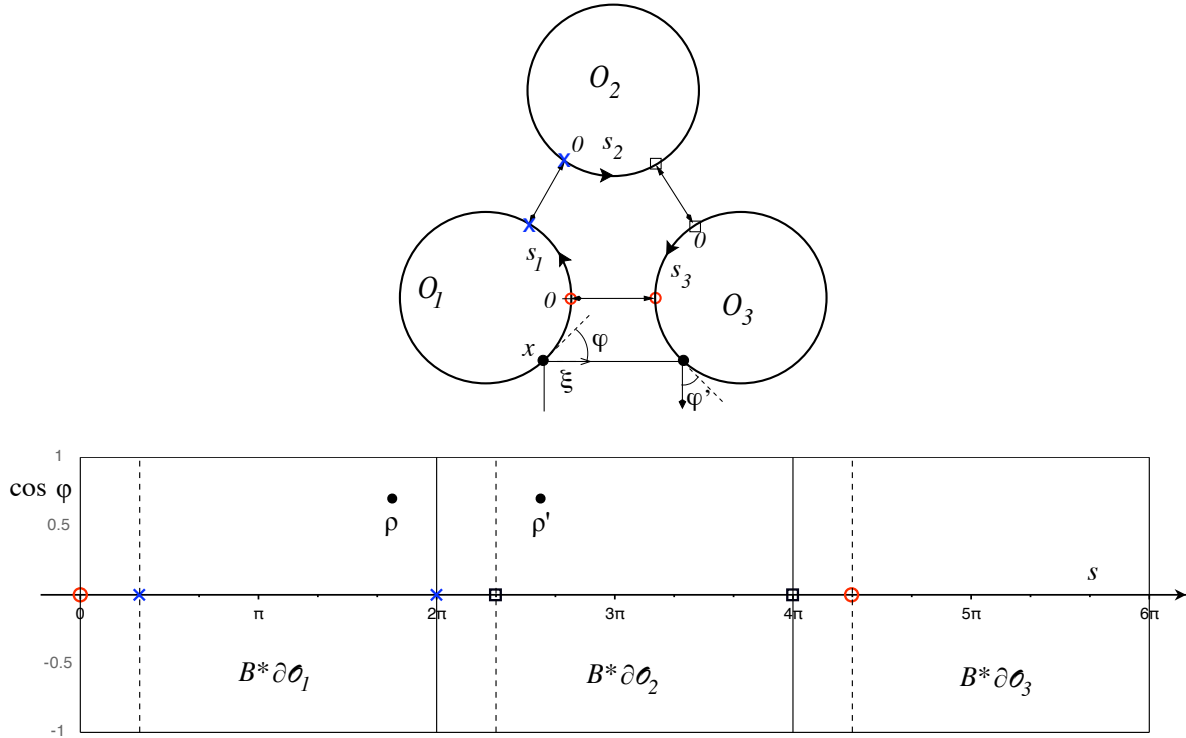
FIGURE 3.   (after [31]) Reduction on the boundary for the symmetric three disk scattering problem (the distance D between the centers is 3, the radius of the disks is 1). Top: the trajectories hitting the obstacles can be parametrized by position of the impact along the circles $\partial\mathcal{O}_i$, (length coordinate $s_i \in [0, 2\pi)$) and the angle between the velocity after impact and the tangent to the circle (momentum coordinate $\cos\varphi \in [-1, 1]$). We show three short periodic orbits, and a transient orbit. Bottom: reduced phase space $B^*\partial\mathcal{O} = \sqcup_{i=1}^{3} B^*\partial\mathcal{O}_i$ of the obstacles (we concatenate the three length coordinates into a single parameter $s = s_i + 2\pi(i-1)$, $i = 1, 2, 3$). Each of the 3 periodic orbits is represented by 2 points on the horizontal axis with the same symbol as in the above picture (circles, crosses, squares), while the transient orbit is represented by the successive points ρ, $\rho' = F(\rho)$. The partial phase space $\{s_i \in [0, \pi/3]\}$ shown in Fig. 4 corresponds to the 3 thin vertical strips delimitied by successive solid and dashed lines.

the operators $H_j(z)$. However, Ikawa's condition (1.1) guarantees that none of these rays belongs to the trapped set:

$$(1.8) \qquad\qquad\qquad \mathcal{T} \cap S^*\partial\mathcal{O} = \emptyset \,.$$
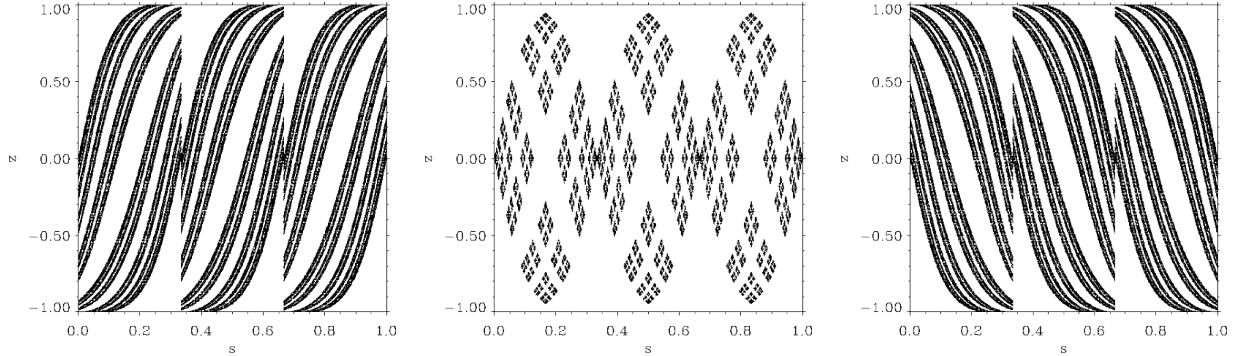
FIGURE 4. The unstable manifold, $\mathcal{T}_+$, the trapped set, $\mathcal{T}$, and the stable manifold $\mathcal{T}_-$ and for the system of Fig. 3. Compared with Fig. 3 (bottom), we only plot the union of the 3 thin vertical strips $\{s_i \in [0, \pi/3]\}$, represented by a single coordinate $\tilde{s} = s_i/\pi + (i-1)/3$, $i = 1, 2, 3$. The sets $\mathcal{T}_\pm$ are smooth along the unstable/stable direction and fractal transversely, while $\mathcal{T}$ has the structure of the product of two Cantor sets. The left strip corresponds to the one indicated in Fig. 3. (the figures are from [31]).

At the quantum level, the operator $\mathcal{M}(z, h)$ has the properties of a Fourier integral operator associated with the billiard map away from the glancing rays, but its structure near these rays is more complicated due to diffraction effects. Fortunately, the property (1.8) implies that, as far as the poles of $(I - \mathcal{M}(z, h))^{-1}$ are concerned, these annoying glancing rays are irrelevant. Indeed, we will show in §6 that the operator $\mathcal{M}(z, h)$ can be replaced by a truncated operator of the form

$$(1.9) \qquad M(z, h) = \Pi_h e^{-g^w} \mathcal{M}(z, h) e^{g^w} \Pi_h + \mathcal{O}_{L^2(\partial\mathcal{O}) \to L^2(\partial\mathcal{O})}(h^N)) \Pi_h \,,$$

where $g(x, \xi)$ is an appropriate escape function, and Π_h is an orthogonal projector of finite rank comparable with h^{1-n}, microlocally corresponding to a compact neighbourhood of $\mathcal{T}$ not containing the glancing rays. The operator $M(z, h)$ is thus "nice": it is of the same type as the *hyperbolic quantum monodromy operators* constructed in [28] to study semiclassical scattering problems such as $-h^2\Delta + V(x)$, with $V \in C_c^\infty(\mathbb{R}^n)$. The study of the poles of $(I - M(z, h))^{-1}$ can then be pursued in parallel for both types of problems (see §5).

For simplicity let us assume that the trapped set $\mathcal{T}$ is of pure dimension $\dim \mathcal{T} = 2\mu$. The fractal Weyl upper bound (1.3) corresponds to the statement that, for any fixed $r > 0$, there exists $C_r > 0$ such that

$$(1.10) \qquad -\frac{1}{2\pi i} \operatorname{tr} \int_{\{|z|=r\}} (I - M(z, h))^{-1} \partial_z M(z, h) \, dz \le C_r \, h^{-\mu} \,.$$

To prove it we want to further modify the monodromy operator $M(z, h)$, and replace it by

$$(1.11) \qquad \widetilde{M}(z, h) = \Pi_W \big(e^{-G^w} M(z, h) e^{G^w} + \mathcal{O}(\epsilon) \big) \Pi_W \,, \qquad 0 < \epsilon \ll 1 \,.$$

Here G is a finer escape function, and $\widetilde{\Pi}_W$ is a finer finite rank projector, now associated with a much thinner neighbourhood of $\mathcal{T}$, of diameter $\sim h^{\frac{1}{2}}$, and therefore of volume comparable to $h^{\frac{1}{2}(2(n-1)-2\mu)}$ (see the definition of the dimension in (5.22) below). This projector, and the escape function G, will be constructed in §5 using symbols in an exotic pseudodifferential class, and the associated symbol calculus. The rank of an operator microlocalized to a set in $T^*\mathbb{R}^{n-1}$ of volume v is estimated using the uncertainty principle by $\mathcal{O}(h^{-n+1}v)$. Hence, when $v \sim h^{\frac{1}{2}(2(n-1)-2\mu)}$ we obtain the bound $\mathcal{O}(h^{-\mu})$ for the rank of Π_W. This in turn shows that

$$\log|\det(I - \widetilde{M}(z,h))| = \mathcal{O}(h^{-\mu}).$$

We also show that at some fixed point z_0, the above expression is bounded from *below* by $-Ch^{-\mu}$. Jensen's inequality then leads to (1.10).

In the case where $\mathcal{T}$ is not of pure dimension, one just needs to replace μ by $\mu+0$ in the above estimates.

The strategy for proving fractal upper bounds for obstacle problems using the reduction to the boundary applies to more general situations. The operator $\widetilde{M}(z,h)$ can be replaced by an abstract object: a *hyperbolic open quantum map*. Such quantum maps are of independent interest — see [29] and references given there. In [28] we have shown that the study of scattering resonances for classically hyperbolic systems with *topologically* one dimensional trapped sets can be reduced to the study of such quantum maps (the topological condition on the trapped set is necessary for the construction of the quantum map). Hence, in addition to handling boundary value problems the current approach provides a different proof of the bounds in [43] in this topologically one dimensional case. We should stress that trapped sets which are not topologically one dimensional can occur in interesting situations, most notably in the setting of hyperbolic quotients recently studied in [4] (see for instance the case of quasifuchsian quotients described in [4, App. A]).

The paper is organized as follows. In §2 we give a general definition of hyperbolic open quantum maps and quantum monodromy operators. This definition will be compatible with both the case of several convex obstacles, and the case of semiclassical potential scattering, after the reduction of [28]. The technical preliminaries are given in §3 where various facts about semiclassical microlocal analysis are presented. In particular, we investigate the properties of exotic symbol classes, which will be necessary to analyze the weight (escape function) G used to conjugate the monodromy operator in (1.11), and to construct the projection Π_W. The weight G is constructed in §4, using our dynamical assumptions. In §5 we construct the projection Π_W, and prove the resulting fractal upper bounds on the number of resonances in the general framework of hyperbolic quantum monodromy operators. Finally, §6 is devoted to a detailed study of the obstacle scattering problem: we show that the reduction to the boundary operator $\mathcal{M}(z)$, combined with propagation of singularities, leads to a hyperbolic quantum monodromy operator (1.9), to which the results of the earlier sections can be applied.

2. Hyperbolic open quantum maps

In this section we start from a hyperbolic open map, and quantize it into an open quantum map. The quantum monodromy operators constructed in [28] have the same structure, but also holomorphically depend on a complex parameter z.

Let $Y_j \Subset \mathbb{R}^d$, $j = 1, \cdots J$, be open contractible sets, and let

$$Y \overset{\text{def}}{=} \bigsqcup_{j=1}^{J} Y_j \subset \bigsqcup_{j=1}^{J} \mathbb{R}^d \, ,$$

be their disjoint union. We also define a local phase space,

$$\mathcal{U} \overset{\text{def}}{=} \bigsqcup_{j=1}^{J} U_j \subset \bigsqcup_{j=1}^{J} T^* \mathbb{R}^d \, , \quad U_j \quad \text{open}, \quad U_j \Subset T^* Y_j \, .$$

Let $\mathcal{T} \Subset \mathcal{U}$ be a compact subset and suppose that

$$f \ : \ \mathcal{T} \longrightarrow \mathcal{T}$$

is an invertible transformation which satisfies

(2.1) $f \subset F \subset \mathcal{U} \times \mathcal{U} \, , \quad (f \text{ is identified with its graph}),$

where F is a *smooth Lagrangian relation* with boundary, and $\partial F \cap f = \emptyset$. We assume that F is locally the restriction of a smooth symplectomorphism (see below for a more precise statement). In particular, F is at most single valued:

$$(\rho', \rho), (\rho'', \rho) \in F \implies \rho' = \rho'' \, ,$$

and similarly for F^{-1}. By a slight abuse of notation, we will sometimes replace the graph notation, $(\rho', \rho) \in F$, by the map notation, $\rho' = F(\rho)$. As a result of (2.1), for any $\rho \in \mathcal{T}$ the tangent map $df_\rho : T_\rho \mathcal{U} \longrightarrow T_{f(\rho)} \mathcal{U}$ is well defined, and so is its inverse.

Such a relation F can be considered as an *open canonical transformation* on $\mathcal{U}$. Here, open means that the map F (F^{-1}) is only defined on a subset $\pi_R(F)$ ($\pi_L(F)$, respectively) of $\mathcal{U}$; the complement $\mathcal{U} \setminus \pi_R(F)$ can be thought of as a hole through which particles escape to infinity.

We now make a stringent dynamical assumption on the transformation f, by assuming it to be *hyperbolic* (equivalently, one says that $\mathcal{T}$ is a hyperbolic set for F). This means that, at every $\rho \in \mathcal{T}$, the tangent space $T_\rho \mathcal{U}$ decomposes as

(2.2)
$$\begin{aligned}
&i) \ T_\rho \mathcal{U} = E_\rho^+ \oplus E_\rho^- \, , \quad \dim E_\rho^\pm = d \, , \quad \text{with the properties} \\
&ii) \ df_\rho(E_\rho^\pm) = E_{f(\rho)}^\pm \, , \\
&iii) \ \exists \, 0 < \theta < 1 \, , \ \forall \, v \in E_\rho^\mp \, , \ \forall n \geq 0 \, , \quad \|df_\rho^{\pm n}(v)\| \leq C\theta^n \|v\| \, .
\end{aligned}$$

This decomposition is assumed to be continuous in ρ, and the parameters θ, C can be chosen independent of ρ. It is then a standard fact that [19, §6.4,§19.1]

$$iv) \ \mathcal{T} \ni \rho \longmapsto E_\rho^\pm \subset T_\rho(\mathcal{U}) \ \text{ is Hölder-continuous}$$

(2.3) $v)$ any $\rho \in \mathcal{T}$ admits local stable($-$)/unstable($+$) manifolds $W_{\text{loc}}^\pm(\rho)$,

$$\text{tangent to } E_\rho^\pm$$

Let us now describe the relation F more precisely. It is a disjoint union of symplecto-morphisms

$$F_{ik} \ : \ \widetilde{D}_{ik} \subset U_k \longrightarrow F_{ik}(\widetilde{D}_{ik}) = \widetilde{A}_{ik} \subset U_i \,,$$

where $\widetilde{A}_{ik} \subset U_i$ and $\widetilde{D}_{ik} \subset U_k$ are open neighbourhoods of the arrival and departure subsets of $\mathcal{T}$,

$$A_{ik} \overset{\text{def}}{=} \{\rho \in \mathcal{T}_i \ : \ f^{-1}(\rho) \in \mathcal{T}_k\} = \mathcal{T}_i \cap f(\mathcal{T}_k) \,, \quad \mathcal{T}_j \overset{\text{def}}{=} \mathcal{T} \cap U_j \,,$$

$$D_{ik} \overset{\text{def}}{=} \{\rho \in \mathcal{T}_k \ : \ f(\rho) \in \mathcal{T}_i\} = \mathcal{T}_k \cap f^{-1}(\mathcal{T}_i) \,.$$

(see Fig. 6 for a plot of the sets $\widetilde{A}_{ik}$ and $\widetilde{D}_{ik}$ for the scattering by three disks, and Fig. 4, center, for the trapped set). We also write

$$\widetilde{D}_k \overset{\text{def}}{=} \bigcup_j \widetilde{D}_{jk} \,, \quad \widetilde{D} \overset{\text{def}}{=} \bigsqcup_k \widetilde{D}_k \,,$$

and similarly for the other sets above. Notice that we have $\widetilde{D} = \pi_R(F)$, $\widetilde{A} = \pi_L(F)$.

On the quantum level we associate to F *hyperbolic open quantum maps* defined as follows:

Definition 2.1. *A hyperbolic open quantum map $M = M(h)$, is an h-Fourier integral operator quantizing a smooth Lagrangian relation F of the type described above,*

$$M \in I_{0+}(Y \times Y, F') \,,$$

(here F' is the twisting of the relation F so that F' becomes Lagrangian in $T^*(Y \times Y)$ – see §3.4 for the definition of this class of operators). In particular, M is microlocalized in the interior of F': its semiclassical wavefront set (see (3.25)) satisfies

$$\text{WF}_h(M) \cap \partial F' = \emptyset \,.$$

We also assume that there exists some $a_M \in \mathcal{C}_c^\infty(T^*Y)$, with $\text{supp}\, a_M$ contained in a compact neighbourhood $\mathcal{W}$ of $\mathcal{T}$, $\mathcal{W} \Subset \pi_R(F)$, and $a_M(\rho) \equiv 1$ in a smaller neighbourhood $\mathcal{W}'$ of $\mathcal{T}$, such that

(2.4) $$M(I - A_M) = \mathcal{O}_{L^2 \to L^2}(h^{N_0}) \,, \qquad A_M = \text{Op}_h^w(a_M) \,,$$

with $N_0 \gg 1$, independent of h. This means that $M(h)$ is very small, microlocally outside $\mathcal{W}$.

Assume Π_h is an orthogonal projector of finite dimension comparable with h^{-d}, with Π_h equal to the identity microlocally near $\mathcal{W}$. We will then also call open quantum map (or truncated open quantum map) the finite rank operator

$$\widetilde{M}(h) = \Pi_h\, M(h)\, \Pi_h = M(h) + \mathcal{O}_{L^2 \to L^2}(h^{N_0})\,.$$

A hyperbolic quantum monodromy operator is a family of hyperbolic open quantum maps $\{M(z,h)\}$ (or their finite rank version $\widetilde{M}(z,h)$) associated with the same relation F, which depend holomorphically on z in

$$(2.5) \qquad \Omega = \Omega(h) \stackrel{\text{def}}{=} [-R(h), R_1] + i[-R_1, R_1]\,, \quad R(h) \stackrel{h \to 0}{\longrightarrow} \infty\,, \quad R_1 > 0\,,$$

as operators $L^2 \to L^2$. Furthermore, we assume that there exists a decay rate $\tau_M > 0$ such that

$$(2.6) \qquad \|M(z,h)\| \le C\, e^{\tau_M \operatorname{Re} z}\,, \quad h < h_0,\ z \in \Omega(h)\,.$$

We will also consider truncated monodromy operator $\widetilde{M}(z,h)$. The cutoff A_M, the projector Π_h and the estimates (2.4) are assumed uniform with respect to $z \in \Omega(h)$.

This long definition is tailored to include the monodromy operator $\widetilde{M}(z,h)$ constructed for open hyperbolic flows with topologically one dimensional trapped sets, through a Grushin reduction — see [28, §2.2, Proposition 2.1] for a discussion and references. That construction shows that the exponent N_0 in (2.4) can be taken arbitrary large, and that we can take domain in (2.5) with $R(h) = C\log(1/h)$, for some $C > 0$, and R_1 large but fixed.

For the scattering by several convex bodies described in the introduction, we will be able (using propagation of singularities) to transform the boundary operator $\mathcal{M}(z,h)$ of (1.5) into a monodromy operator $\widetilde{M}(z,h)$ of the form above, with arbitrary N_0, see §6.

3. Preliminaries

The general preliminary material and notation for this paper are the same as in [28, §2]. We specifically present properties of exotic symbols and weights necessary to construct the escape function G and the projector Π_W in (1.11), and analyze their interaction with Fourier integral operators. Some of the material is taken directly from [28, §2], [43, §3], and some developed specifically for our needs.

3.1. **Semiclassical pseudodifferential calculus.** We recall the following class of symbols on $T^*\mathbb{R}^d$ (here $m, k \in \mathbb{R}$, $\delta \in [0, 1/2]$):

$$S_\delta^{m,k}(T^*\mathbb{R}^d) = \Big\{ a \in \mathcal{C}^\infty(T^*\mathbb{R}^d \times (0,1]) : \ \forall \alpha, \beta \in \mathbb{N}^n,$$

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi;h)| \le C_\alpha h^{-m - \delta(|\alpha| + |\beta|)} \langle \xi \rangle^{k - |\beta|} \Big\}\,,$$

where we use the standard notation $\langle \bullet \rangle = (1 + \bullet^2)^{1/2}$.

The Weyl quantization $a^w(x, hD)$ of such a symbol is defined as follows: for any wavefunction u in the Schwartz space $\mathscr{S}(\mathbb{R}^d)$,

$$a^w u(x) = a^w(x, hD)u(x) = [\mathrm{Op}_h^w(a)u](x)$$

(3.1)
$$\stackrel{\mathrm{def}}{=} \frac{1}{(2\pi h)^d} \int \int a\left(\frac{x+y}{2}, \xi\right) e^{i\langle x-y, \xi\rangle / h} u(y) dy d\xi,$$

see [6, Chapter 7] for a detailed discussion of semiclassical quantization, and [39, Appendix], [55, Chapter 14] for the semiclassical calculus for the symbol classes given above, and its implementation on manifolds. When $\delta = 0$ or $m = k = 0$ we will generally omit to indicate those indices. We denote by $\Psi_\delta^{m,k}(\mathbb{R}^d)$, $\Psi^{m,k}(\mathbb{R}^d)$, or $\Psi(\mathbb{R}^d)$ the corresponding classes of pseudodifferential operators.

For a given symbol $a \in S(T^*\mathbb{R}^d)$ we follow [41] and say that its *essential support* is contained a given compact set $K \Subset T^*\mathbb{R}^d$,

$$\mathrm{ess\text{-}supp}_h\, a \subset K \Subset T^*\mathbb{R}^d,$$

if and only if

$$\forall \chi \in S(T^*\mathbb{R}^d),\ \mathrm{supp}\,\chi \cap K = \emptyset \implies \chi a \in h^\infty \mathscr{S}(T^*\mathbb{R}^d).$$

Here $\mathscr{S}$ denotes the Schwartz space. The essential support of a is then the intersection of all such K's.

For $A \in \Psi(\mathbb{R}^d)$, $A = \mathrm{Op}_h^w(a)$, we call

(3.2)
$$\mathrm{WF}_h(A) = \mathrm{ess\text{-}supp}_h\, a$$

the semiclassical wavefront set of A. (In this paper we are concerned with a purely semiclassical theory and will only need to deal with *compact* subsets of $T^*\mathbb{R}^d$.)

Let $u = u(h)$, $\|u(h)\|_{L^2} = \mathcal{O}(h^{-N})$ (for some fixed N) be a wavefunction microlocalized in a compact set in $T^*\mathbb{R}^d$, in the sense that for some $\chi \in \mathcal{C}_c^\infty(T^*\mathbb{R}^d)$, one has $u = \chi^w u + \mathcal{O}_{\mathscr{S}}(h^\infty)$. The semiclassical wavefront set of u is then defined as:

(3.3) $\quad \mathrm{WF}_h(u) = \complement\{(x, \xi) \in T^*\mathbb{R}^d\ :\ \exists a \in S(T^*\mathbb{R}^d),\ a(x, \xi) = 1,\ \|a^w u\|_{L^2} = \mathcal{O}(h^\infty)\}.$

For future reference we record the following simple consequence of this definition:

Lemma 3.1. *If $u(h) = \mathcal{O}_{L^2(\mathbb{R}^n)}(h^{-N})$ is a wavefunction microlocalized in a compact subset of $T^*\mathbb{R}^n$ and*

$$v(h)(x') \stackrel{\mathrm{def}}{=} u(h)(0, x'),\quad (x_1, x') \in \mathbb{R}^n$$

then $v(h) = \mathcal{O}_{L^2(\mathbb{R}^n)}(h^{-N-1/2})$ and $v(h)$ is microlocalized in a compact subset of $T^*\mathbb{R}^{n-1}$. In addition we have

(3.4)
$$\mathrm{WF}_h(v) \subset \{(x', \xi') \in T^*\mathbb{R}^{n-1}\ :\ \exists \xi_1 \in \mathbb{R},\ (0, x', \xi_1, \xi') \in \mathrm{WF}_h(u)\}.$$

We point out that unlike in the case of classical wave front sets we do not have to make the assumption that $\xi' \neq 0$ when $(x', 0, \xi_1, \xi') \in \mathrm{WF}_h(u)$.

Proof. Let $\chi \in \mathcal{C}_{\mathrm{c}}^{\infty}(T^*\mathbb{R}^n)$ such that $u = \chi^w u + \mathcal{O}_{\mathscr{S}}(h^\infty)$ – it exists by the assumption that χ is microlocalized in a compact set.

By choosing $\psi \in \mathcal{C}^\infty(\mathbb{R})$ such that $\psi(\xi_1) = 1$ for $(x, \xi) \in \operatorname{supp} \chi$ we have $\psi(hD_{x_1})u = u + \mathcal{O}_{\mathscr{S}}(h^\infty)$ and we can simply replace u by $\psi(hD_{x_1})$. Then

$$|v(x')|^2 = \frac{1}{\sqrt{2\pi h}} \left| \int_{\mathbb{R}} \psi(\xi_1)(\mathcal{F}_h)_{x_1 \mapsto \xi_1} u(\xi_1, x') d\xi_1 \right|^2 \leq \frac{C_\psi}{\sqrt{h}} \|(\mathcal{F}_h)_{x_1 \mapsto \xi_1} u(\bullet, x')\|_{L^2(\mathbb{R})}^2 \,,$$

where $\mathcal{F}_h$ is the unitary semiclassical Fourier transform (see [55, Chapter 2]). Integrating in x' gives the bound $\|v\|_{L^2(\mathbb{R}^{n-1})} = \mathcal{O}(h^{-N-1/2})$.

Similar arguments prove the remaining statements in the lemma.   □

Semiclassical Sobolev spaces, $H_h^s(X)$ are defined using the norms

(3.5)
$$\|u\|_{H_s(\mathbb{R}^n)} = \|(I - h^2\Delta_{\mathbb{R}^n})^{s/2}u\|_{L^2(\mathbb{R}^n)} \,, \quad X = \mathbb{R}^n \,,$$
$$\|u\|_{H_s(X)} = \|(I - h^2\Delta_g)^{s/2}u\|_{L^2(X)} \,, \quad X \text{ a compact manifold,}$$

for any choice of Riemannian metric g.

3.2. $S_{\frac{1}{2}}$ spaces with two parameters.

We now refine the symbol classes $S_{\frac{1}{2}}^{m,k}$, by introducing a second small parameter, $\tilde{h} \in (0, 1]$, independent of h. Following [43, §3.3] we define the symbol classes:

(3.6)
$$a \in S_{\frac{1}{2}}^{m,\tilde{m},k}(T^*\mathbb{R}^d) \iff |\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq C_{\alpha\beta} h^{-m} \tilde{h}^{-\tilde{m}} \left(\frac{\tilde{h}}{h}\right)^{\frac{1}{2}(|\alpha|+|\beta|)} \langle\xi\rangle^{k-|\beta|} \,,$$

where in the notation we suppress the dependence of a on h and $\tilde{h}$. When working on $\mathbb{R}^d$ or in fixed local coordinates, we will use the simpler classes

(3.7) $$a \in \widetilde{S}(T^*\mathbb{R}^d) \iff |\partial^\alpha a| \leq C_\alpha \,, \qquad a \in \widetilde{S}_{\frac{1}{2}}(T^*\mathbb{R}^d) \iff |\partial^\alpha a| \leq C_\alpha (\tilde{h}/h)^{\frac{1}{2}|\alpha|} \,.$$

We denote the corresponding classes of operators by $\Psi_{\frac{1}{2}}^{m,\tilde{m},k}(\mathbb{R}^d)$ or $\widetilde{\Psi}_{\frac{1}{2}}$.

We recall [43, Lemma 3.6] which provides explicit error estimates on remainders in the product formula:

Lemma 3.2. *Suppose that $a, b \in \widetilde{S}_{\frac{1}{2}}$, and that $c^w = a^w \circ b^w$. Then for any integer $N > 0$ we expand*

(3.8) $$c(x, \xi) = \sum_{k=0}^{N} \frac{1}{k!} \left(\frac{ih}{2}\sigma(D_x, D_\xi; D_y, D_\eta)\right)^k a(x, \xi)b(y, \eta)\lceil_{x=y, \xi=\eta} + e_N(x, \xi) \,.$$

The remainder e_N is bounded as follows: for some integer M independent of N,

$$|\partial^\alpha e_N| \leq C_N h^{N+1} \times$$

(3.9)
$$\sum_{\alpha_1 + \alpha_2 = \alpha} \sup_{T^*\mathbb{R}^d \times T^*\mathbb{R}^d} \sup_{\beta \in \mathbb{N}^{4d}, |\beta| \leq M} \left| (h^{\frac{1}{2}}\partial_{(x,\xi;y,\eta)})^\beta (i\sigma(D)/2)^{N+1} \partial^{\alpha_1} a(x,\xi) \partial^{\alpha_2} b(y,\eta) \right|,$$

where

$$\sigma(D) = \sigma(D_x, D_\xi; D_y, D_\eta) \overset{\text{def}}{=} \langle D_\xi, D_y \rangle - \langle D_\eta, D_x \rangle,$$

is the symplectic form on $T^*\mathbb{R}^d \times T^*\mathbb{R}^d$.

Notice that, due to the growth of the derivatives of a, b, the expression (3.8) is really an expansion in powers of $\tilde{h}$ rather than h. On the other hand, if $a \in \widetilde{S}_{\frac{1}{2}}(T^*\mathbb{R}^d)$ and b is in the more regular class $S(T^*\mathbb{R}^d)$, then

$$c(x,\xi) = \sum_{k=0}^N \frac{1}{k!} \left(ih\sigma(D_x, D_\xi; D_y, D_\eta) \right)^k a(x,\xi)b(y,\eta)\rceil_{x=y, \xi=\eta} + \mathcal{O}(h^{\frac{N+1}{2}}\tilde{h}^{\frac{N+1}{2}}).$$

We also recall [43, Lemma 3.5] which is an easy adaptation of the semiclassical Beals's lemma — see [6, Chapter 8] and [55, §8.1]. Because of a small modification of the statement we present the reduction to the standard case:

Lemma 3.3. *Suppose that $A : \mathscr{S}(\mathbb{R}^d) \to \mathscr{S}'(\mathbb{R}^d)$. Then $A = \mathrm{Op}_h^w(a)$ with $a \in \widetilde{S}_{\frac{1}{2}}$ if and only if, for any $N \geq 0$ and any sequence $\{\ell_j\}_{j=1}^N$ of smooth functions which are linear outside a compact subset of $T^*\mathbb{R}^d$,*

(3.10)
$$\| \mathrm{ad}_{\mathrm{Op}_h^w(\ell_1)} \circ \cdots \mathrm{ad}_{\mathrm{Op}_h^w(\ell_N)} Au \|_{L^2(\mathbb{R}^d)} \leq C h^{N/2}\tilde{h}^{N/2} \|u\|_{L^2(\mathbb{R}^d)},$$

for any $u \in \mathscr{S}(\mathbb{R}^d)$.

Proof. We use the standard rescaling to eliminate h:

$$(\tilde{x}, \tilde{\xi}) = (\tilde{h}/h)^{\frac{1}{2}}(x, \xi),$$

and implement it through the following unitary operator on $L^2(\mathbb{R}^d)$:

(3.11)
$$U_{h/\tilde{h}}u(\tilde{x}) = (\tilde{h}/h)^{\frac{d}{4}}u((h/\tilde{h})^{\frac{1}{2}}\tilde{x}) = (\tilde{h}/h)^{\frac{d}{4}}u(x).$$

One can easily check that

$$\mathrm{Op}_h^w(a) = U_{h/\tilde{h}}^{-1}\mathrm{Op}_{\tilde{h}}^w(\tilde{a})U_{h/\tilde{h}}, \quad \text{where} \quad \tilde{a}(\tilde{x}, \tilde{\xi}) = a((h/\tilde{h})^{\frac{1}{2}}(\tilde{x}, \tilde{\xi})).$$

Notice that the symbol $\tilde{a} \in \widetilde{S}(T^*\mathbb{R}^d)$ if $a \in \widetilde{S}_{\frac{1}{2}}(T^*\mathbb{R}^d)$. In the rescaled coordinates, the condition (3.10) concerns $\tilde{h}$-pseudodifferential operators: it reads

(3.12)
$$\| \mathrm{ad}_{\mathrm{Op}_{\tilde{h}}^w(\tilde{\ell}_1)} \circ \cdots \circ \mathrm{ad}_{\mathrm{Op}_{\tilde{h}}^w(\tilde{\ell}_N)} \mathrm{Op}_{\tilde{h}}^w(\tilde{a})u \|_{L^2} \leq C\tilde{h}^N \|u\|_{L^2}.$$

Let us prove the statement in the case where the ℓ_j's are linear: then

$$\tilde{\ell}_j = (\tilde{h}/h)^{\frac{1}{2}} \ell_j$$

are also linear, and Beals's lemma for $\tilde{h}$-pseudodifferential operators [6, Prop. 8.3] states that (3.12) for any N is equivalent with $\tilde{a} \in \widetilde{S}(1)$.

We finally want to show that, if $\tilde{a} \in \widetilde{S}_{\frac{1}{2}}$, then (3.12) holds for ℓ_j's which are compactly supported. Actually, for $\ell_j \in S(1)$ we may use Lemma 3.2 to see that

$$\mathrm{ad}_{\mathrm{Op}_{\tilde{h}}^w(\tilde{\ell}_j)} \mathrm{Op}_{\tilde{h}}^w(\tilde{a}) = \mathcal{O}_{L^2 \to L^2}(h^{\frac{1}{2}} \tilde{h}^{\frac{1}{2}}),$$

which implies (3.10). $\qquad\square$

The rescaling (3.11) can be used to obtain analogues of other standard results. Here is one which we will need below.

Lemma 3.4. *Suppose $a \in \widetilde{S}_{\frac{1}{2}}$ and that $\sup_{T^*\mathbb{R}^d} |a| > c > 0$, with c independent of h and $\tilde{h}$. Then*

$$\|\mathrm{Op}_h^w(a)\|_{L^2 \to L^2} \leq \sup_{T^*\mathbb{R}^d} |a| + \mathcal{O}(\tilde{h}).$$

Proof. We first apply the rescaling (3.11), so that $\mathrm{Op}_h^w(a)$ is unitarily equivalent to $\mathrm{Op}_{\tilde{h}}^w(\tilde{a})$, where $\tilde{a} \in \widetilde{S}(T^*\mathbb{R}^d)$, and $\sup|\tilde{a}| = \sup|a|$. We then note that

$$\mathrm{Op}_{\tilde{h}}^w(\tilde{a}^w)^* \mathrm{Op}_{\tilde{h}}^w(\tilde{a}) = \mathrm{Op}_{\tilde{h}}^w(|\tilde{a}|^2) + \mathcal{O}_{L^2 \to L^2}(\tilde{h}),$$

and that the sharp Gårding inequality (see for instance [6, 7.12] or [55, Theorem 4.32]) shows that

$$(\sup|a|)^2 - \mathrm{Op}_{\tilde{h}}^w(\tilde{a})^* \mathrm{Op}_{\tilde{h}}^w(\tilde{a}) \geq -C\tilde{h},$$

from which the lemma follows. $\qquad\square$

A stronger result given in [55, Theorem 13.13] could also be used in place of this simple lemma.

3.3. Exponentiation and quantization.

As in [43], we will need to consider operators of the form $\exp(G^w(x, hD))$, where $G \in S_{\frac{1}{2}}^{0,0,-\infty}(T^*\mathbb{R}^d)$. To understand the properties of the conjugated operators,

$$\exp(-G^w(x, hD)) \, P \, \exp(G^w(x, hD)),$$

we will use a special case of a result of Bony and Chemin [1, Théorème 6.4] — see [43, Appendix] or [55, §8.2]

To state it we need to recall a more general class of pseudodifferential operators defined using *order functions*. A function $m : T^*\mathbb{R}^d \to \mathbb{R}_+$ is an order function in the sense of [6] iff for some $N \geq 0$ and $C > 0$, we have

$$(3.13) \qquad\qquad \forall \rho, \rho' \in T^*\mathbb{R}^d, \qquad m(\rho) \leq C m(\rho') \langle \rho - \rho' \rangle^N.$$

The class of symbols corresponding to m, denoted by $\widetilde{S}(m)$, is defined as

$$a \in \widetilde{S}(m) \iff |\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \le C_{\alpha\beta}\, m(x,\xi)$$

(in this notation $\widetilde{S}(1)$ is the class of symbols we had called $\widetilde{S}(T^*\mathbb{R}^d)$). If m_1 and m_2 are order functions in the sense of (3.13), and $a_j \in S(m_j)$ then (we put $h = 1$ here),

$$a_1^w(x,D)\, a_2^w(x,D) = b^w(x,D)\,, \quad b \in \widetilde{S}(m_1 m_2)\,,$$

with b given by the usual formula

(3.14)
$$\begin{aligned} b(x,\xi) &= a_1 \,\sharp\, a_2(x,\xi) \\ &\stackrel{\text{def}}{=} \exp(i\sigma(D_{x^1}, D_{\xi^1}; D_{x^2}, D_{\xi^2})/2)\, a_1(x^1,\xi^1)\, a_2(x^2,\xi^2)\!\restriction_{x^1=x^2=x,\xi^1=\xi^2=\xi}\ . \end{aligned}$$

Note that here we do not have a small parameter h, so $a_1 \sharp a_2$ cannot be expanded as a power series. The value of the following proposition lies in the calculus based on order functions. A special case of [1, Théorème 6.4], see [43, Appendix], gives

Proposition 3.5. *Let m be an order function in the sense of* (3.13), *and suppose that $g \in \mathcal{C}^\infty(T^*\mathbb{R}^n; \mathbb{R})$ satisfies, uniformly in $(x,\xi) \in T^*\mathbb{R}^d$,*

(3.15)
$$g(x,\xi) - \log m(x,\xi) = \mathcal{O}(1)\,, \quad \partial_x^\alpha \partial_\xi^\beta g(x,\xi) = \mathcal{O}(1)\,, \quad |\alpha| + |\beta| \ge 1\,.$$

Then for any $t \in \mathbb{R}$,

(3.16)
$$\exp(tg^w(x,D)) = B_t^w(x,D)\,, \quad B_t \in \widetilde{S}(m^t)\,.$$

Here $\exp(tg^w(x,D))u$, $u \in \mathscr{S}(\mathbb{R}^d)$, is constructed by solving

$$\partial_t u(t) = g^w(x,D)u(t)\,, \quad u(0) = u\,.$$

The estimates on $B_t \in \widetilde{S}(m^t)$ depend only *on the constants in* (3.15) *and in* (3.13). *In particular they are independent of the support of g.*

Since m^t is the order function $\exp(t \log m(x,\xi))$, we can say that, on the level of order functions, quantization commutes with exponentiation.

This proposition will be used after applying the rescaling (3.11) to the above formalism. For the class $\widetilde{S}_{\frac{1}{2}}$, the order functions are defined by demanding that for some $N \ge 0$,

(3.17)
$$m(\rho) \le C\, m(\rho') \Big\langle \frac{\rho - \rho'}{(h/\tilde{h})^{\frac{1}{2}}} \Big\rangle^N\,.$$

The corresponding class is defined by

$$a \in \widetilde{S}_{\frac{1}{2}}(m) \iff |\partial^\alpha a(\rho)| \le C_\alpha (\tilde{h}/h)^{|\alpha|/2}\, m(\rho)\,.$$

We will consider order functions satisfying

(3.18)
$$m \in \widetilde{S}_{\frac{1}{2}}(m)\,, \quad \frac{1}{m} \in \widetilde{S}_{\frac{1}{2}}\left(\frac{1}{m}\right)\,.$$

This is equivalent to the fact that the function

(3.19) $$G(x, \xi) \overset{\text{def}}{=} \log m(x, \xi)$$

satisfies

(3.20) $$\frac{\exp G(\rho)}{\exp G(\rho')} \leq C \Big\langle \frac{\rho - \rho'}{(h/\tilde{h})^{\frac{1}{2}}} \Big\rangle^N , \quad \partial^\alpha G = \mathcal{O}((h/\tilde{h})^{-|\alpha|/2}), \; |\alpha| \geq 1 .$$

Using the rescaling (3.11), we see that Proposition 3.5 implies that

(3.21)
$$\exp(G^w(x, hD)) = B^w(x, hD), \quad B \in \widetilde{S}_{\frac{1}{2}}(m),$$
$$a \in \widetilde{S}_{\frac{1}{2}}(m) \iff \mathrm{Op}_h^w(a) = e^{G^w(x,hD)} \, \mathrm{Op}_h^w(a_0), \quad a_0 \in \widetilde{S}_{\frac{1}{2}} .$$

For future reference we also note the following fact: for $A \in \Psi_\delta(\mathbb{R}^d)$,

(3.22) $$A - e^{-G^w(x,hD)} \, A \, e^{G^w(x,hD)} = h^{\frac{1}{2}(1-2\delta)} \tilde{h}^{\frac{1}{2}} a_1^w(x, hD), \quad a_1 \in \widetilde{S}_{\frac{1}{2}} .$$

The following lemma will also be useful when applying these weights to Fourier integral operators.

Lemma 3.6. *Let $U \Subset T^*\mathbb{R}^d$, and let $\chi \in \mathcal{C}_{\mathrm{c}}^\infty(U)$. Take G_1, G_2 two weight functions as in (3.19), such that*

$$(G_1 - G_2)\!\restriction_U = 0 .$$

Then,

$$e^{G_1^w(x,hD)} \chi^w = e^{G_2^w(x,hD)} \chi^w + \mathcal{O}_{\mathscr{S}' \to \mathscr{S}}(h^\infty) ,$$
$$\chi^w \, e^{G_1^w(x,hD)} = \chi^w \, e^{G_2^w(x,hD)} + \mathcal{O}_{\mathscr{S}' \to \mathscr{S}}(h^\infty) .$$

Proof. We just give the proof of the first identity, the second being very similar. Let us differentiate the operator $e^{-tG_2^w(x,hD)} e^{tG_1^w(x,hD)} \chi^w$:

$$\frac{d}{dt} e^{-tG_2^w(x,hD)} e^{tG_1^w(x,hD)} \chi^w = e^{-tG_2^w(x,hD)} (G_1^w - G_2^w) e^{tG_1^w(x,hD)} \chi^w$$

For each $t \in [0, 1]$, the operator $e^{tG_1^w(x,hD)} \chi^w \in \widetilde{\Psi}_{\frac{1}{2}}(m_1^t)$ is bounded on L^2, and has its semiclassical wavefront set contained in $\mathrm{supp}\,\chi$. Due to the support property of $G_1 - G_2$, we get

$$(G_1^w - G_2^w) e^{tG_1^w(x,hD)} \chi^w = \mathcal{O}_{\mathscr{S}' \to \mathscr{S}}(h^\infty) ,$$

and the same estimate holds once we apply $e^{-tG_2^w(x,hD)}$ on the left. As a result,

$$e^{-G_2^w(x,hD)} e^{G_1^w(x,hD)} \chi^w - \chi^w = \mathcal{O}_{\mathscr{S}' \to \mathscr{S}}(h^\infty) ,$$

from which the statement follows.   □

3.4. **Fourier integral operators.** We now follow [55, Chapter 10],[41] and review some aspects of the theory of semiclassical Fourier integral operators. Since we will deal with operators in $\widetilde{\Psi}_{\frac{1}{2}}$, the rescaling to $\tilde{h}$-semiclassical calculus, as in the proof of Lemma 3.3, involves dealing with large h-dependent sets. It is then convenient to have a global point view, which involves suitable extensions of locally defined canonical transformations and relations.

3.4.1. *Local symplectomorphisms.* We start with a simple fact about symplectomorphisms of open sets in $T^*\mathbb{R}^d$.

Proposition 3.7. *Let U_0 and U_1 be open neighbourhoods of $(0,0)$ and $\kappa : U_0 \to U_1$ a symplectomorphism such that $\kappa(0,0) = (0,0)$. Suppose also that U_0 is star shaped with respect to the origin, that is, if $\rho \in U_0$, then $t\rho \in U_0$ for $0 \le t \le 1$.*

Then there exists a continuous, piecewise smooth family

$$\{\kappa_t\}_{0 \le t \le 1}$$

of symplectomorphisms $\kappa_t : U_0 \to U_t = \kappa_t(U_0)$, such that

(3.23)
$$
\begin{aligned}
&\text{(i)} \quad \kappa_t(0,0) = (0,0)\,, \quad 0 \le t \le 1\,, \\
&\text{(ii)} \quad \kappa_1 = \kappa\,, \quad \kappa_0 = \mathrm{id}_{U_0}\,, \\
&\text{(iii)} \quad \frac{d}{dt}\kappa_t = (\kappa_t)_* H_{q_t}\,, \qquad 0 \le t \le 1\,,
\end{aligned}
$$

where $\{q_t\}_{0 \le t \le 1}$ is a continuous, piecewise smooth family of $\mathcal{C}^\infty(U_0)$ functions.

The last condition means that κ_t is generated by the (time dependent) Hamiltonian vector field H_{q_t} associated with the Hamiltonian q_t[1].

Sketch of the proof: Since the group of linear symplectic transformations is connected, we only need to deform κ to a linear transformation. That is done by taking

$$\tilde{\kappa}_t(\rho) \stackrel{\mathrm{def}}{=} \kappa(t\rho)/t\,, \quad t \in [0,1]\,,$$

which requires the condition that U_0 is star shaped. It satisfies $\tilde{\kappa}_0 = d\kappa(0,0)$. $\square$

As a consequence we have the possibility to globalize a locally defined symplectomorphism:

Proposition 3.8. *Let U_0 and U_1 be open precompact sets in $T^*\mathbb{R}^d$ and let $\kappa : U_0 \to U_1$ be a symplectomorphism which extends to $\widetilde{U}_0 \supseteq U_0$, an open star shaped set. Then κ extends to a symplectomorphism*

$$\tilde{\kappa} \; : \; T^*\mathbb{R}^d \to T^*\mathbb{R}^d\,,$$

which is equal to the identity outside of a compact set. $\tilde{\kappa}$ can be deformed to the identity with q_t's in (iii) of (3.23) supported in a fixed compact set.

---

[1]This generation can also be expressed through the more usual form $\frac{d\kappa_t}{dt} = H_{q'_t}$, where $q'_t = q_t \circ \kappa_t^{-1}$.

Proof. Let $\tilde{\kappa}^0$ be the extension of κ to $\widetilde{U}_0$. By Proposition 3.7 we can deform $\tilde{\kappa}^0$ to the identity, with $(d/dt)\tilde{\kappa}_t^0 = (\tilde{\kappa}_t^0)_* H_{\tilde{q}_t}$, $\tilde{q}_t \in \mathcal{C}^\infty(\widetilde{U}_0)$. If we replace $\tilde{q}_t$ by $q_t = \chi\tilde{q}_t$ where $\chi \in \mathcal{C}_c^\infty(\widetilde{U}_0)$, and $\chi = 1$ in U_0, the family of symplectomorphisms of $T^*\mathbb{R}^d$ generated by $(q_t)_{0 \le t \le 1}$ satisfies

$$\kappa_t\!\restriction_{U_0} = \tilde{\kappa}_t^0\!\restriction_{U_0}, \quad \kappa_0 = \mathrm{id}, \quad \kappa_1\!\restriction_{U_0} = \kappa, \quad \tilde{\kappa}_t\!\restriction_{\complement\,\widetilde{U}_0} = \mathrm{id}_{\complement\,\widetilde{U}_0}.$$

Then $\tilde{\kappa} = \kappa_1$ provides the desired extension of κ, and the family κ_t the deformation with compactly supported q_t's. $\qquad\square$

The proposition means that, as long as we have some geometric freedom in extending our symplectomorphisms, we can consider local symplectomorphisms as restrictions of global ones which are isotopic to the identity with compactly supported Hamiltonians. We denote the latter class by $\mathcal{K}$:

$$\mathcal{K} \overset{\text{def}}{=} \{\kappa \; : \; T^*\mathbb{R}^d \to T^*\mathbb{R}^d, \quad (3.23) \text{ holds with compactly supported } q_t.\}\,.$$

We note that, except for $d = 1, 2$, it is not known whether every κ which is equal to the identity outside a compact set is in $\mathcal{K}$.

For $\kappa \in \mathcal{K}$ we now define a class of (semiclassical) Fourier integral operators associated with the graph of κ. It fits in the Heisenberg picture of quantum mechanics – see [49, §8.1] for a microlocal version and [55, §§10.1,10.2] for a detailed presentation on the semiclassical setting.

Definition 3.9. *Let $\kappa \in \mathcal{K}$ as above, and let $0 \le \delta < 1/2$. The operator $U : \mathscr{S}(\mathbb{R}^d) \to \mathscr{S}'(\mathbb{R}^d)$ belongs to the class of h-Fourier integral operators*

$$I_\delta(\mathbb{R}^d \times \mathbb{R}^d, C'), \quad C' = \{(x, \xi; y, -\eta) \; : \; (x, \xi) = \kappa(y, \eta)\}\,,$$

if and only if there exists $U_0 \in \Psi_\delta(\mathbb{R}^d)$, such that $U = U(1)$, where

$$(3.24) \qquad hD_t U(t) + U(t)Q(t) = 0, \quad Q(t) = \mathrm{Op}_h^w(q_t), \quad U(0) = U_0\,.$$

Here the time dependent Hamiltonian $q_t \in \mathcal{C}_c^\infty(T^*\mathbb{R}^d)$ satisfies (iii) *of* (3.23).

We will write

$$I_{0+}(\mathbb{R}^d \times \mathbb{R}^d, C') \overset{\text{def}}{=} \bigcap_{\delta > 0} I_\delta(\mathbb{R}^d \times \mathbb{R}^d, C')\,.$$

For $A \in I_{0+}(\mathbb{R}^d \times \mathbb{R}^d, C')$ we define its h-wavefront set

$$(3.25) \qquad \mathrm{WF}_h(A) \overset{\text{def}}{=} \{(x, \xi; y, -\eta) \in C' \; : \; (y, \eta) \in \mathrm{WF}_h(U_0)\} \subset T^*\mathbb{R}^d \times T^*\mathbb{R}^d\,,$$

where $\mathrm{WF}_h(U_0)$ is defined in (3.2).

We recall Egorov's theorem in this setting — see [14, Appendix a] or [55, Theorem 10.10].

Proposition 3.10. *Suppose that $U = U(1)$ for (3.24) with $U_0 = I$, and that $A = \mathrm{Op}_h^w(a)$, $a \in S(T^*\mathbb{R}^d)$. Then*

$$U^{-1}AU = B, \quad B = \mathrm{Op}_h^w(b), \quad b - \kappa^*a \in h^2 S(T^*\mathbb{R}^d).$$

An analogous result holds for $a \in S^{m,k}(T^*\mathbb{R}^d)$.

More generally, if $T \in I_\delta(\mathbb{R}^d \times \mathbb{R}^d, C')$ and A as above, then

$$AT = TB + T_1B_1,$$

where $B = \mathrm{Op}_h^w(b)$ is as above, $T_1 \in I_\delta(\mathbb{R}^d \times \mathbb{R}^d, C')$, and $B_1 \in h^{1-2\delta}\Psi_\delta(\mathbb{R}^d)$.

Remark. The additional term T_1B_1 is necessary, due to the fact that T may not be elliptic.

The main result of this subsection is an extension of this Egorov property to operators A in more exotic symbol classes.

Proposition 3.11. *Suppose that $\kappa \in \mathcal{K}$ and $T \in I_\delta(\mathbb{R}^d \times \mathbb{R}^d; C')$, where C is the graph of κ. Take $A = e^{G^w(x,hD)}\mathrm{Op}_h^w(a_0)$, where $a_0 \in \widetilde{S}_{\frac{1}{2}}(1)$ and G satisfies (3.20). Then*

$$AT = TB + T_1B_1,$$

(3.26)
$$B = e^{(\kappa^*G)^w(x,hD)}\mathrm{Op}_h^w(b_0), \quad b_0 - \kappa^*a_0 \in h^{\frac{1}{2}}\tilde{h}^{\frac{3}{2}}\widetilde{S}_{\frac{1}{2}}(1),$$

$$B_1 = h^{\frac{1}{2}(1-2\delta)}\tilde{h}^{\frac{1}{2}}\mathrm{Op}_h^w(b_1), \quad b_1 \in \widetilde{S}_{\frac{1}{2}}(e^{\kappa^*G}), \quad T_1 \in I_\delta(\mathbb{R}^d \times \mathbb{R}^d, C').$$

The proof is based on two lemmas. The first one is essentially Proposition 3.11 with $G = 1$, $\delta = 0$, and T invertible:

Lemma 3.12. *Suppose $U = U(1)$, where $U(t)$ solves (3.24) with $U(0) = I$. Then for $A = \mathrm{Op}_h^w(a)$, $a \in \widetilde{S}_{\frac{1}{2}}(1)$,*

(3.27)
$$U^{-1}AU = B, \quad B = \mathrm{Op}_h^w(b), \quad b - \kappa^*a \in h^{\frac{1}{2}}\tilde{h}^{\frac{3}{2}}\widetilde{S}_{\frac{1}{2}}(1).$$

Proof. We will use the $\widetilde{S}_{\frac{1}{2}}$ variant of Beals's lemma given in Lemma 3.3 above. Let ℓ_j be as in (3.10), $j = 1, \cdots, N$, and denote

$$\mathrm{Op}_h^w(\ell_j(t)) \stackrel{\text{def}}{=} U(t)\mathrm{Op}_h^w(\ell_j)U(t)^{-1}.$$

We first claim that $\ell_j(t)$ are, to leading order, linear outside of a compact set:

(3.28)
$$\ell_j(t) = (\kappa_t^{-1})^*\ell_j + h^2r_t, \quad r_t \in S(1).$$

To prove it, we see that the evolution (3.24) gives

(3.29)
$$hD_t\ell_j(t)^w = [\widetilde{Q}(t), \ell_j(t)^w] \stackrel{\text{def}}{=} L_j(t),$$

where

$$\widetilde{Q}(t) \stackrel{\text{def}}{=} U(t)\mathrm{Op}_h^w(q_t)U(t)^{-1} = \mathrm{Op}_h^w(\tilde{q}_t + \mathcal{O}_{S(1)}(h^2)), \quad \tilde{q}_t = (\kappa_t^{-1})^*q_t.$$

Then, using Lemma 3.2,

$$L_j(t) - (h/i)\mathrm{Op}_h^w(H_{\tilde{q}_t}\ell_j(t)) \in h^3 S(1).$$

Since $(d/dt)\kappa_t = H_{\tilde{q}_t}$, (3.29) becomes

$$\partial_t(\kappa_t^*\ell_j(t)) = \mathcal{O}_{S(1)}(h^2),$$

which implies (3.28).

For A in the statement of the lemma let us define $A(t)$ as

$$A(t) \overset{\text{def}}{=} U(t)^{-1}AU(t), \quad hD_tA(t) = [Q(t), A(t)], \quad A(0) = A.$$

We want to show that $A(t) = \mathrm{Op}_h^w(a(t))$ where $a(t) \in \widetilde{S}_{\frac{1}{2}}$. We will prove that using (3.10):

(3.30) $$\mathrm{ad}_{\mathrm{Op}_h^w(\ell_1)} \circ \cdots \mathrm{ad}_{\mathrm{Op}_h^w(\ell_N)} A(t) = U(t)^{-1}\left(\mathrm{ad}_{\mathrm{Op}_h^w(\ell_1(t))} \circ \cdots \mathrm{ad}_{\mathrm{Op}_h^w(\ell_N(t))} A\right)U(t).$$

In view of (3.28) and the assumptions on A we see that the right hand side is

$$\mathcal{O}(h^{N/2}\tilde{h}^{N/2}) : L^2(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^d),$$

and thus $A(t) = \mathrm{Op}_h^w(a(t))$, $a(t) \in \widetilde{S}_{\frac{1}{2}}$. One has $B = A(1)$.

We now need to show that $a(t) - \kappa_t^*a \in h^{\frac{1}{2}}\tilde{h}^{\frac{3}{2}}\widetilde{S}_{\frac{1}{2}}$. For that define

$$\widetilde{A}(t) := \mathrm{Op}_h^w(\kappa_t^*a).$$

As in the proof of Egorov's theorem, we calculate

$$\begin{aligned}
hD_t\widetilde{A}(t) &= \frac{h}{i}\mathrm{Op}_h^w\left(\frac{d}{dt}\kappa_t^*a\right) = \frac{h}{i}\mathrm{Op}_h^w(H_{q_t}\kappa_t^*a) \\
&= \frac{h}{i}\mathrm{Op}_h^w(\{q_t, \kappa_t^*a\}) = [Q(t), \widetilde{A}(t)] + h^{\frac{3}{2}}\tilde{h}^{\frac{3}{2}}E(t),
\end{aligned}$$

and Lemma 3.2 shows that $E(t) = \mathrm{Op}_h^w(e(t))$, where $e(t) \in \widetilde{S}_{\frac{1}{2}}$. A calculation then shows that

$$hD_t(U(t)\widetilde{A}(t)U(t)^{-1}) = U(t)E(t)U(t)^{-1},$$

and consequently,

(3.31) $$A(t) - \widetilde{A}(t) = ih^{\frac{1}{2}}\tilde{h}^{\frac{3}{2}}\int_0^t U(t)^{-1}U(s)E(s)U(s)^{-1}U(t)ds.$$

We have already shown that conjugation by $U(s)$ preserves the class $\widetilde{S}_{\frac{1}{2}}$, hence the integral above is in $\widetilde{S}_{\frac{1}{2}}$. Then $A(t) - \tilde{A}(t) \in h^{\frac{1}{2}}\tilde{h}^{\frac{3}{2}}\widetilde{S}_{\frac{1}{2}}$. $\square$

Remark. The use of Weyl quantization is essential for getting an error of size $h^{\frac{1}{2}}\tilde{h}^{\frac{3}{2}}$. To see this we take U to be a metaplectic transformation – see for instance [6, Appendix to Chapter 7]. The rescaling (3.11) simply changes it to the same metaplectic tranformation, $\tilde{U}$, with $\tilde{h}$ as the new Planck constant. Then

$$\tilde{U}^{-1}\tilde{a}^w(\tilde{x}, \tilde{h}D_{\tilde{x}})\tilde{U} = (\tilde{\kappa}^*\tilde{a})^w(\tilde{x}, \tilde{h}D_{\tilde{x}}),$$

that is we have no error term. Had we used right quantization, $a(x, hD)$, we would have acquired error terms of size $\tilde{h}$, which could not be eliminated after rescaling back. For the invariance of the $\widetilde{S}_{\frac{1}{2}}$ calculus see [54, §3.3].

The arguments of the previous lemma can be extended to encompass the weight function G of (3.19), which may not be in $\widetilde{S}_{\frac{1}{2}}$ (indeed, G may be unbounded), but which has bounded derivatives.

Lemma 3.13. *Let U be as in Lemma 3.12. For G satisfying* (3.20) *we define*

$$\mathcal{G}_1 \stackrel{\text{def}}{=} U^{-1}G^w(x, hD)U \; : \; \mathscr{S} \longrightarrow \mathscr{S}.$$

Then

$$\mathcal{G}_1 = G_1^w(x, hD), \quad G_1 - \kappa^*G \in h^{\frac{1}{2}}\tilde{h}^{\frac{3}{2}}\widetilde{S}_{\frac{1}{2}}(1).$$

Proof. Since $U : \mathscr{S} \to \mathscr{S}$, the operator $\mathcal{G}_1$ maps $\mathscr{S}$ to $\mathscr{S}$. We now proceed as in the proof of Lemma 3.12, noting that

$$\partial^\alpha(\kappa_t^*G) \in (h/\tilde{h})^{-|\alpha|/2)}\widetilde{S}_{\frac{1}{2}}, \quad |\alpha| \geq 1, \quad \text{uniformly for } t \in [0, 1].$$

Lemma 3.2 shows that only terms involving derivatives appear in the expansions, hence the same arguments apply. $\qquad\Box$

We now combine these various lemmas.

Proof of Proposition 3.11: Suppose that $T \in I_\delta(\mathbb{R}^d \times \mathbb{R}^d, C')$, meaning that $T = U_0U$, where $U = U(1)$ satisfies (3.24) with $U(0) = I$. By Egorov we may write $T = UU_1$, with $U_1 \in \Psi_\delta$. Then, in the notation of Lemma 3.13,

$$AT = U(U^{-1}AU)U_1 = U \exp\mathcal{G}_1 U^{-1}A_0UU_1 = TB + UB_1,$$

where, using Lemma 3.13,

$$B = \exp\mathcal{G}_1 U^{-1}A_0U = \exp((\kappa^*G)^w)B_0, \quad B_0 \in \widetilde{\Psi}_{\frac{1}{2}},$$

and, using (3.22),

$$\begin{aligned}
B_1 &= [\exp((\kappa^*G)^w), U_1]B_0 + \exp((\kappa^*G)^w)[B_0, U_1] \\
&= \exp((\kappa^*G)^w)(U_1 - \exp(-(\kappa^*G)^w)U_1\exp((\kappa^*G)^w) + [B_0, U_1]) \\
&\in h^{\frac{1}{2}(1-2\delta)}\tilde{h}^{\frac{1}{2}}\exp((\kappa^*G)^w)\widetilde{\Psi}_{\frac{1}{2}}.
\end{aligned}$$

$\qquad\Box$

3.4.2. *Lagrangian relations.* We are now ready to extend the above semiglobal theory ("semi" because of our special class of symplectomorphisms $\mathcal{K}$) into a construction of h-Fourier integral operators associated with an arbitrary smooth Lagrangian relation on $F \subset T^*Y \times T^*Y$ (as introduced in §2). This construction will naturally be done by splitting F into local symplectomorphisms defined on (small) star shaped sets.

Eventually, we want consider the full setup of §2, that is taking F the disjoint union of $F_{ij} \subset U_i \times U_j$, defining our Fourier integral operator $T \in I_\delta(Y \times Y, F')$ as a matrix of operators,

$$T = (T_{ij})_{1 \leq i,j \leq J}, \quad T_{ij} \in I_\delta(Y_i \times Y_j, F'_{ij}),$$

and finally take $I_{0+}(Y \times Y, F') = \bigcup_{\delta > 0} I_{0+}(Y \times Y, F')$.

To avoid too cumbersome notations, we will omit the indices i, j, and consider a single Lagrangian relation $F \subset U' \times U$, where $U \Subset T^*Y \Subset T^*\mathbb{R}^d$, $U' \Subset T^*Y' \subset T^*\mathbb{R}^d$ two open sets, and define the classes $I_\delta(Y' \times Y)$.

Fix some small $\varepsilon > 0$. On U we introduce two open covers of U,

$$U \subset \bigcup_{\ell=1}^{L} U_\ell, \quad U_\ell \Subset \widetilde{U}_\ell,$$

such that each $\widetilde{U}_\ell$ is star shaped around one of its points, and has a diameter $\leq \varepsilon$. We also introduce a smooth partition of unity $(\chi_\ell)_{\ell=1,\dots,L}$ associated with the cover (U_ℓ):

$$(3.32) \qquad \sum_\ell \chi_\ell(\rho) = 1, \qquad \rho \in \mathrm{neigh}(U), \quad \chi_\ell \in \mathcal{C}_\mathrm{c}^\infty(U_\ell, [0,1]).$$

F can be seen as a canonical map defined on the departure subset $\pi_R(F) \subset U$, with range $\pi_L(F) \subset U'$. Let us call $\widetilde{F}_\ell = F\!\restriction_{\widetilde{U}_\ell}$ its restriction to $\widetilde{U}_\ell$.

The set of interior indices ℓ such that $\widetilde{U}_\ell \subset \pi_R(F)$ will be denoted by $\mathcal{L}$.

For each interior index ℓ, the symplectomorphism $\widetilde{F}_\ell$ is the extension of $F_\ell = F \restriction_{U_\ell}$, so we may apply Proposition 3.7, and produce a global symplectomorphism $\widetilde{\kappa}_\ell \in \mathcal{K}$, which coincides with F_ℓ on the set U_ℓ. The previous section provides the family of h-Fourier integral operators $I_\delta(\mathbb{R}^d \times \mathbb{R}^d, C'_\ell)$, where C_ℓ is the graph of $\widetilde{\kappa}_\ell$. For each interior index ℓ we consider a Fourier integral operator $\widetilde{T}_\ell \in I_\delta(\mathbb{R}^d \times \mathbb{R}^d, C'_\ell)$, and use the partition of unity (3.32) to define

$$T_\ell \overset{\mathrm{def}}{=} \widetilde{T}_\ell \, \chi_\ell^w(x, hD).$$

Due to the support properties of χ_ℓ, the operator T_ℓ is actually associated with the restriction F_ℓ of $\widetilde{\kappa}_\ell$ on U_ℓ. The sum

$$T^{\mathbb{R}} \overset{\mathrm{def}}{=} \sum_{\ell \in \mathcal{L}} T_\ell$$

defines a Fourier integral operator on $\mathbb{R}^d$, microlocalized inside $\pi_L(F) \times \pi_R(F)$, which we call $I_\delta(\mathbb{R}^d \times \mathbb{R}^d, F')$.

Finally, since we want the wavefunctions to be defined on the open sets Y, Y' rather than on the whole of $\mathbb{R}^d$, we use cutoffs $\Psi \in \mathcal{C}_c^\infty(Y, [0,1])$, $\Psi \in \mathcal{C}_c^\infty(Y, [0,1])$ such that $\Psi(x) = 1$ on $\pi(U)$, $\Psi'(x) = 1$ on $\pi(U')$.

We will say that $T : \mathcal{D}'(Y) \to \mathcal{C}^\infty(\overline{Y}')$ belongs to the class

$$I_\delta(Y' \times Y, F')$$

iff

$$\Psi' T \Psi = \Psi' T^{\mathbb{R}} \Psi \quad \text{for some} \quad T^{\mathbb{R}} \in I_\delta(\mathbb{R}^d \times \mathbb{R}^d, F'),$$

and

$$T - \Psi' T \Psi = \mathcal{O}(h^\infty) \ : \ \mathcal{D}'(Y) \to \mathcal{C}^\infty(\overline{Y}') \,.$$

We notice that $\pi_R(\mathrm{WF}_h(T))$ is automatically contained in the support of $\sum_{\ell \in \mathcal{L}} \chi_\ell$, a strict subset of $\pi_R(F)$: in this sense, the above definition of $I_\delta(Y' \times Y, F')$ depends on the partition of unity (3.32). However, for any subset of $W \Subset F$, one can always choose a cover $(\widetilde{U}_\ell)$ such that $\bigcup_{j \in \mathcal{L}} \widetilde{U}_\ell \supset \pi_R(W)$. In particular, the assumption $\mathcal{T} \cap \partial F = \emptyset$ we made in §2 shows that such a subset W may contain the trapped set $\mathcal{T}$.

3.4.3. *Conjugating global Fourier integral operators by weights.* By linearity one can generalize Prop. 3.11 to a Fourier integral operator $T \in I_{0+}(Y' \times Y, F')$, and by linearity to the full setup of §2. The observable a_0 and weight G are now functions on T^*Y' or on $T^*\mathbb{R}^d$.

If G is supported inside $\pi_L(F) \subset T^*Y'$, then $F^*G = G \circ F$ is a smooth function on $\pi_R(F)$, which can be smoothly extended (by zero) outside. In each U_ℓ we apply Prop. 3.11 to T_ℓ, and obtain on the right hand side terms of the form

$$T_\ell \, e^{\widetilde{\kappa}_\ell^* G} \, \mathrm{Op}_h^w(b_0) + T_1 B_1 \,, \quad T_1 \in I_\delta(\mathbb{R}^d \times \mathbb{R}^d, F'), \quad B_1 \in h^{1/2-\delta} \tilde{h}^{1/2} \widetilde{\Psi}_{\frac{1}{2}}(e^{\widetilde{\kappa}_\ell^* G}) \,,$$

and $\mathrm{WF}_h(T_1) \subset \mathrm{WF}_h(T)$.

Since $\pi_R(\mathrm{WF}_h(T_\ell)) \Subset U_\ell$, Lemma 3.6 shows that only the part of $\widetilde{\kappa}_\ell^* G$ inside U_ℓ is relevant to the above operator, that is a part where $\widetilde{\kappa}_\ell \equiv F$. Therefore we have

$$T_\ell \, e^{(\widetilde{\kappa}_\ell^* G)^w} = T_\ell \, e^{(F^* G)^w} + \mathcal{O}(h^\infty) \,, \qquad B_1 \in h^{1/2-\delta} \tilde{h}^{1/2} \widetilde{\Psi}_{\frac{1}{2}}(e^{F^* G}) + \mathcal{O}(h^\infty) \,.$$

This proves the generalization of Prop. 3.11 to the setting of the relation $F \subset T^*Y' \times T^*Y$, in case $\mathrm{supp}\, G \subset \pi_L(F)$.

In case G is not supported on $\pi_L(F)$, the notation $e^{(F^* G)^w}$ still makes sense microlocally inside $\pi_R(F)$. Indeed, take $\chi, \tilde{\chi} \in \mathcal{C}_c^\infty(\pi_L(F))$, $\chi \equiv 1$ near $\pi_L(\mathrm{WF}_h(T))$, $\tilde{\chi} \equiv 1$ near $\mathrm{supp}\,\chi$. Lemma 3.6, with $V = \pi_L(F)$ implies that

$$e^{G^w} T = e^{\tilde{G}^w} T + \mathcal{O}(h^\infty) \,, \qquad \tilde{G} \stackrel{\text{def}}{=} \tilde{\chi}\, G.$$

The above generalization of Prop. 3.11 then shows that

$$e^{\tilde{G}^w} a_0^w T + \mathcal{O}(h^\infty) = T \, e^{F^* \tilde{G}^w} b_0^w + T_1 B_1 \,.$$

The weight $F^*\tilde{G}$ is only relevant on $\pi_R(\mathrm{WF}_h(T)) \Subset \pi_R(F)$, so it makes sense to write the first term on the above right hand side as

$$T\, e^{F^*G^w}\, b_0^w \overset{\mathrm{def}}{=} T\, e^{F^*\tilde{G}^w}\, b_0^w\,,$$

emphasizing that this operator does not depend (modulo $\mathcal{O}_{\mathscr{S}'\to\mathscr{S}}(h^\infty)$) of the way we have truncated G into $\tilde{G}$.

For the same reason, the symbol class $\widetilde{S}_{\frac{1}{2}}(e^{F^*G})$ makes sense if we assume that the symbols are essentially supported inside $\pi_R(F)$. We have just proved the following generalization of Prop. 3.11:

Proposition 3.14. *Take F a Lagrangian relation as described in §2, and $T \in I_\delta(Y \times Y, F')$ as defined above. Take $G \in \mathcal{C}_c^\infty(T^*Y)$ a weight function satisfying (3.20), and $A = e^{G^w(x,hD)}\mathrm{Op}_h^w(a_0)$, where the symbol $a_0 \in \widetilde{S}_{\frac{1}{2}}(1)$, ess-supp $a_0 \Subset \pi_L(F)$.*

Then the following Egorov property holds:

$$AT = TB + T_1 B_1\,,$$

(3.33) $B = e^{(F^*G)^w(x,hD)}\mathrm{Op}_h^w(b_0)\,, \quad b_0 - F^*a_0 \in h^{\frac{1}{2}}\tilde{h}^{\frac{3}{2}}\widetilde{S}_{\frac{1}{2}}(1)\,, \quad \mathrm{ess\text{-}supp}\, b_0 \Subset \pi_R(F)\,,$

$$B_1 = h^{\frac{1}{2}(1-2\delta)}\tilde{h}^{\frac{1}{2}}\mathrm{Op}_h^w(b_1)\,, \quad b_1 \in \widetilde{S}_{\frac{1}{2}}(e^{F^*G})\,, \quad T_1 \in I_\delta(Y \times Y, F')\,.$$

This proposition is the main result of our preliminary section on exotic symbols and weights. Our task in the next section will be to construct an explicit weight G, adapted to the hyperbolic Lagrangian relation F.

4. Construction of escape functions

Escape functions are used to conjugate our quantum map (or monodromy operator), so that the conjugated operator has nicer microlocal properties than the original one, even though it has the same spectrum. More precisely, an escape function $G(x,\xi)$ should have the property to strictly increase along the dynamics, away from the trapped set. An escape function $g(x,\xi)$ has already been used in to construct the monodromy operators associated with the scattering problems in [28]: its effect was indeed to damp the monodromy operator by a factor $\sim h^{N_0}$ outside a fixed neighbourhood of $\mathcal{T}$. Our aim in this section is to construct a more refined escape function, the rôle of which is to damp the monodromy operator outside a semiclassically small neighbourhood of $\mathcal{T}$, namely an $h^{1/2}$-neighbourhood. For this aim, it is necessary to use the calculus on symbol classes $\widetilde{S}_{\frac{1}{2}}(m)$ we have presented in §3.

Our construction will be made in two steps: first in the vicinity the trapped set $\mathcal{T}$, following [43, §7] (where it was partly based on [37, §5]), and then away from the trapped set, following an adaptation of the arguments of [12, Appendix]. For the case of relations F as in §2 which arise from Poincaré maps of smooth flows, we could alternatively use the

flow escape functions given in [43, Proposition 7.7]. However, the general presentation for open hyperbolic maps is simpler than that for flows, and will also apply to the monodromy operators obtained from the broken geodesic flow of the obstacle scattering problem.

4.1. Regularized escape function near the trapped set.

Let $\mathcal{T}_\pm$ be the outgoing and incoming tails given by (1.7) in the case of the obstacle scattering. For an open map F with properties described in §2, these sets are defined by

$$\mathcal{T}_\pm = \{\rho \,:\, F^{\mp n}(\rho) \in \mathcal{U}, \ \forall n \geq 0\},$$

where $F^{\mp n} = F^\mp \circ \cdots \circ F^\mp$ denotes the usual composition of relations. We note that $\mathcal{T}_\pm$ are closed subsets of $\mathcal{U}$, and due to the hyperbolicity of the flow, they are unions of unstable/stable manifolds $W^\pm(\rho)$, $\rho \in \mathcal{T}$.

Remark Before entering the construction, let us consider the simplest model of hyperbolic map, namely the linear dilation $(x, \xi) \mapsto (\Lambda x, \Lambda^{-1}\xi)$ on $T^*\mathbb{R}$, with $\Lambda > 1$. In that case, the trapped set is reduced to one point, the origin, and the sets $\mathcal{T}_\pm$ coincide with the position and momentum axes. In this case, a simple escape function is given by

$$(4.1) \qquad G(x, \xi) = x^2 - \xi^2 = d(\rho, \mathcal{T}_-)^2 - d(\rho, \mathcal{T}_+)^2,$$

and it can be used (after some modification) to analyse scattering flows with a single hyperbolic periodic orbit [11, 12].

The construction of $G(x, \xi)$ in the case of a more complex, but still hyperbolic, trapped set, inspires itself from the expression (4.1) [37]. Our first Lemma is a construction of two functions related with, respectively, the outgoing and incoming tails. It is a straightforward adaptation of [43, Prop. 7.4]. For a moment we will use a small parameter $\epsilon > 0$, which will eventually be taken equal to $h/\tilde{h}$.

Lemma 4.1. *Let $\widetilde{\mathcal{V}}$ be a small neighbourhood of $\mathcal{T}$ and $\widetilde{F} : \widetilde{\mathcal{V}} \to \widetilde{F}(\widetilde{\mathcal{V}})$ be the symplectomorphic restriction of F. Then, there exists $C_0 > 0$ and a neighbourhood $\mathcal{V} \Subset \widetilde{\mathcal{V}}$ of the trapped set, such that the following holds.*

For any small $\epsilon > 0$ there exist functions $\widehat{\varphi}_\pm \in \mathcal{C}^\infty(\mathcal{V} \cup \widetilde{F}(\mathcal{V}); [\epsilon, \infty))$ such that

$$(4.2) \qquad \begin{aligned}
\widehat{\varphi}_\pm(\rho) &\sim d(\rho, \mathcal{T}_\pm)^2 + \epsilon, \\
\pm(\widehat{\varphi}_\pm(\rho) - \widehat{\varphi}_\pm(\widetilde{F}(\rho))) + C_0\epsilon &\sim \widehat{\varphi}_\pm(\rho), \quad \rho \in \mathcal{V}, \\
\partial^\alpha \widehat{\varphi}_\pm(\rho) &= \mathcal{O}(\widehat{\varphi}_\pm(\rho)^{1-|\alpha|/2}), \\
\widehat{\varphi}_+(\rho) + \widehat{\varphi}_-(\rho) &\sim d(\rho, \mathcal{T})^2 + \epsilon.
\end{aligned}$$

Here and below, $a \sim b$ means that there exists a constant $C \geq 1$ (independendent of ϵ) such that $b/C \leq a \leq Cb$.

To prove this lemma we need two preliminary results.

Lemma 4.2. *Suppose* $\Gamma \subset \mathbb{R}^m$ *is a closed set. For any* $\epsilon > 0$ *there exists* $\varphi \in \mathcal{C}^\infty(\mathbb{R}^m)$, *such that*

(4.3) $$\varphi \geq \epsilon \,, \quad \varphi \sim \epsilon + d(\bullet, \Gamma)^2 \,, \qquad \partial^\alpha \varphi = \mathcal{O}(\varphi^{1-|\alpha|/2}) \,,$$

where the estimates are uniform on $\mathbb{R}^m$.

Proof. For reader's convenience we recall the proof (see [43, Lemma 7.2]) based on a Whitney covering argument (see [15, Example 1.4.8, Lemma 1.4.9]). For $\delta \ll 1$ choose a maximal sequence $x_j \in \mathbb{R}^m \setminus \Gamma$ such that $d(x_j, x_i) \geq \delta d(x_i, \Gamma)$ (here d is the Euclidean distance). We claim that

(4.4) $$\bigcup_j B(x_j, d(x_j, \Gamma)/8) = \mathbb{R}^m \setminus \Gamma \,.$$

In fact, if x is not in the sequence then, for some j, $d(x, x_j) < \delta d(x_j, \Gamma)$ or $d(x, x_j) < \delta d(x, \Gamma)$. In the first case $x \in B(x_j, d(x_j, \Gamma)/8)$ if $\delta < 1/8$. In the second case, $d(x, x_j) < \delta(d(x_j, \Gamma) + d(x, x_j))$ which means that $x \in B(x_j, d(x_j, \Gamma)/8)$, if $\delta/(1 - \delta) < 1/8$. Hence (4.4) holds provided $\delta < 1/9$.

We now claim that every $x \in \mathbb{R}^m \setminus \Gamma$ lies in at most $N_0 = N_0(\delta, m)$ balls $B(x_j, d(x_j, \Gamma)/2)$. To see this consider x and $i \neq j$ such that $d(x, x_j) \leq d(x_j, \Gamma)/2$ and $d(x, x_i) \leq d(x_j, \Gamma)/2$. Then simple applications of the triangle inequality show that

$$d(x_i, x_j) \geq 2\delta d(x, \Gamma)/3 \,, \quad d(x_j, \Gamma) \leq d(x, \Gamma)/2 \,.$$

Hence

$$B(x_i, \delta d(x, \Gamma)/3) \cap B(x_j, \delta d(x, \Gamma)/3) = \emptyset \,,$$
$$B(x_\ell, \delta d(x, \Gamma)/3) \subset B(x, 4d(x, \Gamma)/3) \,, \quad \ell = i, j \,.$$

Comparison of volumes shows that the maximal number of such $\ell's$ is $(4/\delta)^m$.

Let $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^m; [0, 1])$ be supported in $B(0, 1/4)$, and be identically one in $B(0, 1/8)$. We define

$$\varphi_\epsilon(x) \overset{\text{def}}{=} \epsilon + \sum_{d(x_j, \Gamma) > \sqrt{\epsilon}} d(x_j, \Gamma)^2 \chi\left(\frac{x - x_j}{d(x_j, \Gamma) + \sqrt{\epsilon}}\right)$$

We first note that the number non-zero terms in the sum is uniformly bounded by N_0. In fact, $d(x_j, \Gamma) + \sqrt{\epsilon} < 2d(x_j, \Gamma)$, and hence if $\chi((x - x_j)/(d(x_j, \Gamma) + \sqrt{\epsilon})) \neq 0$ then

$$1/4 \geq |x - x_j|/(d(x_j, \Gamma) + \sqrt{\epsilon}) \geq (1/2)|x - x_j|/d(x_j, \Gamma) \,,$$

and $x \in B(x_j, d(x_j, d(x_j, \Gamma))/2)$. This shows that $\varphi_\epsilon(x) \leq 2N_0(\epsilon + d(x, \Gamma)^2)$, and

$$\partial^\alpha \varphi_\epsilon(x) = \mathcal{O}((d(x, \Gamma)^2 + \epsilon)^{1-|\alpha|/2}) \,,$$

uniformly on compact sets.

To see the lower bound on φ_ϵ we first consider the case when $d(x, \Gamma) \leq C\sqrt{\epsilon}$.

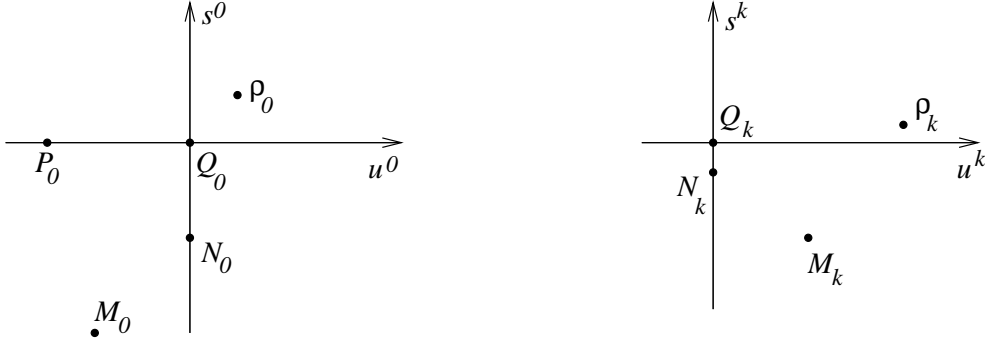$$\varphi_\epsilon(x) \geq \epsilon \geq (\epsilon + d(x, \Gamma)^2)/C' \,.$$

FIGURE 5.

If $d(x,\Gamma) > C\sqrt{\epsilon}$ then for at least one j, $\chi((x - x_j)/(d(x_j,\Gamma) + \sqrt{\epsilon})) = 1$ (since the balls $B(x_j, d(x_j,\Gamma)/8)$ cover the complement of Γ, and $\chi(t) = 1$ if $|t| \leq 1/8$). Thus

$$\varphi_\epsilon(x) \geq \epsilon + d(x_j,\Gamma)^2 \geq (\epsilon + d(x,\Gamma)^2)/C,$$

which concludes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

The second preliminary result is essentially standard in the dynamical systems literature, resulting from the hyperbolicity of the map f on $\mathcal{T}$.

Lemma 4.3. *For $\widetilde{\mathcal{V}}$ a small enough neighbourhood of $\mathcal{T}$, there exist $0 < \theta_1 < 1$ and $C > 0$ such that, for any $K \geq 0$ and $\rho \in \widetilde{\mathcal{V}}$ such that $\widetilde{F}^k(\rho)$ remains in $\widetilde{\mathcal{V}}$ for all $0 \leq k \leq K$, we have*

$$d(\widetilde{F}^k(\rho), \mathcal{T}_+) \leq C\,\theta_1^k\, d(\rho, \mathcal{T}_+), \quad 0 \leq k \leq K.$$

The same property holds in the backwards evolution with $\mathcal{T}_+$ replaced by $\mathcal{T}_-$.

Proof. We want to use the fact that the map $\widetilde{F}$ is strictly contracting in the direction transerse to $\mathcal{T}_+$ (unstable manifold). To state this contractivity it is convenient to choose coordinate charts adapted to the dynamics, and containing the points $\widetilde{F}^k(\rho)$. Let us assume that $\widetilde{\mathcal{V}}$ is an ε-neighbourhood of $\mathcal{T}$, with $\varepsilon > 0$ small. By assumption, each $\rho_k = \widetilde{F}^k(\rho_0)$, $\rho_0 = \rho$, lies in an ε-size neighbourhood of some $M_k \in \mathcal{T}$, $k \leq K$. As a result, the sequence (M_k) satisfies $d(F(M_k), M_{k+1}) \leq C\varepsilon$: it is a $C\varepsilon$-pseudoorbit. From the *shadowing lemma* [19, §18.1], there exists an associated orbit

$$N_k = F(N_{k-1}) \in \mathcal{T}, \quad d(N_k, M_k) \leq \delta,$$

with δ small if $C\varepsilon$ is small. Hence, $d(N_k, \rho_k) \leq \delta + \varepsilon$.

Besides, the distance $d(\rho_0, \mathcal{T}_+)$ is equal to the distance between ρ_0 and a certain local unstable leaf $W^+_{loc}(P_0)$, with $P_0 \in \mathcal{T} \cap \widetilde{\mathcal{V}}$. We will consider the point $Q_0 = W^-_{loc}(N_0) \cap$

$W^+_{\mathrm{loc}}(P_0) \in \mathcal{T}$ and its images $Q_k = \widetilde{F}(Q_0)$ to construct our coordinate charts (u^k, s^k), such that the local stable and unstable manifolds $W^\pm_{\mathrm{loc}}(Q_k)$ (see [19, §6.2]) are given by

$$W^-_{\mathrm{loc}}(Q_k) = \{(0, s^k)\}, \quad W^+_{\mathrm{loc}}(Q_k) = \{(u^k, 0)\}.$$

From the uniform transversality of stable/unstable manifolds, these coordinates can be chosen such that for the Euclidean norm we have

$$\|u^k\|^2 + \|s^k\|^2 \sim d(Q_k, \rho^k)^2,$$

uniformly for $0 \leq k \leq K$. We also have $\|u^k\|, \|s^k\| \leq C(\varepsilon + \delta)$. See Fig. 5 for a schematic representation. In this coordinates, the point $\rho_k = (u^k, s^k)$ is mapped into

$$(4.5) \qquad \widetilde{F}(u^k, s^k) = (A_k u^k + \alpha_k(u^k, s^k), {}^t A_k^{-1} s^k + \beta_k(u^k, s^k))$$

with α_k, β_k smooth functions, $\alpha_k(0, s) = \beta_k(u, 0) = 0$, $d\alpha_k(0, 0) = d\beta_k(0, 0) = 0$, and the contraction property $\|A_k^{-1}\| \leq \nu < 1$.

This contraction implies that

$$\|s^k\| \leq (\nu + C(\delta + \varepsilon))^k \|s^0\|, \quad 0 \leq k \leq K.$$

We can choose ε, δ small enough such that $\theta_1 \stackrel{\mathrm{def}}{=} \nu + C(\delta + \varepsilon) < 1$. Finally, $d(\rho_k, W^+_{\mathrm{loc}}(Q_k)) \sim \|s^k\|$ satisfies

$$d(\rho_k, \mathcal{T}_+) \leq d(\rho_k, W^+_{\mathrm{loc}}(Q_k)) \leq C \theta_1^k d(\rho_0, \mathcal{T}_+).$$

$\square$

Proof of Lemma 4.1. We adapt the proof of [43, Proposition 7.4] to the setting of a discrete dynamical system. Let $\varphi_\pm$ be the functions provided by Lemma 4.2, respectively for $\Gamma = \mathcal{T}_\pm$. As above we call $\widetilde{F}$ the restriction of F on $\widetilde{\mathcal{V}}$, and similarly call $\widetilde{F}^{-1} = F^{-1}|_{\widetilde{\mathcal{V}}}$. For $K \geq 1$ to be determined below, we consider the following neighbourhood of $\mathcal{T}$:

$$(4.6) \qquad \mathcal{V} = \bigcap_{k=-K-1}^{K+1} \widetilde{F}^k(\widetilde{\mathcal{V}}), \quad \text{and define } \mathcal{V}' \stackrel{\mathrm{def}}{=} \mathcal{V} \cup \widetilde{F}(\mathcal{V}).$$

In words, $\mathcal{V}$ is the set of points $\rho \in \widetilde{\mathcal{V}}$, whose orbit remains in $\widetilde{\mathcal{V}}$ in the time interval $[-K-1, K+1]$. The following functions are then well-defined on $\mathcal{V}'$:

$$\widehat{\varphi}_\pm(\rho) \stackrel{\mathrm{def}}{=} \sum_{k=0}^{K} \varphi_\pm(\widetilde{F}^{\pm k}(\rho)).$$

Lemma 4.3 shows that if $\widetilde{\mathcal{V}}$ (and therefore $\mathcal{V}$) is small enough, then there exist $\theta_1 \in (0, 1)$ and $C > 1$, such that

$$(4.7) \qquad d(\widetilde{F}^{\pm k}(\rho), \mathcal{T}_\pm) \leq C \theta_1^k d(\rho, \mathcal{T}_\pm), \qquad 0 \leq k \leq K.$$

It thus follows that

$$(4.8) \qquad \widehat{\varphi}_\pm \sim \varphi_\pm(\rho) \sim d(\rho, \mathcal{T}_\pm)^2 + \epsilon,$$

with implicit constants independent of K. This establishes the first statement in (4.2). To obtain the second statement we see that, for any $\rho \in \mathcal{V}$,

$$\widehat{\varphi}_+(\rho) - \widehat{\varphi}_+(\widetilde{F}(\rho)) = \varphi_+(\rho) - \varphi_+(\widetilde{F}^{K+1}(\rho))$$
$$\widehat{\varphi}_-(\widetilde{F}(\rho)) - \widehat{\varphi}_-(\rho) = \varphi_-(\widetilde{F}(\rho)) - \varphi_-(\widetilde{F}^{-K}(\rho)) \,.$$

In view of (4.8), we find

$$\varphi_+(\rho) - \varphi_+(\widetilde{F}^{K+1}(\rho)) = \varphi_+(\rho) + \mathcal{O}(d(\widetilde{F}^{K+1}(\rho), \mathcal{T}_+)^2 + \epsilon)$$
$$= \varphi_+(\rho) + \mathcal{O}(\theta_1^K d(\rho, \mathcal{T}_+)^2 + \epsilon)$$
$$= \varphi_+(\rho)(1 + \mathcal{O}(\theta_1^K)) + \mathcal{O}(\epsilon) \,,$$

and similarly for φ_-. Taking K large enough so that $\mathcal{O}(\theta_1^K) \leq 1/2$, we obtain, for some $C_0 > 0$, the required estimates:

$$\pm(\widehat{\varphi}_\pm(\rho) - \widehat{\varphi}_\pm(\widetilde{F}(\rho))) + C_0\epsilon \sim \widehat{\varphi}_\pm(\rho) \,.$$

The estimate

$$\partial^\alpha \widehat{\varphi}_\pm(\rho) = \mathcal{O}(\widehat{\varphi}_\pm(\rho)^{1-|\alpha|/2}) \,,$$

follows from the properties of $\varphi_\pm$ stated in Lemma 4.2. It remains to show that

(4.9)
$$\widehat{\varphi}_+(\rho) + \widehat{\varphi}_-(\rho) \sim d(\rho, \mathcal{T})^2 + \epsilon \,.$$

This results [37, 43] from the uniform transversality of the stable and unstable manifolds (near the trapped set). Indeed, this transversality implies that, for any two nearby points $\rho_1, \rho_- \in \mathcal{T}$ and ρ near them, we have

(4.10)
$$d(\rho, W_{\text{loc}}^+(\rho_1) \cap W_{\text{loc}}^-(\rho_2))^2 \sim d(\rho, W_{\text{loc}}^+(\rho_1))^2 + d(\rho, W_{\text{loc}}^-(\rho_2))^2 \,.$$

Besides, since $\mathcal{T}_+$ $(\mathcal{T}_-)$ is a union of local unstable (stable, respectively) manifolds, for any ρ near $\mathcal{T}$ the distance $d(\rho, \mathcal{T}_\pm)$ is equal to $d(\rho, W_{\text{loc}}^+(\rho_\pm))$ for some nearby points $\rho_\pm$. We thus get

(4.11)
$$d(\rho, \mathcal{T}_+)^2 + d(\rho, \mathcal{T}_-)^2 \sim d(\rho, W_{\text{loc}}^+(\rho_+) \cap W_{\text{loc}}^-(\rho_-))^2 \geq d(\rho, \mathcal{T})^2 \,.$$

On the other hand, $d(\rho, \mathcal{T}) = d(\rho, \rho_0)$ for some $\rho_0 \in \mathcal{T}$, so that

$$d(\rho, \mathcal{T})^2 = d(\rho, W_{\text{loc}}^+(\rho_0) \cap W_{\text{loc}}^-(\rho_0))^2$$
$$\sim d(\rho, W_{\text{loc}}^+(\rho_0))^2 + d(\rho, W_{\text{loc}}^-(\rho_0))^2 \geq d(\rho, \mathcal{T}_+) + d(\rho, \mathcal{T}_-) \,.$$

We have thus proven the transversality result

(4.12)
$$d(\rho, \mathcal{T})^2 \sim d(\rho, \mathcal{T}_+)^2 + d(\rho, \mathcal{T}_-)^2 \,, \qquad \rho \text{ near } \mathcal{T} \,.$$

The statement (4.9) then directly follows from (4.8). $\qquad\square$

From the properties of Lemma 4.1, and in view of the model (4.1), it seems tempting to take the escape function of the form $\widehat{\varphi}_+(\rho) - \widehat{\varphi}_-(\rho)$. Yet, since we want e^G to be an order function, G cannot grow too fast at infinity. For this reason, following [43, §7] we use a logarithmic flattening to construct our escape function:

Lemma 4.4. *Let $\widehat{\varphi}_\pm$ be the functions given by Lemma 4.1. For some $M \gg 1$ independent of ϵ, let us define the function*

$$(4.13) \qquad \widehat{G} \stackrel{\text{def}}{=} \log(M\epsilon + \widehat{\varphi}_-) - \log(M\epsilon + \widehat{\varphi}_+)$$

on the neighbourhood $\mathcal{V}'$ of the trapped set defined in Eq. (4.6).

Then there exists $C_1 > 0$ such that

$$\widehat{G} = \mathcal{O}(\log(1/\epsilon)), \quad \partial_\rho^\alpha \widehat{G} = \mathcal{O}(\min(\widehat{\varphi}_+, \widehat{\varphi}_-)^{-\frac{|\alpha|}{2}}) = \mathcal{O}(\epsilon^{-\frac{|\alpha|}{2}}), \quad |\alpha| \geq 1,$$

$$(4.14) \qquad \partial_\rho^\alpha(\widehat{G}(\widetilde{F}(\rho)) - \widehat{G}(\rho)) = \mathcal{O}(\min(\widehat{\varphi}_+, \widehat{\varphi}_-)^{-\frac{|\alpha|}{2}}) = \mathcal{O}(\epsilon^{-\frac{|\alpha|}{2}}), \quad |\alpha| \geq 0, \quad \rho \in \mathcal{V},$$

$$\rho \in \mathcal{V}, \ d(\rho, \mathcal{T})^2 \geq C_1\epsilon \implies \widehat{G}(\widetilde{F}(\rho)) - \widehat{G}(\rho) \geq 1/C_1.$$

Proof. Only the last property in (4.14) needs to be checked, the others following directly from Lemma 4.1. For this aim we compute

$$(4.15) \quad \widehat{G}(\widetilde{F}(\rho)) - \widehat{G}(\rho) = \log\left(1 + \frac{\widehat{\varphi}_-(\widetilde{F}(\rho)) - \widehat{\varphi}_-(\rho)}{M\epsilon + \widehat{\varphi}_-(\rho)}\right) + \log\left(1 + \frac{\widehat{\varphi}_+(\rho) - \widehat{\varphi}_+(\widetilde{F}(\rho))}{M\epsilon + \widehat{\varphi}_+(\widetilde{F}(\rho))}\right).$$

Using (4.12), the condition $d(\mathcal{T}, \rho)^2 \geq C_1\epsilon$ implies that $d(\mathcal{T}_+, \rho)^2 \geq C_2\epsilon$ or $d(\mathcal{T}_-, \rho)^2 \geq C_2\epsilon$, and C_2 can be taken as large as we wish if C_1 is chosen large enough.

Let us take care of the first term in (4.15). For this we need to bound from below the ratio

$$(4.16) \qquad R_-(\rho) \stackrel{\text{def}}{=} \frac{\widehat{\varphi}_-(\widetilde{F}(\rho)) - \widehat{\varphi}_-(\rho)}{M\epsilon + \widehat{\varphi}_-(\rho)}.$$

Let us call $C_3 \geq 1$ a uniform constant for the equivalences in (4.2). The second equivalence shows that

$$\widehat{\varphi}_-(\widetilde{F}(\rho)) - \widehat{\varphi}_-(\rho) \geq \widehat{\varphi}_-(\rho)/C_3 - C_0\epsilon.$$

Since the function $x \mapsto \frac{x/C_3 - C_0}{x + M}$ is increasing for $x \geq 0$, the ratio (4.16) satisfies $R_-(\rho) \geq -C_0/M$. If we take M large enough, we ensure that

$$\log(1 + R_-(\rho)) \geq -2C_0/M.$$

Furthermore, in the region where $d(\rho, \mathcal{T}_-)^2 \geq C_2\epsilon$, the first statement in (4.2) shows that $\widehat{\varphi}_-(\rho) \geq \frac{C_2\epsilon}{C_3}$, so that

$$R_-(\rho) \geq \frac{C_2/C_3^2 - C_0}{C_2/C_3 + M} \stackrel{\text{def}}{=} C_4.$$

If we take C_1 (and thus C_2) large enough, C_4 is nonnegative, and $\log(1 + R_-(\rho)) \geq C_4/2$. By increasing M and C_1 if necessary, we can assume that $C_4 > 6C_0/M$.

The same inequalities hold for the second term in (4.15) (the condition is now $d(\rho, \mathcal{T}_+)^2 \geq C_2\epsilon$).

Finally, if $d(\rho, \mathcal{T})^2 \geq C_1 \epsilon$, we find the inequality

$$\widehat{G}(\widetilde{F}(\rho)) - \widehat{G}(\rho) \geq \frac{C_4}{2} - \frac{2C_0}{M} > \frac{C_0}{M}.$$

Increasing C_1 if necessary, the right hand side is $\geq 1/C_1$. □

4.2. **Final construction of the escape function.**

We now set up the escape function away from the trapped set. We recall that $\widetilde{D}$ is the departure, resp. arrival sets of the open relation F.

Lemma 4.5. *Let $\mathcal{W}_2$ be an arbitrary small neighbourhood of the trapped set, and $\mathcal{W}_3 \Subset \widetilde{D}$ large enough (in particular, we require that $\mathcal{W}_3 \supseteq \operatorname{supp} a_M$, where a_M is the function in Definition 2.1). Then, there exists $g_0 \in \mathcal{C}_c^\infty(T^*Y)$, and a neighbourhood $\mathcal{W}_1 \Subset \mathcal{W}_2$ of the trapped set, with the following properties:*

$$\begin{aligned}
&\forall \rho \in \mathcal{W}_1, \quad g_0(\rho) \equiv 0, \\
\text{(4.17)} \quad &\forall \rho \in \mathcal{W}_3, \quad g_0(F(\rho)) - g_0(\rho) \geq 0, \\
&\forall \rho \in \mathcal{W}_3 \setminus \mathcal{W}_2, \quad g_0(F(\rho)) - g_0(\rho) \geq 1.
\end{aligned}$$

In [28] such a function g_0 was obtained as the restriction (on the Poincaré section) of an escape function for the scattering flow, the latter being constructed in [12, Appendix].

The proof of this Lemma for a general open map satisfying the assumptions of §2 will be given in the appendix. It is an adaptation of the construction of an escape function near the outgoing tail performed in [5] and [51].

Now we want to glue our escape function $\widehat{G}$ constructed in Lemma 4.4, defined in the small neighbourhood $\mathcal{V}'$ of $\mathcal{T}$, with the escape function g_0 defined away from the trapped set: the final escape function G will be a globally defined function on T^*Y. One crucial thing is to check that this function is the logarithm of an order function for the $\widetilde{S}_{\frac{1}{2}}$ class — see (3.18) and (3.20).

The following construction is directly inspired by [43, Prop. 7.7]

Proposition 4.6. *Let $\mathcal{V}$, $\widehat{G}$ be as in Lemma 4.4, and choose the neighbourhoods $\mathcal{W}_i$ such that $\mathcal{T} \Subset \mathcal{W}_2 \Subset \mathcal{V} \Subset \mathcal{W}_3 \Subset \pi_R(F)$.*

Take $\chi \in \mathcal{C}_c^\infty(\mathcal{V}')$ equal to 1 in $\mathcal{W}_2 \cup F(\mathcal{W}_2) \Subset \mathcal{V}'$. Construct an escape function g_0 as in Lemma 4.5, and define

$$G \stackrel{\text{def}}{=} \chi \widehat{G} + C_5 \log(1/\epsilon) \, g_0 \in \mathcal{C}_c^\infty(T^*Y).$$

Then, provided C_5 is chosen large enough, the function G satisfies the following estimates:

$$|G(\rho)| \leq C_6 \log(1/\epsilon), \quad \partial^\alpha G = \mathcal{O}(\epsilon^{-|\alpha|/2}), \quad |\alpha| \geq 1,$$

(4.18)
$$\rho \in \mathcal{W}_2 \implies G(F(\rho)) - G(\rho) \geq -C_7,$$
$$\rho \in \mathcal{W}_2, \quad d(\rho, \mathcal{T})^2 \geq C_1\epsilon \implies G(F(\rho)) - G(\rho) \geq 1/C_1,$$
$$\rho \in \mathcal{W}_3 \setminus \mathcal{W}_2 \implies G(F(\rho)) - G(\rho) \geq C_8 \log(1/\epsilon).$$

In addition we have

(4.19)
$$\frac{\exp G(\rho)}{\exp G(\mu)} \leq C_9 \left\langle \frac{\rho - \mu}{\sqrt{\epsilon}} \right\rangle^{N_1},$$

for some constants C_9 and N_1.

Eventually we will apply this construction with the small parameter

$$\epsilon = \frac{h}{\tilde{h}}.$$

In particular, the condition (4.19) shows that $\exp G$ is an order function in the sense of (3.18) and (3.20).

Proof of Proposition 4.6: The first three lines of (4.18) are obvious, since $\chi \equiv 1$ on $\mathcal{W}_2 \cup F(\mathcal{W}_2)$, and $g_0 \circ F - g_0 \geq 0$. To check the fourth line, we notice that outside $\mathcal{W}_2$, we have

$$G(F(\rho)) - G(\rho) = \mathcal{O}(\log(1/\epsilon)) + C_5 \log(1/\epsilon)(g_0 \circ F(\rho) - g_0(\rho)) \geq \mathcal{O}(\log(1/\epsilon)) + C_5 \log(1/\epsilon).$$

If C_5 is chosen large enough, the right hand side is bounded from below by $C_8 \log(1/\epsilon)$ for some $C_8 > 0$.

We then need to check (4.19). We first check it for the function $\widehat{G}$: we want to show

(4.20)
$$\frac{\widehat{\varphi}_\pm(\rho) + M\epsilon}{\widehat{\varphi}_\pm(\mu) + M\epsilon} \leq C_1 \left\langle \frac{\rho - \mu}{\sqrt{\epsilon}} \right\rangle^2,$$

with C_1 depending on M. Since $\widehat{\varphi}_\pm + M\epsilon \sim \widehat{\varphi}_\pm$, this is the same as

$$\frac{\widehat{\varphi}_\pm(\rho)}{\widehat{\varphi}_\pm(\mu)} \leq \tilde{C}_1 \left\langle \frac{\rho - \mu}{\sqrt{\epsilon}} \right\rangle^2,$$

and that follows from the triangle inequality and the properties of $\widehat{\varphi}_\pm$:

$$\widehat{\varphi}_\pm(\rho) \leq C(d(\rho, \mathcal{T}_\pm)^2 + \epsilon) \leq C(d(\mu, \mathcal{T}_\pm)^2 + |\mu - \rho|^2 + \epsilon)$$
$$\leq C'(\widehat{\varphi}_\pm(\mu) + |\mu - \rho|^2) = C'(\widehat{\varphi}_\pm(\mu) + \epsilon\langle(\rho - \mu)/\sqrt{\epsilon}\rangle^2)$$
$$\leq 2C'\widehat{\varphi}_\pm(\mu)\langle(\rho - \mu)/\sqrt{\epsilon}\rangle^2.$$

Inserting the definition of $\widehat{G}$, (4.20) gives

$$|\widehat{G}(\rho) - \widehat{G}(\mu)| \leq C + 2\log\langle(\rho - \mu)/\sqrt{\epsilon}\rangle.$$

The estimate for G is essentially the same:

$$|G(\rho) - G(\mu)| \leq |\chi(\rho)\widehat{G}(\rho) - \chi(\mu)\widehat{G}(\mu)| + C_5 \log(1/\epsilon) |g_0(\rho) - g_0(\mu)|$$
$$\leq C|\rho - \mu| \log(1/\epsilon) + C \log\langle(\rho - \mu)/\sqrt{\epsilon}\rangle + C$$
$$\leq C' \log\langle(\rho - \mu)/\sqrt{\epsilon}\rangle + C'.$$

The last estimate follows from

$$x \log \frac{1}{\epsilon} \leq C \log \left\langle \frac{x}{\sqrt{\epsilon}} \right\rangle + C, \qquad 0 \leq x \leq 1.$$

$\square$

The function $G(x, \xi)$ we have constructed will be used to twist the monodromy operator $M(z, h)$, before injecting it inside a Grushin problem. We first recall the structure (and strategy) of Grushin problems.

5. THE GRUSHIN PROBLEM

In this section we will construct a well posed Grushin problem for the operator $I - M$, where $M = M(z, h)$ is an abstract hyperbolic open quantum map (or monodromy operator) as defined in §2. We will treat successively the untruncated operators M, and then the operators $\widetilde{M}$ truncated by the finite rank projector Π_h. The second case applies to the monodromy operators constructed as effective Hamiltonians for open chaotic systems [28], and to the operators constructed in §6 to deal with obstacle scattering.

5.1. **Refresher on Grushin problems.** We recall some linear algebra facts related to the Schur complement formula. For any invertible square matrix decomposed into 4 blocks, we have

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \implies a^{-1} = \alpha - \beta\delta^{-1}\gamma,$$

provided that δ^{-1} exists. As reviewed in [42] this formula can be applied to *Grushin problems*

$$\begin{bmatrix} P & R_- \\ R_+ & 0 \end{bmatrix} : \mathcal{H}_1 \oplus \mathcal{H}_- \longrightarrow \mathcal{H}_2 \oplus \mathcal{H}_+,$$

where P is the operator under investigation and $R_\pm$ are suitably chosen. When this matrix of operators is invertible we say that the Grushin problem is *well posed*. If $\dim \mathcal{H}_- = \dim \mathcal{H}_+ < \infty$, and $P = P(z)$, it is customary to write

$$\begin{bmatrix} P(z) & R_- \\ R_+ & 0 \end{bmatrix}^{-1} = \begin{bmatrix} E(z) & E_+(z) \\ E_-(z) & E_{-+}(z) \end{bmatrix},$$

and the invertibility of $P(z) : \mathcal{H}_1 \to \mathcal{H}_2$ is equivalent to the invertibility of the finite dimensional matrix $E_{-+}(z)$. For this reason, the latter will be called an *effective Hamiltonian*.

This connection is made more precise by the following standard result [42, Proposition 4.1]:

Proposition 5.1. *Suppose that $P = P(z)$ is a family of Fredholm operators depending holomorphically on $z \in \Omega$, where $\Omega \subset \mathbb{C}$ is a simply connected open set. Suppose also that the operators $R_\pm = R_\pm(z)$ are of finite rank, depend holomorphically on $z \in \Omega$, and the corresponding Grushin problem is well posed for $z \in \Omega$. Then for any smooth positively oriented $\gamma = \partial \Gamma$, $\Gamma \Subset \Omega$, on which $P(z)^{-1}$ exists, the operator $\int_\gamma \partial_z P(z) P(z)^{-1} dz$ is of trace class and we have*

$$
\begin{aligned}
(5.1) \qquad \frac{1}{2\pi i} \,\mathrm{tr} \int_\gamma P(z)^{-1} \partial_z P(z) \, dz &= \frac{1}{2\pi i} \,\mathrm{tr} \int_\gamma E_{-+}(z)^{-1} \partial_z E_{-+}(z) \, dz \\
&= \#\{z \in \Gamma : \det E_{-+}(z) = 0\} \,,
\end{aligned}
$$

where the zeros are counted according to their multiplicities.

5.2. **A well posed Grushin problem.** Let $Y \Subset \bigsqcup_{j=1}^J \mathbb{R}^d$, $\mathcal{U} \Subset T^*Y$, $F \subset \mathcal{U} \times \mathcal{U}$, be a hyperbolic Lagrangian relation, and $M = M(z,h) \in I_{0+}(Y \times Y, F')$ be an associated hyperbolic quantum monodromy operator as in Definition 2.1. In this section it will be more convenient to see M as a Fourier integral operator acting on $L^2(\mathbb{R}^d)^J$. Let G be the escape function constructed in Proposition 4.6, and $G^w(x, hD)$ the corresponding pseudo-differential operator.

We will construct a well posed Grushin problem for the operator $P(z) = I - M_{tG}(z)$, where

$$
(5.2) \qquad M_{tG}(z) \stackrel{\text{def}}{=} e^{-tG^w(x,hD)} \, M(z) \, e^{tG^w(x,hD)} \,, \quad t > 0 \,.
$$

For this aim we will need a finite dimensional subspace of $L^2(\mathbb{R}^d)^J$ microlocally covering an $(h/\tilde{h})^{\frac{1}{2}}$ neighbourhood of the trapped set $\mathcal{T}$. We will construct that subspace by using an auxiliary pseudodifferential operator.

Proposition 5.2. *Let $\Gamma \Subset T^*\mathbb{R}^d$ be a compact set. For the order function*

$$
(5.3) \qquad m(x,\xi) = h/\tilde{h} + d((x,\xi), \Gamma)^2 \,,
$$

there exists $q \in \widetilde{S}_{\frac{1}{2}}(m)$, so that

$$
(5.4) \qquad q(x,\xi) \sim m(x,\xi) \,, \qquad \partial^\alpha q = \mathcal{O}(q^{1-|\alpha|/2}) \,,
$$

and such that, for $\tilde{h}$ small enough, the operator $Q \stackrel{\text{def}}{=} q^w(x, hD)$ satisfies $Q = Q^* \geq h/2\tilde{h}$.

Proof. Let φ be as in Lemma 4.2 with $\epsilon = h/\tilde{h}$. We will take $q = \varphi$, $Q = \mathrm{Op}_h^w(q)$. From the reality of q, this operator is symmetric on $L^2(\mathbb{R}^d)$. The estimates (5.4) are automatic, as well as the uniform bound $q(\rho) \geq h/\tilde{h}$. Taking into account the compactness of Γ, the estimates (4.3) easily imply that $q \in \widetilde{S}_{\frac{1}{2}}(m)$.

To prove the lower bound on Q, we use the rescaling (3.11):

$$\tilde{\varphi}(\tilde{\rho}) = (\tilde{h}/h)\varphi\big((h/\tilde{h})^{\frac{1}{2}}\tilde{\rho}\big)\,, \qquad \tilde{\Gamma} = (\tilde{h}/h)^{\frac{1}{2}}\Gamma\,.$$

Our aim is to show that $\mathrm{Op}_{\tilde{h}}^w(\tilde{\varphi}) \geq 1/2$ for $\tilde{h}$ small enough. This is a form of Gårding inequality, but for an unbounded symbol. To prove it, we draw from (4.3)

$$\frac{\tilde{\varphi}(\tilde{\rho}_1)}{\tilde{\varphi}(\tilde{\rho}_2)} \leq C\frac{1 + d(\tilde{\rho}_1, \tilde{\Gamma})^2}{1 + d(\tilde{\rho}_2, \tilde{\Gamma})^2} \leq C(1 + d(\tilde{\rho}_1, \tilde{\rho}_2)^2)\,,$$

which shows that $\tilde{\varphi}$ is an order function in the sense of (3.13), and $\tilde{\varphi} \in \widetilde{S}(\tilde{\varphi})$. Similarly, the uniform bound $\tilde{\varphi} \geq 1$ implies that

$$(\tilde{\varphi} - \lambda)^{-1} \in \widetilde{S}(1/\tilde{\varphi})\,, \quad \lambda < \frac{2}{3}\,,$$

and hence for $\tilde{h}$ small enough, $\mathrm{Op}_{\tilde{h}}^w(\tilde{\varphi}) - \lambda$ is invertible on $L^2(\mathbb{R}^d)$, uniformly for $\lambda \leq 1/2$ As a result, for $\tilde{h}$ small enough we have the bound

$$\mathrm{Op}_{\tilde{h}}^w(\tilde{\varphi}) \geq 1/2\,.$$

Recalling that

$$\mathrm{Op}_h^w(\varphi) = (h/\tilde{h})\, U_{h,\tilde{h}}^{-1}\, \mathrm{Op}_{\tilde{h}}^w(\tilde{\varphi})\, U_{h,\tilde{h}}\,,$$

for $\tilde{h}$ small enough we obtain the stated lower bound on $Q = \mathrm{Op}_h^w(q) = \mathrm{Op}_h^w(\varphi)$. $\qquad\square$

The above construction can be straightforwardly extended to the case where $\Gamma = \mathcal{T} \Subset \mathcal{U} \Subset (T^*\mathbb{R}^d)^J$: the symbol $q = (q_j)_{j=1,\dots,J}$ is then a vector of symbols $q_j \in \widetilde{S}_{\frac{1}{2}}(m)$, and similarly $Q = (Q_j)_{j=1,\dots,J}$ is an operator on $L^2(\mathbb{R}^d)^J$.

Let $K \gg 1$ be a constant to be chosen later. We define the following finite dimensional Hilbert space and the corresponding orthogonal projection:

$$(5.5) \quad V \overset{\text{def}}{=} \bigoplus_{j=1}^J V_j, \quad V_j \overset{\text{def}}{=} \mathbb{1}_{Q_j \leq Kh/\tilde{h}} L^2(\mathbb{R}^d)\,, \qquad \Pi_V \overset{\text{def}}{=} \mathrm{diag}(\mathbb{1}_{Q_j \leq Kh/\tilde{h}}) : L^2(\mathbb{R}^d)^J \overset{\perp}{\longrightarrow} V\,.$$

Due to the ellipticity of Q away from $\mathcal{T}$, see (5.3), we have

$$\dim V < \infty\,.$$

Lemma 5.4 below will give a more precise statement.

The operators $R_\pm$ to inject in the Grushin problem are then defined as follows:

$$(5.6) \qquad\qquad R_+ = \Pi_V\ :\ L^2(\mathbb{R}^d)^J \longrightarrow V\,, \qquad R_-\ :\ V \hookrightarrow L^2(\mathbb{R}^d)^J\,.$$

Before stating the result about the well posed Grushin problem for the operator M_{tG}, we prove a crucial lemma based on the analysis in previous sections:

Lemma 5.3. *Let $M_{tG}(z)$ be given by (5.2), $z \in \Omega(h)$, where $\Omega(h)$ is given in (2.5). Then for Π_V given above, and any $\varepsilon > 0$, there exists $t = t(\varepsilon)$, sufficiently large, and $\tilde{h} = \tilde{h}(t, \varepsilon)$, sufficiently small (independently of $h \to 0$) so that, provided N_0 from (2.4) and C_6 from (5.9) satisfy $N_0 > 4C_6 t$, then for $h > 0$ small enough one has*

$$(5.7) \qquad \|(I - \Pi_V) M_{tG}\|_{L^2(\mathbb{R}^d)^J \to L^2(\mathbb{R}^d)^J} < \varepsilon, \quad \|M_{tG}(I - \Pi_V)\|_{L^2(\mathbb{R}^d)^J \to L^2(\mathbb{R}^d)^J} < \varepsilon.$$

Proof. Since $M(z)$ is uniformly bounded for $z \in \Omega(h)$ – see (2.6) – we will work with a fixed z and normalize $\|M(z)\|_{L^2 \to L^2}$ to be 1. That is done purely for notational convenience.

First, we can choose $\psi \in \mathcal{C}_c^\infty([0, 3K/4), [0, 1])$, $\psi\restriction_{[0,K/2]} \equiv 1$, so that

$$\Pi_V \psi(\tilde{h}Q/h) = \psi(\tilde{h}Q/h), \qquad \psi(\tilde{h}Q/h) \in \widetilde{\Psi}_{\frac{1}{2}},$$

and hence,

$$\|M_{tG}(I - \Pi_V)\|_{L^2 \to L^2} = \|M_{tG}(I - \psi(\tilde{h}Q/h))(I - \Pi_V)\|_{L^2 \to L^2}$$
$$\leq \|M_{tG}(I - \psi(\tilde{h}Q/h))\|_{L^2 \to L^2}.$$

All we need to prove then is (5.7) with Π_V replaced by the smooth cutoff $\psi(\tilde{h}Q/h)$, which now puts the problem in the setting of §3.4.

Before applying the weights, we split $M = MA_M + M(1 - A_M)$, using the cutoff A_M of Definition 2.1. We then apply Proposition 3.14 to the Fourier integral operator $T = MA_M$, with the function $a_0 \equiv 1$ and replacing G with $-tG$. We then get

$$M_{tG} = (MA_M)_{tG} + (M(I - A_M))_{tG} = MA_M \, e^{-t(F^*G)^w(x, hD)} \, e^{tG^w(x, hD)}$$
$$(5.8) \hspace{4cm} + M_1 \, h^{\frac{1}{2}(1-\delta)} \tilde{h}^{\frac{1}{2}} \widetilde{\Psi}_{\frac{1}{2}}(e^{t(G - F^*G)})$$
$$\hspace{4cm} + \mathcal{O}_{L^2 \to L^2}(h^{N_0 - 4C_6 t}),$$

where $M_1 \in I_{0+}(Y \times Y, F')$, $\mathrm{WF}_h(M_1) \subset F(\operatorname{supp} a_M) \times \operatorname{supp} a_M$.

The error term $\mathcal{O}_{L^2 \to L^2}(h^{N_0 - 4C_6 t})$ corresponds to $\|(M(I - A_M))_{tG}\|$, it comes from (2.4) together with the bound

$$(5.9) \qquad e^{\pm tG^w(x, hD)} = \mathcal{O}_{L^2 \to L^2}(h^{-tC_6(1 + \mathcal{O}(\tilde{h}))}) = \mathcal{O}_{L^2 \to L^2}(h^{-2tC_6}), \quad \tilde{h} \text{ small enough},$$

due to the first property in (4.18).

By contrast, the second line in (4.18) shows that $\exp(-t(F^*G - G)(\rho)) \leq e^{tC_7}$ for $\rho \in \mathcal{W}_3$. We have assumed that $\mathcal{W}_3 \ni \operatorname{supp} a_M$: this implies that for any $t \geq 0$, we have

$$\|A_M \, e^{-t(F^*G)^w} \, e^{tG^w}\|_{L^2 \to L^2}, \quad \|M_1 \mathrm{Op}_h^w(e^{t(G - F^*G)})\|_{L^2 \to L^2} \leq C \, e^{2C_7 t} \quad \text{uniformly in } h.$$

This is a crucial application of Proposition 3.5 and the properties of the escape function.

It remains to estimate the norm of

$$MA_M \, e^{-t(F^*G)^w} \, e^{tG^w} (I - \psi(\tilde{h}Q/h)) = M \, b_t(x, hD), \quad b_t \in \widetilde{S}_{\frac{1}{2}}(1),$$

$$b_t = a_M \, e^{-t(F^*G - G)} (1 - \psi(\tilde{h}q/h)) + \tilde{h} \, \widetilde{S}_{\frac{1}{2}}(1),$$

where q is as in (5.4) and a_M is the symbol of A_M.

Fixing $\varepsilon > 0$, we first choose t large enough so that

$$e^{-t/C_1} < \varepsilon/4$$

where C_1 is the constant appearing in (4.18).

At this point, we can select the constant $K > 1$ in the definition (5.5). In view of the estimates (5.4), we choose it large enough, so that

$$q(\rho) \geq \frac{K}{2}(h/\tilde{h}) \implies d(\rho, \mathcal{T})^2 \geq 2\,C_1\,(h/\tilde{h})\,,$$

(5.10) $$\rho \in \pi_R(F),\ q \circ F(\rho) \geq \frac{K}{2}(h/\tilde{h}) \implies d(\rho, \mathcal{T})^2 \geq 2\,C_1\,(h/\tilde{h})\,.$$

As a consequence, all points $\rho \in \mathrm{supp}(1 - \psi(\tilde{h}q/h))$ satisfy $d(\rho, \mathcal{T})^2 \geq 2\,C_1(h/\tilde{h})$, and therefore

$$\forall \rho \in \mathcal{W}_3, \quad e^{-t(F^*G-G)(\rho)}(1 - \psi(\tilde{h}q(\rho)/h)) \leq e^{-t/C_1} < \varepsilon/4\,.$$

It then follows, see Lemma 3.4, that

$$\|M\,b_t^w(x, hD)\|_{L^2 \to L^2} \leq \frac{\varepsilon}{2} + C(t)\tilde{h}\,.$$

Altogether, we have obtained

$$\|M_{tG}(I - \Pi_V)\|_{L^2 \to L^2} \leq \|Mb_t(x, hD)\|_{L^2 \to L^2} + \mathcal{O}(h^{\frac{1}{2}}\tilde{h}^{\frac{1}{2}}) + \mathcal{O}(h^{N_0 - 4C_6 t})$$
$$\leq \varepsilon/2 + \mathcal{O}_t(\tilde{h}) + \mathcal{O}(h^{\frac{1}{2}}\tilde{h}^{\frac{1}{2}}) + \mathcal{O}(h^{N_0 - 4C_6 t})\,.$$

The assumption $N_0 > 4C_6 t$ ensures that, once we take $\tilde{h} < \tilde{h}_0(t, \varepsilon)$ and take $h > 0$ small enough, the above right hand side is $< \varepsilon$.

A similar proof provides the estimate for $(I - \Pi_V)M_{tG}$, replacing Π_V by $\psi(\tilde{h}Q/h)$, using Proposition 3.14 to bring $\psi(\tilde{h}Q/h)$ to the right of M, and using the assumption (5.10) to bound from above $(1 - \psi(\tilde{h}q \circ F/h))e^{-t(F^*G-G)}$. $\qquad\square$

The invertibility of the Grushin problem is now a matter of linear algebra:

Theorem 2. *Suppose that $M = M(z)$ is a hyperbolic quantum monodromy operator in the sense of Definition 2.1, and M_{tG} the conjugated operator (5.2). We use the auxiliary operators of (5.6) to define the Grushin problem*

(5.11) $$\mathcal{M}_{tG} \overset{\mathrm{def}}{=} \begin{pmatrix} I - M_{tG} & R_- \\ R_+ & 0 \end{pmatrix}\ :\ L^2(\mathbb{R}^d)^J \oplus V \longrightarrow L^2(\mathbb{R}^d)^J \oplus V\,,$$

If t is large enough, $\tilde{h}$ is small enough, and N_0 from (2.4), C_6 from (5.9) satisfy $N_0 > 4C_6 t$, then for h small enough the above Grushin problem is invertible. Its inverse, $\mathcal{E}_{tG}$, is

uniformly bounded as $h \to 0$, and the effective Hamiltonian reads

$$(5.12) \qquad E_{-+} = -(I_V - \Pi_V M_{tG}) + \sum_{k=1}^{\infty} \Pi_V M_{tG}[(I - \Pi_V)M_{tG}]^k \; : \; V \longrightarrow V \,,$$

where the convergence of the series is guaranteed by (5.7).

Proof. We first construct an approximate inverse,

$$\mathcal{E}_{tG}^0 \overset{\text{def}}{=} \begin{pmatrix} I - \Pi_V & R_- \\ \Pi_V & -(I - \Pi_V M_{tG}) \end{pmatrix} ,$$

for which we check that

$$(5.13) \qquad \mathcal{M}_{tG}\,\mathcal{E}_{tG}^0 = I_{L^2(\mathbb{R}^d)^J \oplus V} - \begin{pmatrix} M_{tG}(I - \Pi_V) & (I - \Pi_V)M_{tG} \\ 0 & 0 \end{pmatrix} .$$

The theorem now follows from Lemma 5.3 and the Neumann series inversion:

$$\mathcal{M}_{tG}\,\mathcal{E}_{tG} = I_{L^2(\mathbb{R}^d)^J \oplus V} \,,$$

$$\mathcal{E}_{tG} = \mathcal{E}_{tG}^0 \begin{pmatrix} I + R & (I + R)(I - \Pi_V)M_{tG} \\ 0 & I_V \end{pmatrix} , \quad R \overset{\text{def}}{=} \sum_{k=1}^{\infty}[M_{tG}(I - \Pi_V)]^k \,.$$

We finally show that the operator $\Pi_V M_{tG}$, and thus the whole inverse $\mathcal{E}_{tG}$, is bounded uniformly in h. Consider a cutoff $\psi_1 \in \mathcal{C}_c^\infty(T^*Y)$ supported inside a small neighbourhood of $\mathcal{T}$, equal to unity in a smaller neighbourhood of $\mathcal{T}$. In particular, using the notations of Proposition 4.6 we assume that $\operatorname{supp} F^*\psi_1 \Subset \mathcal{W}_2$. Since Π_V is microlocalized in a semiclassically thin neighbourhood of $\mathcal{T}$, we have

$$\Pi_V \psi_1^w(x, hD) = \Pi_V + \mathcal{O}_{L^2 \to L^2}(h^\infty) \,.$$

We are then led to estimate the norm of the operator $\psi_1^w M_{tG}$. Using the decomposition (5.8) and the fact that $(F^*G - G)(\rho) \geq -C_7$ for $\rho \in \mathcal{W}_2$, we obtain for $\tilde{h}$, h small enough:

$$(5.14) \qquad \forall t > 0, \quad \|\psi_1^w M_{tG}\|_{L^2 \to L^2} \leq e^{3C_7 t} \implies \|\Pi_V M_{tG}\|_{L^2 \to L^2} \leq C' e^{3C_7 t} \,.$$

$\square$

This achieves the reduction of the monodromy operator $M(z, h)$ to the finite rank operator $E_{-+}(z)$. In the next section we perform the same task, starting from a monodromy operator $\widetilde{M}(z, h)$ which is already of finite rank.

5.3. **Quantum monodromy operators acting on finite dimensional spaces.** So far we have been considering quantum maps or monodromy operators given by smooth h-Fourier integral operators of infinite rank. The quantum monodromy operator constructed in [28] and providing an effective Hamiltonian for operators $(i/h)P - z$, $z \in D(0, R)$, was

given by the restriction of such a Fourier integral operator to a finite dimensional space $W \subset L^2(\mathbb{R}^d)^J$ microlocalized on some bounded neighbourhood of the trapped set.

$$
(5.15) \qquad
\begin{aligned}
W \subset L^2(\mathbb{R}^d)^J, \quad \dim W < \infty, \quad \Pi_W : L^2(\mathbb{R}^d)^J \xrightarrow{\perp} W, \\
M_W = \Pi_W M \Pi_W + \mathcal{O}_{W \to W}(h^{N_0}),
\end{aligned}
$$

where $M \in I_{0+}(Y \times Y, F')$ is a smooth Fourier integral operator. Compared with the notations in Definition 2.1, we have $\Pi_W = \Pi_h$ and $M_W = \widetilde{M} + \mathcal{O}_{W \to W}(h^{N_0})$.

Let us now consider the Grushin problem for $(I_W - M_W)$.

Theorem 3. *Suppose that M_W is given by (5.15), with M a monodromy operator as defined in Definition 2.1. The space V, and the auxiliary operators $R_\pm$ are as in Theorem 2. We construct a weight G as in Prop. 4.6, and such that $\Pi_W \equiv I$ near $\operatorname{supp} G$.*

If t is large enough, $\tilde{h}$ is small enough, and N_0 satisfies $N_0 > 4C_6 t$, with C_6 from (5.9), then the operator

$$
\widetilde{\mathcal{M}}_{tG} \overset{\text{def}}{=}
\begin{pmatrix}
I_W - M_W & \Pi_W e^{tG^w(x,hD)} R_- \\
R_+ e^{-tG^w(x,hD)} \Pi_W & 0
\end{pmatrix}
: W \oplus V \longrightarrow W \oplus V,
$$

is invertible, with the inverse

$$
\widetilde{\mathcal{E}}_{tG} =
\begin{pmatrix}
\widetilde{E} & \widetilde{E}_+ \\
\widetilde{E}_- & \widetilde{E}_{-+}
\end{pmatrix}
= \mathcal{O}(h^{-4tC_6}) \; : \; W \oplus V \longrightarrow W \oplus V.
$$

Furthermore, the effective Hamiltonian is uniformly bounded:

$$
\|\widetilde{E}_{-+}\|_{V \to V} = \mathcal{O}_\varepsilon(1).
$$

Proof. We start by proving three estimates showing that the projector Π_W does not interfere too much with the Grushin problem. Firstly, in the Definition 2.1 we have assumed that Π_W is equal to the identity, microlocally near the support of a_M: this has for consequence the estimate

$$
(5.16) \qquad A_M = \Pi_W A_M + \mathcal{O}_{L^2 \to L^2}(h^\infty) = A_M \Pi_W + \mathcal{O}_{L^2 \to L^2}(h^\infty).
$$

Secondly, let us notice that in Proposition 4.6 we required the weight to satisfy $\operatorname{supp} a_M \Subset \mathcal{W}_3 \Subset \operatorname{supp} G$. Since we also know that $\Pi_W \equiv I$ some neighbourhood $\mathcal{W}_4$ of $\operatorname{supp} a_M$, it is indeed possible construct the weight G such that $\mathcal{W}_3 \Subset \operatorname{supp} G \Subset \mathcal{W}_4$. This has for consequence the estimate

$$
(5.17) \qquad \forall t \geq 0, \qquad e^{-tG^w(x,hD)} \Pi_W e^{tG^w(x,hD)} = \Pi_W + \mathcal{O}_{L^2 \to L^2}(h^\infty).
$$

(the fact that we are dealing with $\widetilde{S}_{\frac{1}{2}}$ symbol classes does not affect the result, see for instance [55, Theorem 4.24]). Thirdly, the definition of the subspace V in (5.5) shows that

the projector Π_V is microlocalized inside an $h^{1/2}$ neighbourhood of $\mathcal{T}$, while $\Pi_W \equiv I$ in a fixed neighbourhood. This induces the estimate

$$(5.18) \qquad \Pi_V \Pi_W = \Pi_V + \mathcal{O}_{L^2 \to L^2}(h^\infty).$$

We now want to solve the problem

$$\widetilde{\mathcal{M}}_{tG} \begin{pmatrix} u \\ u_- \end{pmatrix} = \begin{pmatrix} v \\ v_+ \end{pmatrix}.$$

We first consider the approximate solution

$$\begin{pmatrix} u^0 \\ u^0_- \end{pmatrix} = \mathcal{E}^0_{tG} \begin{pmatrix} v \\ v_+ \end{pmatrix}, \qquad \mathcal{E}^0_{tG} \stackrel{\text{def}}{=} \begin{pmatrix} e^{tG^w} & 0 \\ 0 & I_V \end{pmatrix} \mathcal{E}_{tG} \begin{pmatrix} e^{-tG^w} & 0 \\ 0 & I_V \end{pmatrix},$$

where $\mathcal{E}_{tG}$ is the inverse of the Grushin problem in Theorem 2. In particular,

$$(I - M)u^0 + e^{tG^w(x,hD)} R_- u^0_- = v.$$

The estimate (5.16) implies that

$$\Pi_W M \Pi_W = M + \mathcal{O}_{L^2 \to L^2}(h^{N_0}),$$

and hence ($v = \Pi_W v$ as v is assumed to be in W),

$$(5.19) \qquad (I_W - M_W)\Pi_W u^0 + \Pi_W e^{tG^w(x,hD)} u^0_- = v + \mathcal{O}_W(h^{N_0} \|u^0\|_{L^2}).$$

Since $R_+ e^{-tG^w(x,hD)} u^0 = v_+$, the definition of $R_+ = \Pi_V$ and the estimates (5.17), (5.18) imply that

$$(5.20) \qquad \begin{aligned} R_+ e^{-tG^w(x,hD)} \Pi_W u^0 &= \Pi_V \left(e^{-tG^w(x,hD)} \Pi_W e^{tG^w(x,hD)} \right) e^{-tG^w(x,hD)} u^0 \\ &= v_+ + \mathcal{O}_V(h^{N_0} \|e^{-tG^w(x,hD)} u^0\|_{L^2}). \end{aligned}$$

Since $\mathcal{E}_{tG}$ is uniformly bounded, we obtain the bound

$$\begin{aligned} \|e^{-tG^w} u^0\|_{L^2} &\leq \|\mathcal{E}_{tG}\| \left(\|e^{-tG^w} v\| + \|v_+\| \right) \\ &\leq C \left(h^{-2C_6 t} \|v\|_W + \|v_+\|_V \right) \\ \Longrightarrow \|u^0\|_{L^2} &\leq C \|e^{tG^w}\|_{L^2 \to L^2} \|e^{-tG^w} u^0\|_{L^2} \\ &\leq C(h^{-4C_6 t} \|v\|_W + h^{-2C_6 t} \|v_+\|_V). \end{aligned}$$

This bound, together with (5.19) and (5.20), gives

$$\widetilde{\mathcal{M}}_{tG} \begin{pmatrix} \Pi_W u^0 \\ u^0_- \end{pmatrix} = \begin{pmatrix} I_W + \mathcal{O}(h^{N_0 - 4C_6 t}) & \mathcal{O}(h^{N_0 - 2C_6 t}) \\ \mathcal{O}(h^{N_0 - 2C_6 t}) & I_V + \mathcal{O}(h^{N_0}) \end{pmatrix} \begin{pmatrix} v \\ v_+ \end{pmatrix}.$$

The assumption $N_0 > 4C_6 t$ implies that the operator on the right hand side can be inverted for h small enough, and proves the existence of

$$\widetilde{\mathcal{E}}_{tG} = \begin{pmatrix} \Pi_W & 0 \\ 0 & I_V \end{pmatrix} \mathcal{E}^0_{tG} \begin{pmatrix} I_W + \mathcal{O}(h^{N_0 - 4C_6 t}) & \mathcal{O}(h^{N_0 - 2C_6 t}) \\ \mathcal{O}(h^{N_0 - 2C_6 t}) & I_V + \mathcal{O}(h^{N_0}) \end{pmatrix}.$$

From this expression, we deduce the estimate for $\|\widetilde{\mathcal{E}}_{tG}\|$, as well as the uniform boundedness of

$$\text{(5.21)} \qquad \widetilde{E}_{-+} = E_{-+} + \mathcal{O}_{V \to V}(h^{N_0 - 4C_6 t})\,.$$

$\square$

Remarks. 1) The projector Π_W is typically obtained by taking $\Pi_W = \text{diag}(\Pi_{W,j})$, $\Pi_{W,j} = \mathbb{1}_{Q_{0,j} \leq C}$ where $Q_{0,j} = \text{Op}_h^w(q_{0,j}) \in \Psi(\mathbb{R}^d)$ is elliptic. This way, Π_W is a microlocal projector associated with the compact region

$$\mathcal{W}_5 \overset{\text{def}}{=} \sqcup_{j=1}^J \{\rho \in T^*\mathbb{R}^d \ : \ q_{0,j}(\rho) \leq C\}\,.$$

Then the estimates (5.16), (5.17) hold if $\text{supp}\, a_M \Subset \text{supp}\, G \Subset \mathcal{W}_5$.

2) The requirement that $N_0 > 4C_6 t$, where the constant C_6 depends on G, and $t = t(\varepsilon)$ has to be chosen large enough, seems awkward in the abstract setting (N_0 is the power appearing in (2.4)). In practice, when constructing the monodromy operator M we can take N_0 arbitrary large, independently of the weight G (see [28], or the application presented in §6).

5.4. **Upper bounds on the number of resonances.** Let us first recall the definition of the *box* or *Minkowski* dimension of a compact subset $\Gamma \Subset \mathbb{R}^k$:

$$\text{(5.22)} \qquad \dim_M \Gamma = 2\mu_0 \overset{\text{def}}{=} k - \sup\{\gamma \ : \ \limsup_{\epsilon \to 0} \epsilon^{-\gamma} \text{vol}(\{\rho \in \mathbb{R}^k \ : \ d(\rho, \Gamma) < \epsilon\}) < \infty\}\,.$$

The set Γ is said to be of pure dimension if

$$\limsup_{\epsilon \to 0} \epsilon^{-k+2\mu_0} \text{vol}(\{\rho \in \mathbb{R}^k \ : \ d(\rho, \Gamma) < \epsilon\}) < \infty\,.$$

In other words, for ϵ small

$$\text{vol}(\{\rho \in \mathbb{R}^k \ : \ d(\rho, \Gamma) < \epsilon\}) \leq C\epsilon^{k-2\mu}\,, \quad \mu > \mu_0\,,$$

with μ replaceable by μ_0 when Γ is of pure dimension.

In the case of a compact set $\Gamma = \sqcup \Gamma_j \Subset \sqcup_{j=1}^J \mathbb{R}^k$, its Minkowski dimension is simply

$$\text{(5.23)} \qquad \dim_M \Gamma = \max_{j=1,\dots,J} \dim_M \Gamma_j\,.$$

The following lemma expresses the intuitive idea that a domain in $T^*\mathbb{R}^d$ of symplectic volume v can support at most $h^{-d}v$ quantum states.

Lemma 5.4. *Let Γ be a compact subset of $T^*\mathbb{R}^d$, of Minkowski dimension $2\mu_0$. Using the operator Q constructed in Proposition 5.2, we take $K \gg 1$ and define the subspace*

$$V \overset{\text{def}}{=} \mathbb{1}_{Q \leq Kh/\tilde{h}} L^2(\mathbb{R}^d)\,.$$

Then, for any $\mu > \mu_0$ there exists $C = C_\mu$, such that

$$(5.24) \qquad\qquad \dim V \le C\tilde{h}^{-d}\left(\frac{\tilde{h}}{h}\right)^{\mu}.$$

When Γ is of pure dimension we can take $\mu = \mu_0$ in (5.24).

Proof. Since the order function $m(\rho) \to \infty$ as $|\rho| \to \infty$ and $Q \in \widetilde{\Psi}_{\frac{1}{2}}(m)$, the selfadjoint operator Q has a discrete spectrum, hence V is finite dimensional, and

$$\dim V = \#\{\lambda \le Kh/\tilde{h} \ : \ \lambda \in \mathrm{Spec}(Q)\}.$$

The usual min-max arguments — see for instance [37] or [55, Theorem C.11] — show that $\dim V \le N$ if there exists $\delta > 0$ and an operator A of rank less than or equal to N, such that

$$(5.25) \qquad \langle Qu, u\rangle + \mathrm{Re}\langle Au, u\rangle \ge (Kh/\tilde{h} + \delta)\|u\|^2, \qquad u \in \mathcal{C}_c^\infty(\mathbb{R}^d).$$

To construct A, take $a = \psi(\tilde{h}q/h)$, where $\psi \in \mathcal{C}_c^\infty(\mathbb{R})$, $\psi(t) \equiv 1$ for $|t| \le 3K$ and $\psi(t) = 0$ for $|t| \ge 4K$. At the symbolic level, $a \in \widetilde{S}_{\frac{1}{2}}$ and $a(\rho) = 1$ in the region where $q(\rho) \le 3Kh/\tilde{h}$. Taking into account the fact that $q \ge h/\tilde{h}$ everywhere, we have

$$q(\rho) + 2Kha(\rho)/\tilde{h} \ge (2K+1)h/\tilde{h}, \qquad \rho \in T^*\mathbb{R}^d.$$

The arguments presented in the proof of Proposition 5.2 show that, at the operator level, we have for $\tilde{h}$ small enough

$$(5.26) \qquad Q + A_0 \ge 2Kh/\tilde{h}, \qquad \text{for} \quad A_0 \overset{\text{def}}{=} 2Kha^w(x, hD)/\tilde{h}.$$

This inequality obviously implies (5.25), with A replaced by the (selfadjoint) operator A_0. Our task is thus to replace A_0 in (5.25) by a finite rank operator. We do that as in [43, Proposition 5.10], by using a *locally finite* open covering of a neighbourhood of $\mathcal{T}$:

$$W_{h,\tilde{h}} \overset{\text{def}}{=} \{\rho \ : \ d(\rho, \Gamma)^2 \le 4Kh/\tilde{h}\} \subset \bigcup_{k=1}^{N(h,\tilde{h})} U_k, \quad \mathrm{diam}\,(U_k) \le (h/\tilde{h})^{\frac{1}{2}}.$$

The definition of the box dimension implies that we can choose this covering to be of cardinality

$$(5.27) \qquad\qquad N(h, \tilde{h}) \le C_{K,\mu}(\tilde{h}/h)^\mu,$$

for any $\mu > \mu_0$, and for $\mu = \mu_0$ if Γ is of pure dimension.

To the cover $\{U_k\}$, we associate a partition of unity on $W_{h,\tilde{h}}$,

$$\sum_{k=1}^{N(h,\tilde{h})} \chi_k = 1 \quad \text{on } W_{h,\tilde{h}}, \quad \mathrm{supp}\,\chi_k \subset U_k, \quad \chi_k \in \widetilde{S}_{\frac{1}{2}},$$

where all seminorms are assumed to be uniform with respect to k. The condition on the support of ψ in the definition of a and the pseudodifferential calculus in Lemma 3.2 show that

$$(5.28) \qquad \big(I - \sum_{k=1}^{N(h,\tilde{h})} \chi_k^w(x, hD)\big) a^w(x, hD) \in \tilde{h}^\infty \widetilde{S}_{\frac{1}{2}} \,.$$

Hence it suffices to show that for each $k = 1, \ldots, N(h, \tilde{h})$, there exists an operator R_k such that

$$\chi_k^w(x, hD) a^w(x, hD) - R_k \in \tilde{h}^\infty \widetilde{\Psi}_{\frac{1}{2}} \,, \quad \operatorname{rank}(R_k) \leq C' \tilde{h}^{-d} \,,$$

with C' and the implied constants independent of k. We can assume that, for some $\rho^k = (x^k, \xi^k) \in T^* \mathbb{R}^d$,

$$U_k \subset B_{\mathbb{R}^{2d}}(\rho^k, (h/\tilde{h})^{\frac{1}{2}}) \,.$$

Then consider the harmonic oscillator shifted to the point ρ^k:

$$H^k \stackrel{\text{def}}{=} \sum_{i=1}^{d} (hD_{x_i} - \xi_i^k)^2 + (x_i - x_i^k)^2 \,.$$

If $\psi_0 \in \mathcal{C}_c^\infty(\mathbb{R})$, $\psi_0(t) = 1$ for $|t| \leq 2$, $\psi_0(t) = 0$ for $|t| \geq 3$, then $\psi_0(\tilde{h}H^k/h) \in \widetilde{\Psi}_{\frac{1}{2}}$, and

$$(5.29) \quad \psi_0(\tilde{h}H^k/h) \chi_k^w(x, hD) a^w(x, hD) - \chi_k^w(x, hD) a^w(x, hD) = \mathcal{R}_k, \qquad \mathcal{R}_k \in \tilde{h}^\infty \widetilde{\Psi}_{\frac{1}{2}} \,,$$

where the implied constants are uniform with respect to k. The properties of the harmonic oscillator (see for instance [55, §6.1]) show that $\psi_0(\tilde{h}H^k/h)$ is a finite rank operator, with rank bounded by $C_d \tilde{h}^{-d}$. Hence for each k we can take

$$R_k \stackrel{\text{def}}{=} \psi_0(\tilde{h}H^k/h) \chi_k^w(x, hD) a^w(x, hD) \,,$$

and define

$$(5.30) \qquad A \stackrel{\text{def}}{=} 2K(h/\tilde{h}) \sum_{k=1}^{N(h,\tilde{h})} R_k \,, \qquad \operatorname{rank}(A) \leq C_d \tilde{h}^{-d} N(h, \tilde{h}) \,.$$

The remainder operators $\mathcal{R}_k$ in (5.29) satisfy, for any $M > 0$,

$$\|\mathcal{R}_k^* \mathcal{R}_{k'}\|_{L^2 \to L^2}, \quad \|\mathcal{R}_k \mathcal{R}_{k'}^*\|_{L^2 \to L^2} \leq C_M \tilde{h}^M \left\langle \frac{d(\rho^k, \rho^{k'})}{\tilde{h}} \right\rangle^{-M}$$

$$\text{uniformly for } k, k' = 1, \ldots, N(h, \tilde{h}) \,.$$

Since the supports of the χ_k's form a locally finite partition, each remainder $\mathcal{R}_k$ effectively interferes with only finitely many other remainders. One can then invoke the Cotlar-Stein

lemma (see [6, Lemma 7.10]) to show that

$$\| \sum_{k=1}^{N(h,\tilde{h})} \mathcal{R}_k \|_{L^2 \to L^2} = \mathcal{O}(\tilde{h}^\infty) \,.$$

Using also (5.28), we obtain

$$A = A_0 + \mathcal{O}_{L^2 \to L^2}(h\tilde{h}^\infty) \,.$$

Consequently, for $\tilde{h}$ small enough the estimate (5.25) holds with $\delta = h/\tilde{h}$. In view of (5.27) and (5.30) the bound we have obtained on the rank of A leads to (5.24). $\qquad\square$

We now consider a monodromy operator as defined in Definition 2.1:

$$(5.31) \qquad\qquad \Omega(h) \ni z \longmapsto M(z,h) \in I^0_+(Y \times Y, F') \,,$$

where the depends on z is holomorphic.

The decay assumption (2.6) implies that there exists $R_0 > 0$ such that, for h small enough,

$$(5.32) \qquad\qquad z \in \Omega(h), \ \operatorname{Re} z \le -R_0 \implies \|M(z,h)\|_{L^2 \to L^2} \le 1/2 \,.$$

Using the analytic Fredholm theory (see for instance [42, §2]), this implies that $(I - M(z,h))^{-1}$ is meromorphic in $\Omega(h)$ with poles of finite rank. The multiplicities of the poles are defined by the usual formula:

$$(5.33) \qquad m_M(z) \stackrel{\text{def}}{=} \inf_{\epsilon > 0} \frac{1}{2\pi i} \operatorname{tr} \oint_{\gamma_\epsilon(z)} (I - M(\zeta))^{-1} \partial_\zeta M(\zeta) d\zeta \,,$$

$$\gamma_\epsilon(z) : t \mapsto z + \epsilon e^{2\pi i t}, \quad t \in [0, 2\pi) \,,$$

see Lemma 6.2 below for the standard justification of taking the trace.

Theorem 4. *Suppose that $\{M(z,h), z \in \Omega(h)\}$, is a hyperbolic quantum monodromy operator, or its truncated version $\widetilde{M}(z,h)$, in the sense of Definition 2.1, and that $\mathcal{T}$ is the trapped set for the associated open relation F. Let $2\mu_0$ be the Minkowski dimension of $\mathcal{T}$, as defined by (5.22,5.23), with $k = 2d$ and $\Gamma = \mathcal{T}$.*

Then for any $R_1 > 0$ and any $\mu > \mu_0$, there exists K_{μ,R_0}, such that

$$(5.34) \qquad\qquad \sum_{z \in D(0,R_1)} m_M(z) \le K_{\mu,R_1} h^{-\mu}, \qquad h \to 0 \,.$$

When $\mathcal{T}$ is of pure dimension we can take $\mu = \mu_0$ in the above equation.

Proof. Let us treat the case of the untruncated monodromy operator $M(z,h)$, the case of the truncated one being similar. We apply Theorem 2 to the family $M(z,h)$. Since

the construction of the Grushin problem (5.11) depends only on the relation F and the estimates (2.4), we see that for any radius $R > 0$, the Grushin problem

(5.35)
$$\mathcal{M}_{tG}(z) \stackrel{\text{def}}{=} \begin{pmatrix} I - M_{tG}(z) & R_- \\ R_+ & 0 \end{pmatrix} : L^2(\mathbb{R}^d)^J \oplus V \longrightarrow L^2(\mathbb{R}^d)^J \oplus V,$$

$$M_{tG}(z) \stackrel{\text{def}}{=} e^{-tG^w(x,hD)} M(z) e^{tG^w(x,hD)},$$

is invertible for $t = t(\varepsilon) > 0$ large enough, and the inverse $\mathcal{E}_{tG}(z)$ is holomorphic in $z \in D(0, R)$, uniformly when $h \to 0$. Using the standard result (see [42, Proposition 4.1] for that, and [42, Proposition 4.2] for a generalization not requiring holomorphy) we obtain

$$m_M(z) = \inf_{\epsilon > 0} \frac{1}{2\pi i} \operatorname{tr} \oint_{\gamma_\epsilon(z)} (I - M_{tG}(\zeta))^{-1} \partial_\zeta M_{tG}(\zeta) d\zeta$$

$$= \inf_{\epsilon > 0} \frac{1}{2\pi i} \operatorname{tr} \oint_{\gamma_\epsilon(z)} E_{-+}(\zeta)^{-1} \partial_\zeta E_{-+}(\zeta) d\zeta.$$

Since $E_{-+}(\zeta)$ is a matrix with holomorphic coefficients, the right hand side is the multiplicity of the zero of $\det E_{-+}(\zeta)$ at z.

Once $0 < \varepsilon < 1/2$ and the parameter $t = t(\varepsilon)$ has been selected, the decay assumption (2.6), together with the norm estimate (5.14), show that there exists a radius $R = R(t) > 0$ such that, for $h < h_0$,

$$z \in \Omega(h), \ \operatorname{Re} z \le -R/4 \implies \|\Pi_V M_{tG}(z, h)\|_{L^2 \to L^2} \le 1/2.$$

Together with the expression (5.12), the bound (5.7) and the assumption $\varepsilon < 1/2$, this shows that $E_{-+}(-R/4)$ is invertible, with $\|E_{-+}(-R/4)^{-1}\|$ uniformly bounded.

We may assume that $R \ge 4R_1$, where R_1 is the radius in the statement of the theorem. The bound (5.34) then follows from estimating the number of those zeros in $D(0, R/4)$. That in turn follows from Jensen's formula, which has a long tradition in estimating the number of resonances (see [24]). Namely, for any function $f(z)$ holomorphic in $z \in D(0, R)$ and nonvanishing at 0, the number $n(r)$ of its zeros in $D(0, r)$ (counted with multiplicity) satisfies, for any $r < R$,

$$\int_0^r \frac{n(x)}{x} dx = \frac{1}{2\pi} \int_0^{2\pi} \log|f(re^{i\theta})| d\theta - \log|f(0)|.$$

Applying this identity to the function $f(\zeta) = \det E_{-+}(\zeta - R/4)$, we see that

$$\sum_{z \in D(0,R/4)} m_M(z) \le n(R/2) \le \frac{1}{\log(3/2)} \int_{R/2}^{3R/4} \frac{n(x)}{x} dx \le \frac{1}{\log(3/2)} \int_0^{3R/4} \frac{n(x)}{x} dx$$

$$\le \frac{1}{\log(3/2)} \left(\max_{z \in D(0,R)} \log|\det E_{-+}(z)| - \log|\det E_{-+}(-R/4)| \right).$$

Since $\|E_{-+}(z)\|_{V \to V}$ is uniformly bounded for $z \in D(0, R)$ and the rank of $E_{-+}(z)$ is bounded by $\dim V$, Lemma 5.4 gives

$$\max_{z \in D(0,R)} \log |\det E_{-+}(z)| \leq C_0 \dim V \leq K h^{-\mu},$$

where μ is as in (5.34). Also,

$$-\log |\det E_{-+}(-R/4)| = \log |\det E_{-+}(-R/4)^{-1}| \leq \dim V \, \log \|E_{-+}(-R/4)^{-1}\|$$
$$\leq K h^{-\mu},$$

where in the last inequality we used the fact that $E_{-+}(-R/4)^{-1}$ is uniformly bounded. This completes the proof of (5.34) in the case of an untruncated monodromy operator.

In the case of a truncated operator $\widetilde{M}(z, h) = M_W(z, h) + \mathcal{O}(h^{N_0})$, we apply Theorem 3 and get an effective Hamiltonian $\widetilde{E}_{-+}(z)$ which is also uniformly bounded. The estimate (5.21) provides a uniform estimate for $\widetilde{E}_{-+}(-R/4)^{-1}$, and the rest of the proof is identical. $\qquad \square$

6. APPLICATION TO SCATTERING BY SEVERAL CONVEX BODIES

We now apply the abstract formalism to a very concrete setting of several convex obstacles. This will prove Theorem 1 stated in §1.

The setting of several convex obstacles has been a very popular testing ground for quantum chaos since the work of Gaspard and Rice [10] but the fractal nature of the distribution of resonances has been missed by the physics community. The optimality of the fractal Weyl laws in that setting was tested numerically using semiclassical zeta function (hence not in the true quantum régime) in [22] – see Fig. 1. The mathematical developments of other aspects of scattering by several convex obstacles can be found in the works of Ikawa, Gérard, Petkov, Stoyanov, and Burq – see [2],[11],[17],[33], and references given there.

6.1. **Resonances for obstacles with several connected components.** We first present some general aspects of scattering by several obstacles. This generalizes and simplifies the presentation of [11, §6].

Let $\mathcal{O}_j \Subset \mathbb{R}^n$, $j = 1, \cdots, J$, be a collection of connected open sets, $\overline{\mathcal{O}}_k \cap \overline{\mathcal{O}}_j = \emptyset$, with smooth boundaries, $\partial \mathcal{O}_j$. Let

$$\Omega \overset{\text{def}}{=} \mathbb{R}^n \setminus \bigcup_{j=1}^{J} \mathcal{O}_j,$$

and let γ be a natural restriction map:

$$\gamma \, : \, H^2(\Omega) \longrightarrow H^{\frac{3}{2}}(\partial \Omega), \quad (\gamma \, u)_j \overset{\text{def}}{=} \gamma_j u \overset{\text{def}}{=} u\!\restriction_{\partial \mathcal{O}_j},$$

where we interpret γ as a column vector of operators.

Let $\Delta_\theta \; : \; H^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ be the complex-scaled Laplacian, in the sense of [40, §3],

$$\Delta_\theta = \left(\sum_{k=1}^{n} \partial_{z_k}^2 \right)\!\upharpoonright_{\Gamma_\theta}, \quad z \in \mathbb{C}^n, \quad \Gamma_\theta \simeq \mathbb{R}^n,$$

with $\Gamma_\theta \cap B_{\mathbb{C}^n}(0,R) = \mathbb{R}^n \cap B_{\mathbb{R}^n}(0,R)$, $\mathcal{O}_j \Subset B_{\mathbb{R}^n}(0,R)$, for all j. Here we identified the functions on $\mathbb{R}^n$ and functions on Γ_θ.

For $z \in D(0,r)$ we define a semiclassical differential operator

$$(6.1) \qquad\qquad\qquad P(z) \stackrel{\text{def}}{=} \frac{i}{h}(-h^2\Delta_\theta - 1) - z,$$

with the domain given by either $H^2(\mathbb{R}^n)$ or $H^2(\Omega) \cap H_0^1(\Omega)$, $P(z)$ is a Fredholm operator and we have two corresponding resolvents:

$$R_0(z) \; : \; L^2(\mathbb{R}^n) \; \longrightarrow \; H^2(\mathbb{R}^n), \quad R(z) \; : \; L^2(\Omega) \; \longrightarrow \; H^2(\Omega) \cap H_0^1(\Omega).$$

Here and below $r > 0$ can be taken arbitrary and fixed as long as h is small enough.

The operator $R_0(z)$ is analytic in $z \in D(0,r)$ (the only problem comes from $i/h + z = 0$) and $R(z)$ is meromorphic with singular terms of finite rank – see [40, Lemma 3.5] and, for a concise discussion from the point of view of boundary layer potentials, [24]. The multiplicity of a pole of $R(z)$ is defined by

$$(6.2) \qquad m_R(z_0) = -\frac{1}{2\pi i} \operatorname{tr}_{L^2(\Omega)} \oint_{\gamma_\epsilon(z)} R(z)dz, \quad \gamma_\epsilon(z_0) : t \mapsto z_0 + \epsilon e^{2\pi i t}, \quad t \in [0, 2\pi),$$

and $\epsilon > 0$ is sufficiently small.

A direct proof of the meromorphic continuation and a reduction to the boundary uses Poisson operators associated to individual obstacles $\mathcal{O}_j$:

$$H_j(z) : H^{\frac{3}{2}}(\partial\mathcal{O}_j) \; \longrightarrow \; H^2(\mathbb{R}^n \setminus \mathcal{O}_j) \stackrel{\text{extension by } 0}{\longrightarrow} L^2(\mathbb{R}^n),$$
$$\left(P(z)H_j(z)f \right)(x) = 0, \quad x \in \mathbb{R}^n \setminus \mathcal{O}_j, \quad \gamma_j H_j(z)f = f.$$

We then define a row vector of operators:

$$(6.3) \qquad\qquad H(z) : H^{\frac{3}{2}}(\partial\Omega) \; \longrightarrow \; L^2(\mathbb{R}^n), \quad H(z)\vec{v} = \sum_{j=1}^{J} H_j(z)v_j,$$

We note that $\gamma H(z)\vec{v} \in H^{3/2}(\partial\Omega)$ is well defined.

The family of operators, $H(z)$, is in general meromorphic with poles of finite rank – see the proof of Lemma 6.1 below.

In this notation the monodromy operator $\mathcal{M}(z)$ defined in (1.5) is simply given by

$$I - \mathcal{M}(z) \stackrel{\text{def}}{=} \gamma H(z) : H^{\frac{3}{2}}(\partial\Omega) \longrightarrow H^{\frac{3}{2}}(\partial\Omega),$$

(6.4)
$$(\mathcal{M}(z))_{ij} = \begin{cases} -\gamma_i H_j(z) & i \neq j, \\ 0 & i = j. \end{cases}$$

We first state the following general lemma:

Lemma 6.1. *For $z \in \mathbb{R} + i(-1/h, 1/h)$, the operator*

$$(I - \mathcal{M}(z))^{-1} : H_h^{\frac{3}{2}}(\partial\Omega) \to H_h^{\frac{3}{2}}(\partial\Omega)$$

is meromorphic with poles of finite rank. For $\operatorname{Re} z < 0$

$$\|\mathcal{M}(z)\| \leq \frac{C}{h|\operatorname{Re} z|},$$

and consequently for $\operatorname{Re} z < -\gamma/h$, with γ sufficiently large, $(I - \mathcal{M}(z))^{-1}$ is holomorphic and

(6.5)
$$\|(I - \mathcal{M}(z))^{-1}\|_{H^{\frac{3}{2}}(\partial\Omega) \to H^{\frac{3}{2}}(\partial\Omega)} \leq C, \quad \operatorname{Re} z < -\gamma/h.$$

Proof. We first consider $\operatorname{Re} z < 0$. Let $R_j(z)$ be the resolvent of the Dirichlet realization of $P(z)$ on $\mathbb{R}^n \setminus \mathcal{O}_j$.

Let $\chi(x) \in \mathcal{C}_c^\infty(\text{neigh}(\partial\mathcal{O}_j)$ be equal to 1 near $\partial\mathcal{O}_j$ and have support in a small neighbourhood of $\partial\mathcal{O}_j$. In particular we can assume that the (signed) distance, $d(\bullet, \partial\mathcal{O}_j)$ is smooth there. Define the extension operator, $\gamma_j T_j^h = I$,

$$T_j^h \stackrel{\text{def}}{=} \chi(x) \exp\left(-d(x, \partial\mathcal{O}_j)^2(I - h^2\Delta_{\partial\mathcal{O}_j})/h^2\right)$$

$$T_j^h = \mathcal{O}(h^{\frac{1}{2}}) : H_h^{\frac{3}{2}}(\partial\mathcal{O}_j) \longrightarrow H_h^2(\text{neigh}(\partial\mathcal{O}_j)).$$

We also note that

$$\gamma_k = \mathcal{O}(h^{-\frac{1}{2}}) : H_h^2(\mathbb{R}^n \setminus \mathcal{O}_j) \longrightarrow H_h^{\frac{3}{2}}(\partial\mathcal{O}_k), \quad \text{uniformly in } h,$$

see Lemma 3.1.

Then

(6.6)
$$H_j(z) = T_j^h - R_j(z)P(z)T_j^h,$$

and

$$\gamma_k H_j(z) = \delta_{jk}I - \gamma_k R_j(z)P(z)T_j^h.$$

The basic properties of complex scaling [40, §3] show that for $\chi_j \in \mathcal{C}_c^\infty(B(0, R))$ which is 1 near $\mathcal{O}_j$ (hence supported away from the complex scaling region) we have

(6.7)
$$\begin{aligned}\chi(\Delta_{j,\theta} - \zeta)^{-1}\chi &= \chi(\Delta_j - \zeta)^{-1}\chi \\ &= \mathcal{O}(1/\operatorname{Im}\zeta) : L^2(\mathbb{R}^n \setminus \mathcal{O}_j) \longrightarrow L^2(\mathbb{R}^n \setminus \mathcal{O}_j), \quad \operatorname{Im}\zeta > 0,\end{aligned}$$

where $\Delta_{j,\theta}$ and Δ_j are the complex scaled and the usual Dirichlet Laplacians on $\mathbb{R}^n \setminus \mathcal{O}_j$. Hence,

$$\gamma_k R_j(z) P(z) T_j^h = \mathcal{O}(1/|h \operatorname{Re} z|) : H_h^{3/2}(\mathbb{R}^n \setminus \mathcal{O}_j) \longrightarrow H_h^{3/2}(\mathbb{R}^n \setminus \mathcal{O}_k), \quad \operatorname{Re} z < 0.$$

This in turn shows that

$$\mathcal{M}(z) = \mathcal{O}(1/|h \operatorname{Re} z|) : H_h^{\frac{3}{2}}(\partial\Omega) \longrightarrow H_h^{\frac{3}{2}}(\partial\Omega), \quad \operatorname{Re} z < 0,$$

and consequently that (6.5) holds.

We know (see for instance [40, Lemma 3.2]) that $R_j(z)$ is meromorphic in $D(0, r)$, and using (6.6) we conclude that so is $I - \mathcal{M}(z)$. Analytic Fredholm theory (see for instance [42, §2]) shows that invertibility of $I - \mathcal{M}(z)$ for $\operatorname{Re} z < -\gamma$ guarantees the meromorphy of its inverse, with poles of finite rank. $\square$

We recall the following standard result which is already behind the definition (6.2):

Lemma 6.2. *Let H_1 and H_2 be Hilbert spaces and let $z \mapsto A(z) \in \mathcal{L}(H_1, H_2)$, $z \mapsto B(z) \in \mathcal{L}(H_2, H_1)$, $z \in D \subset \mathbb{C}$, be holomorphic families of bounded operators. Suppose that $z \mapsto C(z) \in \mathcal{L}(H_2, H_2)$, $z \in D$, is a meromorphic family of bounded operators, with poles of finite rank. Then for any smooth closed curve $\gamma \subset D$, the operator*

$$\oint_\gamma A(z) B(z) C(z) dz,$$

is of trace class and

$$(6.8) \qquad \operatorname{tr}_{H_2} \oint_\gamma A(z) B(z) C(z) dz = \operatorname{tr}_{H_1} \oint_\gamma B(z) C(z) A(z) dz.$$

Proof. Without loss of generality we can assume that the winding number of γ is nonzero with respect to only one pole of $C(z)$, $z_0 \in D$. We can write

$$C(z) = C_0(z) + \sum_{k=1}^K \frac{C_k}{(z - z_0)^k},$$

where $C_0(z)$ is holomorphic near z_0, and C_k are finite rank operators. Consequently,

$$(6.9) \qquad \oint_\gamma A(z) B(z) C(z) dz = \oint_\gamma \left(\sum_{k=1}^K \frac{A(z) B(z) C_k}{(z - z_0)^k} \right) dz,$$

is a finite rank operator, and

$$\operatorname{tr}_{H_2} \oint_\gamma A(z) B(z) C(z) dz = \oint_\gamma \left(\sum_{k=1}^K \frac{\operatorname{tr}_{H_2} A(z) B(z) C_k}{(z - z_0)^k} \right) dz$$

$$= \oint_\gamma \left(\sum_{k=1}^K \frac{\operatorname{tr}_{H_1} B(z) C_k A(z)}{(z - z_0)^k} \right) dz,$$

where we used the cyclicity of the trace: $\operatorname{tr} ST = \operatorname{tr} TS$ when S is of trace class, and T is bounded. Same calculation as in (6.9) gives (6.8). $\square$

The main result of this section is the following

Proposition 6.3. *Suppose that the family of operators, $z \mapsto H(z)$, defined in (6.3) is holomorphic for $z \in D(0, r_0)$, Then the resonances, that is the poles of $R(z)$, agree with multiplicities with the poles of $I - \mathcal{M}(z)$:*

$$(6.10) \qquad m_R(z_0) = -\frac{1}{2\pi i} \operatorname{tr}_{L^2(\partial\Omega)} \oint_{\gamma_\epsilon(z)} (I - \mathcal{M}(z))^{-1} \frac{d}{dz} \mathcal{M}(z) dz \,,$$

where $\gamma_\epsilon(z_0) : t \mapsto z_0 + \epsilon e^{2\pi i t}$, $t \in [0, 2\pi)$, for sufficiently small $\epsilon > 0$, and the multiplicity $m_R(z_0)$ is defined by (6.2).

Proof. We first recall, for instance from [11, §6], that $R(z)$ can be expressed using the inverse of $I - \mathcal{M}(z)$:

$$(6.11) \qquad R(z) = \mathbb{1}_\Omega R_0(z) - \mathbb{1}_\Omega H(z)(I - \mathcal{M}(z))^{-1} \gamma R_0(z) \,,$$

where $R_0(z)$ acts on functions $L^2(\Omega) \hookrightarrow L^2(\mathbb{R}^n)$ extended by 0. Indeed,

$$P \left(\mathbb{1}_\Omega R_0 f - \mathbb{1}_\Omega H (I - \mathcal{M})^{-1} \gamma R_0 f \right) = P \mathbb{1}_\Omega R_0 f = f \,, \quad \text{in } \Omega \,.$$

The Dirichlet boundary condition is satisfied as

$$\gamma \left(\mathbb{1}_\Omega R_0 a - \mathbb{1}_\Omega H (I - \mathcal{M})^{-1} \gamma R_0 \right) = \gamma R_0 - \gamma H (\gamma H)^{-1} \gamma R_0 = 0 \,.$$

Hence by the uniqueness of the outgoing solution (using complex scaling that is simply the Fredholm property of $P(z)$) (6.11) holds.

Since $R_0(z)$ is holomorphic in z, (6.11) shows that

$$(6.12) \qquad m_R(z) = \frac{1}{2\pi i} \operatorname{tr} \oint_{\gamma_\epsilon(z)} \mathbb{1}_\Omega H(z)(I - \mathcal{M}(z))^{-1} \gamma R_0(z) \, dz$$

We want to compare it to the right hand side in (6.10) and for that we will use the following

Lemma 6.4. *With the notation above we have*

$$(6.13) \qquad \frac{d}{dz} \mathcal{M}(z) = -\gamma R_0(z) H(z) + (I - \mathcal{M}(z)) \operatorname{diag} \left(\gamma_k R_0(z) H_k(z) \right) \,.$$

Proof. The definition (6.4) gives $(d/dz)\mathcal{M}(z) = -\gamma H'(z)$, and by differentiating

$$P(z) H_k(z) = 0 \,, \quad \gamma_k H_k(z) = I \,,$$

we obtain

$$P(z) H'_k(z) = H_k(z) \quad \text{in } \mathbb{R}^n \setminus \mathcal{O}_k \,, \quad \gamma_k H'_k(z) = 0 \,.$$

Arguing as in the case of (6.11) we obtain

$$H'_k(z) = \mathbb{1}_{\mathbb{R}^n \setminus \mathcal{O}_k} R_0(z) H_k(z) - H_k(z) \gamma_k R_0(z) H_k(z) \,,$$

and hence

$$\frac{d}{dz}\mathcal{M}(z) = -\gamma R_0(z)H(z) + \gamma H(z)\operatorname{diag}\left(\gamma_k R_0(z)H_k(z)\right),$$

which is the same as (6.13). $\square$

Lemmas 6.2 and 6.4, and the assumption of holomorphy of $H(z)$ in $D(0, r_0)$, show that

(6.14)
$$-\operatorname{tr}_{L^2(\partial\Omega)}\oint_{\gamma_\epsilon(z)}(I - \mathcal{M}(z))^{-1}\frac{d}{dz}\mathcal{M}(z)dz =$$

$$\operatorname{tr}_{L^2(\mathbb{R}^n)}\oint_{\gamma_\epsilon(z)}H(z)(I - \mathcal{M}(z))^{-1}\gamma R_0(z)dz,$$

which is awfully close to (6.12). The difference of the right hand sides of (6.14) and (6.12) is equal to

(6.15)
$$\sum_{j=1}^{J}\operatorname{tr}\oint_{\gamma_\epsilon(z)}\mathbb{1}_{\mathcal{O}_j}H(z)(I - \mathcal{M}(z))^{-1}\gamma R_0(z)dz =$$

$$\sum_{j=1}^{J}\operatorname{tr}\oint_{\gamma_\epsilon(z)}H(z)(I - \mathcal{M}(z))^{-1}\gamma R_0(z)\mathbb{1}_{\mathcal{O}_j}\,dz.$$

Now observe that

$$\mathbb{1}_{\mathbb{R}^n\setminus\mathcal{O}_j}R_0(z)\mathbb{1}_{\mathcal{O}_j} = H_j(z)\gamma_j R_0(z)\mathbb{1}_{\mathcal{O}_j},$$

which implies

$$\gamma_k R_0(z)\mathbb{1}_{\mathcal{O}_j} = \gamma_k H_j(z)\gamma_j R_0(z)\mathbb{1}_{\mathcal{O}_j}.$$

This in turn shows that

$$\gamma R_0(z)\mathbb{1}_{\mathcal{O}_j} = \gamma H(z)\pi_j\gamma R_0(z)\mathbb{1}_{\mathcal{O}_j} = (I - \mathcal{M}(z))\pi_j\gamma R_0(z)\mathbb{1}_{\mathcal{O}_j},$$

where $\pi_j : \mathbb{C}^J \to \mathbb{C}^J$ is the orthogonal projection onto $\mathbb{C}e_j$, e_j denoting the jth canonical basis vector in $\mathbb{C}^J$. Hence

$$H(z)(I - \mathcal{M}(z))^{-1}\gamma R_0(z)\mathbb{1}_{\mathcal{O}_j} = H(z)\pi_j\gamma R_0(z)\mathbb{1}_{\mathcal{O}_j},$$

This expression is holomorphic in z since we assumed that $H(z)$ has no poles in the region of interest. That proves that the trace in (6.15) vanishes and completes the proof of (6.10). $\square$

6.2. **Semiclassical structure of the Poisson operator for convex obstacles.** We now review the properties of the operator $H(z)$ given in (6.3) where it is given in terms of Poisson operators $H_j(z)$ for individual convex obstacles. These properties are derived from results on propagation of singularities for diffractive boundary value problems (see [16, §24.4] and references given there) and from semiclassical parametrix constructions [11, A.II], [46, A.2-A.5]. They are based on ideas going back to Keller, Melrose, and Taylor – see [25] and references given there. The main result we need is stated in Proposition 6.7 below.

To orient the reader we first present a brief discussion of a model case and then use the parametrix to prove the general resuls.

6.2.1. *A model case.* We will review this parametrix in a special model case where it is given by an explicit formula. Using Melrose's equivalence of glancing hypersurfaces [23] this model can be used to analyze the general case but due to the presence of the boundary that is quite involved [26, §7.3, Appendix A] (see also [26, Chapter 2] for a concise presentation of diffractive geometry).

The model case (in two dimensions for simplicity) is provided by the *Friedlander model* [9],[16, §21.4]:

$$(6.16) \qquad P_0 = (hD_{x_2})^2 - x_2 + hD_{x_1}, \quad p_0 = \xi_2^2 - x_2 + \xi_1,$$

with the boundary $x_2 = 0$, the Poisson operator H_0. The surface to which $H_0 u$ is restricted to can be written as $x_2 = g(x_1)$. The Hamilton flow of p_0 is explicitely computed to be

$$(x, \xi) \longmapsto (x_1 + t, x_2 - \xi_2^2 + (t + \xi_2)^2; \xi_1, \xi_2 + t),$$

and the trajectories on the energy surface $p_0 = 0$ tangent to the boundary $x_2 = 0$ correspond to $\xi_1 = 0$. The bicharacteristic concavity of a region $q_0(x) > 0$ (modelling the concavity of $\mathbb{R}^n \setminus \partial \mathcal{O}_j$) is given by the condition $H_{p_0}^2 q_0 > 0$: that is automatically satisfied for $q_0(x) = x_2$ and holds for $q_0(x) = g(x_1) - x_2$ if $g''(x_1) > 2$.

For $v \in \mathcal{C}_c^\infty(\mathbb{R}_{x_1})$ the problem

$$(6.17) \qquad P_0 u(x) = 0, \quad x_2 > 0, \quad u(x_1, 0) = v(x_1),$$

has an explicit solution:

$$(6.18) \qquad u(x) = H_0 v(x) \overset{\text{def}}{=} \frac{1}{2\pi h} \iint \frac{A_+(h^{-2/3}(\xi_1 - x_2))}{A_+(h^{-2/3}\xi_1)} e^{\frac{i}{h}(x_1 - y_1)\xi_1} v(y_1) dy_1 d\xi_1,$$

where A_+ is the Airy function solving $A_+''(t) = tA_+(t)$ and having the following asymptotic behaviour:

$$A_+(t) \sim \begin{cases} \pi^{-\frac{1}{2}} t^{-\frac{1}{4}} e^{\frac{2}{3}t^{3/2}} & t \to +\infty, \\ \pi^{-\frac{1}{2}} (-t)^{-\frac{1}{4}} e^{\frac{2}{3}i(-t)^{3/2} + i\frac{\pi}{4}} & t \to -\infty. \end{cases}$$

Different asymptotic behaviours corresponds to different classical regions:

$$\begin{aligned} \xi_1 &< 0, \quad \text{hyperbolic region: trajectories transversal to the boundary,} \\ \xi_1 &= 0, \quad \text{glancing region: trajectories tangent to the boundary,} \\ \xi_1 &> 0, \quad \text{elliptic region: trajectories disjoint from the boundary.} \end{aligned}$$

If v is microlocally concentrated in the hyperbolic region, $\mathrm{WF}_h(v) \Subset \{\xi_1 < 0\}$, then

$$H_0 v(x) = \frac{1}{2\pi h} \iint e^{\frac{i}{h}\varphi(x, \xi_1) - \frac{i}{h} y_1 \xi_1} a(x, \xi_1) v(y_1) dy_1 \, d\xi_1 + \mathcal{O}(h^\infty) \|v\|_{L^2},$$

where $\varphi(x, \xi_1) = \frac{2}{3}(-(-\xi_1)^{\frac{3}{2}} + (-\xi_1 + x_2)^{\frac{3}{2}}) + x_1\xi_1$. That means that H_0 is microlocally an h-Fourier integral operator in the hyperbolic region, with the canonical relation given by

$$\mathcal{C}_0 \stackrel{\text{def}}{=} \{((x_1, x_2; \partial_{x_1}\varphi, \partial_{x_2}\varphi), (\partial_{\xi_1}\varphi, \xi_1)), \ \xi_1 < 0\}$$
$$= \{((x_1, x_2; \xi_1, (-\xi_1 + x_2)^{\frac{1}{2}}), (x_1 - (-\xi_1 + x_2)^{\frac{1}{2}} + (-\xi_1)^{\frac{1}{2}}, \xi_1)), \ \xi_1 < 0\}$$
$$= \{((x_1, (x_1 - y_1 + (-\xi_1)^{\frac{1}{2}})^2 + \xi_1; \xi_1, x_1 - y_1 + (-\xi_1)^{\frac{1}{2}}), (y_1, \xi_1)), \ \xi_1 < 0\}.$$

For a fixed y_1 and ξ_1, this corresponds to outward trajectories starting at the boundary point $(y_1, 0)$. The choice of the Airy function A_+ gives the outgoing solution to (6.17).

The propagation of semiclassical wave front sets is given by taking the closure of this relation which is smooth for $\xi_1 < 0$ only:

$$(6.19) \qquad \mathrm{WF}_h(H_0v) \cap \{x_2 > 0\} = \overline{\mathcal{C}}_0(\mathrm{WF}_h(v) \cap \{\xi_1 \leq 0\}) \cap \{x_2 > 0\}.$$

This can be proved using (6.18). Strictly speaking the wave front set on the left hand side is defined only in the region $\{x_2 > 0\}$ because of the presence of the bounday $x_2 = 0$.

We now consider $\gamma_1 u(x_1) \stackrel{\text{def}}{=} u(x_1, g(x_1))$ and (putting $x = x_1$),

$$(6.20) \qquad \gamma_1 H_0 v(x) = \frac{1}{2\pi h} \iint \frac{A_+(h^{-2/3}(\eta - g(x)))}{A_+(h^{-2/3}\eta)} e^{\frac{i}{h}(x-y)\eta} v(y) dy \, d\eta.$$

When acting on functions with $\mathrm{WF}_h(v) \Subset \{(y, \eta) : \eta < 0\}$, we can again use asymptotics of A_+ and that shows that, microlocally for $\eta < -c < 0$, $\gamma_1 H_0$ is an h-Fourier integral operator with a canonical relation with a fold [16, §21.4],[25, §4]:

$$\mathcal{B}_0 \stackrel{\text{def}}{=} \{(x, \eta + g'(x)(-\eta + g(x))^{\frac{1}{2}}; x + (-\eta)^{\frac{1}{2}} - (-\eta + g(x))^{\frac{1}{2}}, \eta), \ \eta < 0\}.$$

Having the fold singularity means the following: the map $f : \mathcal{B}_0 \to T^*\mathbb{R}$, defined as the projection on the second factor, $f(x, \xi, y, \eta) = (y, \eta)$, at every point at which

$$g'(x) = 2(-\eta + g(x))^{\frac{1}{2}},$$

satisfies $\dim \ker f' = \dim \mathrm{coker}\, f' = 1$, and has a non-zero Hessian,

$$\ker f' \ni X \longmapsto \langle f''X, X\rangle \in \mathrm{coker}\, f'.$$

The Hessian condition is equivalent to $g''(x) \neq 2$ which is satisfied as we assumed that $g''(x) > 2$. This corresponds to the tangency of the trajectory $x_2 = (x_1 - y + (-\eta)^{1/2})^2 + \eta$, to the boundary $x_2 = g(x_1)$. Using either an explicit calculation or general results on folds (see [16, Theorem 21.4.2 and Appendix C.4]) we can write

$$(6.21) \qquad\qquad\qquad \mathcal{B}_0 = \mathcal{B}_0^+ \cup \mathcal{B}_0^-,$$

where $\mathcal{B}_0^{\pm}$ correspond to trajectories entering (+) and leaving (−) $x_2 > g(x_1)$. Using (6.20) one can show a propagation result similar to (6.19):

$$(6.22) \qquad\qquad \mathrm{WF}_h(\gamma_1 H_0 v) = \overline{\mathcal{B}}_0(\mathrm{WF}_h(v) \cap \{\xi_1 \leq 0\}).$$

6.2.2. *Arbitrary convex obstacle.* To handle the general case we introduce the following notation

$$S^*_{\partial \mathcal{O}_k} \mathbb{R}^n = \{(x, \xi) \in T^* \mathbb{R}^n \ : \ x \in \partial \mathcal{O}_k, \ |\xi| = 1\},$$
$$S^* \partial \mathcal{O}_k = \{(y, \eta) \in T^* \partial \mathcal{O}_j, \ |\eta| = 1\},$$
$$B^* \partial \mathcal{O}_k = \{(y, \eta) \in T^* \partial \mathcal{O}_k \ : \ |\eta| \leq 1\}, \ \ \pi_k : S^*_{\partial \mathcal{O}_k}(\mathbb{R}^n) \longrightarrow B^* \partial \mathcal{O}_k.$$

where $|\bullet|$ is the induced Euclidean metric, and π_k is the orthogonal projection.

We first recall a result showing that when considering $\gamma_k H_j(z)$ we can restrict our attention to a neighbourhood of $B^* \partial \mathcal{O}_k \times B^* \partial \mathcal{O}_j$:

Lemma 6.5. *Suppose that $\partial \mathcal{O}_\ell$, $\ell = j, k$, $\mathcal{O}_j \cap \mathcal{O}_k = \emptyset$, are smooth and that $\mathcal{O}_j$ is strictly convex. If $\chi_\ell \in S^{0,0}(T^* \partial \mathcal{O}_\ell)$ satisfy $\chi_\ell \equiv 1$ near $B^* \partial \mathcal{O}_\ell$, $\ell = j, k$, then for $z \in D(0, r)$,*

(6.23)
$$\gamma_k H_j(z)(1 - \chi_j^w) = \mathcal{O}(h^\infty) \ : \ L^2(\partial \mathcal{O}_j) \longrightarrow C^\infty(\partial \mathcal{O}_k),$$
$$(1 - \chi_j^w)\gamma_k H_j(z) = \mathcal{O}(h^\infty) \ : \ L^2(\partial \mathcal{O}_j) \longrightarrow C^\infty(\partial \mathcal{O}_k).$$

Proof. The first estimate follows from the parametrix construction in the elliptic region – see [11, Proposition A.II.9] and [46, §A.-A.5]. From [46, (A.24)-(A.26)] we see that for $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^n \setminus \overline{\mathcal{O}_j})$

$$\psi(x) H_j(z)(1 - \chi_j^w) = \mathcal{O}_{L^2 \to \mathcal{C}^\infty}(h^\infty).$$

and the first estimate in (6.23) follows. As a consequence we can extend $\gamma_k H_j(z) : H^{\frac{3}{2}}(\partial \mathcal{O}_j) \to H^{\frac{3}{2}}(\partial \mathcal{O}_k)$ to

(6.24)
$$\gamma_k H_j(z) \ : \ L^2(\partial \mathcal{O}_j) \longrightarrow L^2(\partial \mathcal{O}_k).$$

Once $\gamma_k H_j(z)$ is defined on L^2, the second part of (6.23) follows from the fact that $(-h^2\Delta - 1)H_j(z)v = 0$ and hence $\mathrm{WF}_h(\psi H_j(z)) \subset \{|\xi| = 1\}$. We simply apply Lemma 3.1. □

The next proposition establishes boundedness properties:

Proposition 6.6. *Suppose that $\partial \mathcal{O}_\ell$, $\ell = j, k$ are smooth and that $\mathcal{O}_j$ is strictly convex. Then*

(6.25)
$$\gamma_k H_j = \mathcal{O}(1/h) \ : \ L^2(\partial \mathcal{O}_j) \longrightarrow L^2(\partial \mathcal{O}_k).$$

We cannot quote the results of [11] directly since for similar estimates in [11, A.II.1] the Lax-Phillips odd dimensional theory is invoked. As in [11] our proof is based on propagation of singularities for the time dependent problem, but it uses the more flexible method due to Vainberg [50].

Proof. Let $H_h^s(\partial \mathcal{O}_\ell)$ denote semiclassical Sobolev spaces defined in (3.5).

As in the proof of Lemma 6.1 we will use the resolvent of the Dirichlet Laplacian on $\mathbb{R}^n \setminus \mathcal{O}_j$, denoted below by Δ_j. We also use the extension operator T_j^h defined there.

Following (6.6) we write

$$\gamma_k H_j = \delta_{jk} - \gamma_k R_j(z) P(z) T_j^h \ : \ H_h^{\frac{3}{2}}(\partial \mathcal{O}_j) \longrightarrow H_h^{\frac{3}{2}}(\partial \mathcal{O}_k) \,,$$

and we need to show that for $z \in [-C_0 \log(1/h), R_0] + i[-R_0, R_0]$ the bound is $\mathcal{O}(1/h)$. In view of the discussion above that means showing that for $\varphi_\ell \in \mathcal{C}_c^\infty(\text{neigh}(\partial \mathcal{O}_\ell))$,

$$(6.26) \qquad \varphi_k R_j(z) \varphi_j = \mathcal{O}(1) : L^2(\mathbb{R}^n \setminus \mathcal{O}_j) \longrightarrow H_h^2(\mathbb{R}^n) \,.$$

We recall that $\varphi_k R_j(z) \varphi_j = \varphi_k P_j(z)^{-1} \varphi_l$, where

$$P_j(z) = ((i/h)(-h^2 \Delta_{j,\theta} - 1) - z) = \mathcal{O}(1/h) : H_h^2(\mathbb{R}^n \setminus \mathcal{O}_j) \longrightarrow L^2(\mathbb{R}^n \setminus \mathcal{O}_j) \,,$$

and $\Delta_{j,\theta}$ is the complex scaled Dirichlet Laplacian on $\mathbb{R}^n \setminus \mathcal{O}_j$ – see (6.1). The rescaling involved in the definition shows that we need

$$(6.27) \qquad \varphi_k((-h^2 \Delta_{j,\theta} - 1) - hz/i)^{-1} \varphi_j = \mathcal{O}(1/h) \ : \ L^2(\mathbb{R}^n \setminus \mathcal{O}_j) \longrightarrow H_h^2(\mathbb{R}^n) \,,$$

which follows from

$$(6.28) \qquad \varphi_k(-\Delta_j - \zeta^2)^{-1} \varphi_j = \mathcal{O}_{L^2 \to L^2}(1/|\zeta|) \,, \quad \text{Im}\, \zeta > -C \,, \quad \text{Re}\, \zeta > C \,,$$

where, as in (6.7) we have replaced $\Delta_{j,\theta}$ by the unscaled operator.

To establish (6.28) we use Vainberg's theory as presented in [38, Section 3] and [48, Section 3]. For that we need results about the wave propagator.

Let $U_j(t) \stackrel{\text{def}}{=} \sin t \sqrt{-\Delta_j} / \sqrt{-\Delta_j}$ be the Dirichlet wave propagator. Then for $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$

$$(6.29) \qquad \chi U_j(t) \chi : L^2(\mathbb{R}^n \setminus \mathcal{O}_j) \longrightarrow \overline{\mathcal{C}}^\infty(\mathbb{R}^n \setminus \mathcal{O}_j) \,, \quad t \geq T_\chi \,,$$

where $\overline{\mathcal{C}}^\infty$ denotes the space of extendable smooth functions. This follows from the singularities propagate along reflected and glancing rays and that there are no trapped rays in the case of one convex obstacle – see [16, §24.4].

Applying [38, (3.35)] or [48, Proposition 3.1] to (6.29) gives (6.28), and thus completes the proof of (6.25). $\qquad \square$

We now set up some notations concerning the symplectic relations associated with our obstacle system. For $i \neq j$ we now define the (open) symplectic relations $\mathcal{B}_{ij}^\pm$, analogues of the relations $\mathcal{B}_0^\pm$ (6.21) in the Friedlander model. For $x \in \partial \mathcal{O}_j$ denote by $\nu_j(x)$ the *outward* pointing normal vector to $\partial \mathcal{O}_j$ at x. Then

$$(\rho, \rho') \in \mathcal{B}_{ij}^\pm \subset B^* \partial \mathcal{O}_i \times B^* \partial \mathcal{O}_j$$

$$\Longleftrightarrow$$

(6.30)
$$\exists\, t > 0 \,, \xi \in \mathbb{S}^{n-1} \,, \ x \in \partial \mathcal{O}_j \,, \ x + t\xi \in \partial \mathcal{O}_i \,, \ \langle \nu_j(x), \xi \rangle > 0 \,,$$
$$\pm \langle \nu_i(x + t\xi), \xi \rangle < 0 \,, \ \pi_j(x, \xi) = \rho' \,, \quad \pi_i(x + t\xi, \xi) = \rho \,,$$

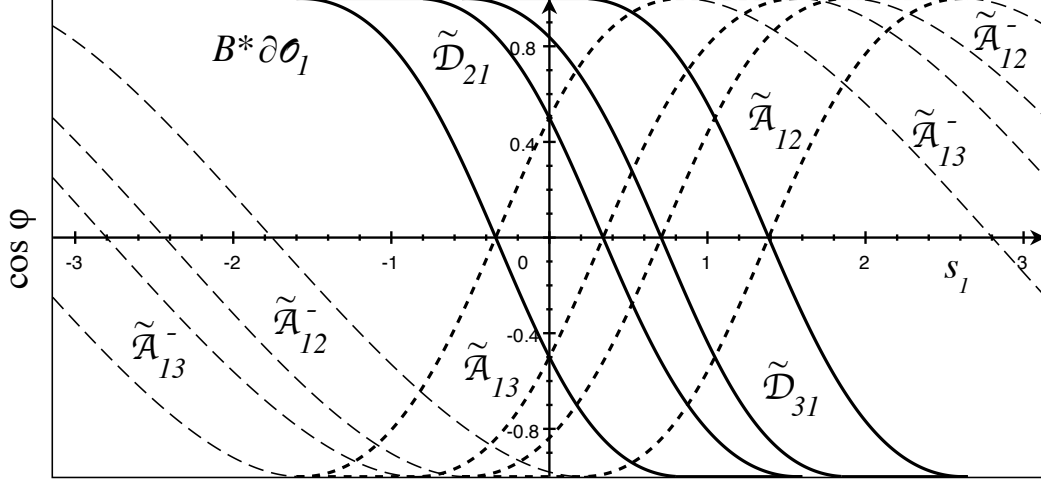Notice that $\mathcal{B}_{ij}^+$ is equal to the billiard relation F_{ij} defined in (1.6).

FIGURE 6.   (after [31]) Partial boundary phase space $B^*\mathcal{O}_1$ for the symmetric three disk scattering shown in Fig. 3. We show the boundaries of the departure ($\widetilde{D}_{i1}$, full lines), arrival ($\widetilde{A}_{1j}$, dashed lines) and shadow arrival ($\widetilde{A}_{1j}^-$, long dashed) sets.

The relations $\mathcal{B}_{ij}^{\pm}$ are singular at their boundaries

$$(6.31) \qquad \partial\mathcal{B}_{ij}^{\pm} = \overline{\mathcal{B}_{ij}^{\pm}} \cap (B^*\partial\mathcal{O}_i \times S^*\partial\mathcal{O}_j \cup S^*\partial\mathcal{O}_i \times B^*\partial\mathcal{O}_j) \,,$$

which corresponds to the glancing rays on $\mathcal{O}_j$ or $\mathcal{O}_i$. We will often use the closures of these relations, $\overline{\mathcal{B}_{ij}^{\pm}}$, which include the glancing rays. The inverse relations ($\mathcal{C}^t \overset{\text{def}}{=} \{(\rho, \rho') : (\rho', \rho) \in \mathcal{C}\}$) are obtained by reversing the momenta:

$$(6.32) \qquad (\mathcal{B}_{ij}^+)^t = \mathcal{J} \circ \mathcal{B}_{ji}^+ \circ \mathcal{J} \,, \quad \mathcal{J}(y, \eta) \overset{\text{def}}{=} (y, -\eta), \quad (y, \eta) \in B^*\partial\mathcal{O} \,.$$

If we define

$$(6.33) \qquad \mathcal{U} \overset{\text{def}}{=} \text{neigh}(B^*\partial\mathcal{O}) = \bigsqcup_{j=1}^{J} \text{neigh}(B^*\partial\mathcal{O}_j) \,,$$

we are in the dynamical setting of §2. Compared with §2, we define the arrival and departure sets from the closure of the relation F_{ij}:

$$\widetilde{A}_{ij} \overset{\text{def}}{=} \overline{\mathcal{B}_{ij}^+}(B^*\partial\mathcal{O}_j) \subset B^*\partial\mathcal{O}_i \,, \qquad \widetilde{D}_{ij} \overset{\text{def}}{=} \overline{(\mathcal{B}_{ij}^+)^t}(B^*\partial\mathcal{O}_i) \subset B^*\partial\mathcal{O}_j \,.$$

From (6.32) we check that

$$\widetilde{D}_{ij} = \mathcal{J}(\widetilde{A}_{ji}) \,.$$

Besides, the *shadow* arrival sets are given by

$$\widetilde{A}_{ij}^- \stackrel{\text{def}}{=} \overline{\mathcal{B}_{ij}^-}(B^*\partial\mathcal{O}_j)\,.$$

Also, let us call

(6.34) $$\widetilde{A}_i^{(-)} \stackrel{\text{def}}{=} \bigcup_{i\neq j}\widetilde{A}_{ij}^{(-)}\,,\quad \widetilde{D}_i \stackrel{\text{def}}{=} \bigcup_{i\neq j}\widetilde{D}_{ji}\,.$$

The subsets of glancing rays are denoted by

$$\widetilde{A}_i^{\mathcal{G}} \stackrel{\text{def}}{=} \widetilde{A}_i \cap S^*\partial\mathcal{O}_i = \widetilde{A}_i^- \cap S^*\partial\mathcal{O}_i,\qquad \widetilde{D}_i^{\mathcal{G}} \stackrel{\text{def}}{=} \widetilde{D}_i \cap S^*\partial\mathcal{O}_i\,.$$

With this notation we can state the most important result of this section:

Proposition 6.7. *For $i \neq j$, let $H_j(z)$ and γ_i be as in §6.1, and assume that $\partial\mathcal{O}_k$, $k = i,j$ are strictly convex. For any tempered $v \in L^2(\mathcal{O}_j)$, we have*

(6.35) $$\mathrm{WF}_h(\gamma_i H_j(z)v) = \left(\overline{\mathcal{B}_{ij}^+ \cup \mathcal{B}_{ij}^-}\right)\left(\mathrm{WF}_h(v) \cap B^*\partial\mathcal{O}_j\right),$$

uniformly for

(6.36) $$z \in \Omega_0 \stackrel{\text{def}}{=} [-C_0\log(1/h), R_0] + i[-R_0, R_0],\quad C_0, R_0 > 0 \quad \text{fixed.}$$

If $Q_k \in \Psi_h^{0,-\infty}(\partial\mathcal{O}_k)$, $k = i,j$, satisfy

(6.37) $$\mathrm{WF}_h(Q_i) \cap \widetilde{A}_i^{\mathcal{G}} = \emptyset\,,\quad \mathrm{WF}_h(Q_j) \cap \widetilde{D}_j^{\mathcal{G}} = \emptyset\,,$$

then

(6.38) $$Q_i\gamma_i H_j(z)Q_j \in I^0(\partial\mathcal{O}_i \times \partial\mathcal{O}_j, (\mathcal{B}_{ij}^+)') + I^0(\partial\mathcal{O}_i \times \partial\mathcal{O}_j, (\mathcal{B}_{ij}^-)')\,.$$

Because of the assumptions on Q_k's, only compact subsets of the open relations $\mathcal{B}_{ij}^{\pm}$ are involved in the definition of the classes I^0.

We also have, for some $\tau > 0$ and z in the above domain, the norm estimate

(6.39) $$\|Q_i\gamma_i H_j(z)Q_j\|_{L^2(\partial\mathcal{O}_j)\to L^2(\partial\mathcal{O}_i)} \le C(R_0)\exp(\tau\,\mathrm{Re}\,z)\,.$$

Although we will never have to use any detailed analysis near the glancing points (that is, points where the trajectories are tangent to the boundary) it is essential that we know (6.35) and that requires the analysis of diffractive effects. In particular, we have to know that there will not be any propagation along the boundary.

Proof. For $z \in \Omega_0$, [11, Theorem A.II.12] gives the wave front set properties of $H_j(z)$. In particular it implies that for $\varphi_k \in \mathcal{C}_c^\infty(\text{neigh}(\partial\mathcal{O}_k))$, $\varphi_k = 0$ near $\mathcal{O}_j$,

$$\mathrm{WF}_h(\varphi_k H_j(z)v) = \big\{(x + t\xi, \xi)\ :\ t > 0\,,\ x + t\xi \in \mathrm{supp}\,\varphi_k\,,\ |\xi| = 1\,,$$
$$(x, \pi_j(\xi)) \in \mathrm{WF}_h(v) \cap B^*\partial\mathcal{O}_j\,,\ \langle\nu_j(x), \xi\rangle \ge 0\big\}\,.$$

This and Lemma 3.1 immediately give (6.35).

The conditions on Q_k's appearing in (6.38) mean that we are cutting off the contributions of rays satisfying $\langle \nu_j(x), \xi \rangle = 0$, that is with $\pi_j(\xi) \in S^* \partial \mathcal{O}_j$ (glancing rays on $\mathcal{O}_j$), as well as the contributions of the rays $\langle \nu_i(x + t\xi), \xi \rangle = 0$ (glancing rays on $\mathcal{O}_i$). This means that the contributions to $Q_i \gamma_i H_j(z) Q_j$ only come from the interior of the hyperbolic regions on the right, $(B^* \partial \mathcal{O}_j)^\circ$, and on the left, $(B^* \partial \mathcal{O}_i)^\circ$. The description of $H_j(z)$ in the hyperbolic region given in [11, Proposition A.II.3] and [46, §A.2] shows that it is a sum of zeroth order Fourier integral operators associated to the relations $\mathcal{B}_{ij}^\pm$. For $\operatorname{Re} z < 0$ the forward solution of the eikonal equation gives the exponential decay of (6.39) (where $0 < \tau \leq d_{ji}$, the distance between $\mathcal{O}_j$ and $\mathcal{O}_i$). As pointed out in [46, §A.2], this decay is valid for z in the logarithmic neighbourhood Ω_0. $\qquad \square$

6.3. The microlocal billiard ball map.

In this section we will show how for several strictly convex obstacles satisfying Ikawa's condition (1.1) the operator $\mathcal{M}(z)$ defined in §6.1 can be replaced by an operator satisfying the assumptions of §2. This follows the outline presented in §1.

Namely, we now show that invertibility of $I - \mathcal{M}(z)$ can be reduced to invertibility of $I - M(z)$ where $M(z)$ satisfies the assumptions of §2. That can only be done after introducing a microlocal weight function.

We first consider a general weight. Suppose that

$$g_0 \in S(T^* \partial \mathcal{O}; \langle \xi \rangle^{-\infty}) \overset{\text{def}}{=} \bigcap_{N \geq 0} S(T^* \partial \mathcal{O}; \langle \xi \rangle^{-N}),$$

or simply that $g_0 \in \mathcal{C}_c^\infty(T^* \partial \mathcal{O})$. Then for a fixed but *large T* consider

$$(6.40) \qquad g \overset{\text{def}}{=} T \log \left(\frac{1}{h} \right) g_0, \quad \exp(\pm g^w(x, hD)) \in \Psi_{0+}^{TC_0, 0}(\partial \mathcal{O}), \quad C_0 = \max_{T^* \partial \mathcal{O}} |g_0|.$$

We remark that the operators $\exp(\pm g^w(x, hD))$ are pseudo-local in the sense that

$$(6.41) \qquad\qquad \operatorname{WF}_h(e^{\pm g^w(x, hD)} v) = \operatorname{WF}_h(v)$$

(we see the inclusion from the pseudodifferential nature of $\exp(\pm g^w)$, and the equality from their invertibility).

For $k \neq j$, we will write

$$(6.42) \qquad\qquad (\mathcal{M}_g)_{kj}(z) \overset{\text{def}}{=} e^{-g_k^w(x, hD)} \mathcal{M}_{kj}(z) e^{g_j^w(x, hD)}.$$

As before, we consider these operators as a matrix acting on $L^2(\partial \mathcal{O}) = \bigoplus_{j=1}^J L^2(\partial \mathcal{O}_j)$. Using Proposition 6.6 we see that $\mathcal{M}(z)$ is bounded by C/h as an operator on $L^2(\partial \mathcal{O})$, uniformly for $z \in \Omega_0$ (see (6.36)). Since $\exp(\pm g^w(x, hD)) = \mathcal{O}_{L^2 \to L^2}(h^{-TC_0})$ the conjugated operator satisfies $\mathcal{M}_g(z) = \mathcal{O}_{L^2 \to L^2}(h^{-2TC_0 - 1})$.

We want to reduce the invertibility of $I - \mathcal{M}_g(z)$ to that of $I - M_g(z)$, where $M(z)$ is a Fourier integral operator. That means eliminating the glancing contributions in $\mathcal{M}(z)$ (see (6.38)).

We start with a simple lemma which shows that Ikawa's condition (1.1) eliminates glancing rays and the restrictions to shadows:

Lemma 6.8. *Suppose that for some $j \neq i \neq k$*

$$\overline{\mathcal{O}}_i \cap \text{convex hull}(\overline{\mathcal{O}}_j \cup \overline{\mathcal{O}}_k) = \emptyset \,.$$

Then, in the notation of Proposition 6.7,

$$(6.43) \qquad\qquad \overline{\mathcal{B}^{\pm}_{ki}} \circ \overline{\mathcal{B}^{-}_{ij}} = \emptyset \,.$$

Proof. Suppose that (6.43) does not hold. Then there exists $x \in \partial \mathcal{O}_j$, $\xi \in \mathbb{S}^{n-1}$, and $0 < t_1 < t_2$, such that $x + t_1 \xi \in \overline{\mathcal{O}}_i$, and $x + t_2 \xi \in \partial \mathcal{O}_k$. But this means that $\overline{\mathcal{O}}_i$ intersects the convex hull of $\partial \mathcal{O}_j$ and $\partial \mathcal{O}_k$.

$\square$

The trapped set $\mathcal{T}$ was defined using (1.7), where we recall that the relation $F = (F_{ij})$ is also given by $F_{ij} = \mathcal{B}^{+}_{ij}$. Lemma 6.8 shows that

$$\mathcal{T} \cap S^* \partial \mathcal{O} = \emptyset \,.$$

Lemma 6.8 implies that (see Fig. 6)

$$(6.44) \qquad \overline{\mathcal{B}^{\pm}_{ki}}(\widetilde{A}^{\mathcal{G}}_i) = \emptyset \,, \quad \text{so that} \quad \widetilde{D}^{\mathcal{G}}_i \cap \widetilde{A}^{\mathcal{G}}_i = \emptyset \,, \quad \text{and similarly} \quad \left(\overline{\mathcal{B}^{\pm}_{ik}}\right)^t (\widetilde{D}^{\mathcal{G}}_i) = \emptyset \,.$$

Since the sets $\widetilde{A}^{\mathcal{G}}_i$ and $\widetilde{D}^{\mathcal{G}}_i$ are closed and disjoint, we can find small neighbourhoods

$$(6.45) \qquad\qquad U^A_i \overset{\text{def}}{=} \text{neigh}(\widetilde{A}^{\mathcal{G}}_i) \,, \quad U^D_i \overset{\text{def}}{=} \text{neigh}(\widetilde{D}^{\mathcal{G}}_i) \,, \quad U^D_i \cap U^A_i = \emptyset \,,$$

so that

$$(6.46) \qquad\qquad \overline{\mathcal{B}^{\pm}_{ki}}(U^A_i) = \emptyset \,, \qquad \left(\overline{\mathcal{B}^{\pm}_{ik}}\right)^t (U^D_i) = \emptyset \,.$$

We can now formulate a Grushin problem which will produce the desired effective Hamiltonian. For that let $\chi_{j,A}, \chi_{j,D} \in \mathcal{C}^{\infty}(T^* \mathcal{O}_j; \mathbb{R})$ satisfy

$$(6.47) \qquad \begin{aligned} &\chi_{j,A}\big\rvert_{\text{neigh}(\widetilde{A}^{\mathcal{G}}_j)} \equiv 1 \,, \quad \text{supp} \, \chi_{j,A} \Subset U^A_j \,, \\ &\chi_{j,D}\big\rvert_{\text{neigh}(\widetilde{D}^{\mathcal{G}}_j)} \equiv 1 \,, \quad \text{supp} \, \chi_{j,D} \Subset U^D_j \,. \end{aligned}$$

We let $\tilde{\chi}_{j,\bullet}$ have the same properties as $\chi_{j,\bullet}$ with $\chi_{j,\bullet} = 1$ on $\text{supp} \, \tilde{\chi}_{j,\bullet}$. In view of (6.35) and (6.46), we have

$$\begin{aligned} \mathcal{M}_{kj}(z) \, \chi^w_{j,A}(x, hD) &= \mathcal{O}_{L^2(\partial \mathcal{O}_j) \to \mathcal{C}^{\infty}(\partial \mathcal{O}_k)}(h^{\infty}) \,, \\ \chi^w_{j,D}(x, hD) \, \mathcal{M}_{ji}(z) &= \mathcal{O}_{L^2(\partial \mathcal{O}_i) \to \mathcal{C}^{\infty}(\partial \mathcal{O}_j)}(h^{\infty}) \,, \end{aligned}$$

for all $k \neq j \neq i$, uniformly for $z \in \Omega_0$. In view of (6.41), and using the notations of (6.42), we have the same properties for the conjugated operator:

$$
(6.48) \quad
\begin{aligned}
(\mathcal{M}_g)_{kj}(z)\, \chi_{j,A}^w(x,hD) &= \mathcal{O}_{L^2(\partial\mathcal{O}_j)\to\mathcal{C}^\infty(\partial\mathcal{O}_k)}(h^\infty)\,, \\
\chi_{j,D}^w(x,hD)\,(\mathcal{M}_g)_{ji}(z) &= \mathcal{O}_{L^2(\partial\mathcal{O}_i)\to\mathcal{C}^\infty(\partial\mathcal{O}_j)}(h^\infty)\,.
\end{aligned}
$$

For $\bullet = A, D$, let $\widetilde{\Pi}_{j,\bullet}$ be an orthogonal finite rank projection on $L^2(\partial\mathcal{O}_j)$ such that

$$
(6.49) \quad
\begin{aligned}
\tilde\chi_{j,\bullet}^w\, \widetilde{\Pi}_{j,\bullet} &= \widetilde{\Pi}_{j,\bullet}\, \tilde\chi_{j,\bullet}^w + \mathcal{O}_{L^2\to\mathcal{C}^\infty}(h^\infty) = \tilde\chi_{j,\bullet}^w + \mathcal{O}_{L^2\to\mathcal{C}^\infty}(h^\infty)\,, \\
\chi_{j,\bullet}^w\, \widetilde{\Pi}_{j,\bullet} &= \widetilde{\Pi}_{j,\bullet}\, \chi_{j,\bullet}^w + \mathcal{O}_{L^2\to\mathcal{C}^\infty}(h^\infty) = \widetilde{\Pi}_{j,\bullet} + \mathcal{O}_{L^2\to\mathcal{C}^\infty}(h^\infty)\,.
\end{aligned}
$$

Such projections can be found by constructing a real valued function $\psi_{j,\bullet} \in \mathcal{C}_c^\infty(T^*\partial\mathcal{O}_j)$ satisfying $\psi_{j,\bullet} \equiv 1$ on $\mathrm{supp}(\tilde\chi_{j,\bullet})$ and $\psi_{j,\bullet} \equiv 0$ on $\mathrm{supp}(1 - \chi_{j,\bullet})$. Then $\widetilde{\Pi}_{j,\bullet} \overset{\mathrm{def}}{=} \mathbb{1}_{\psi_{j,\bullet}^w(x,hD)\geq 1/2}$ provides a desired projection of rank comparable to h^{1-n}.

We need one more orthogonal projector $\Pi_j^\#$, microlocally projecting in a neighbourhood of $B^*\partial\mathcal{O}_j$. Precisely, we assume that for some cutoff $\chi_j \in \mathcal{C}_c^\infty(T^*\partial\mathcal{O}_j)$ with $\chi_j = 1$ near $B^*\partial\mathcal{O}_j \cup U_j^A \cup U_j^D$, this projector satisfies

$$
(6.50) \quad \Pi_j^\#\, \chi_j^w = \chi_j^w\, \Pi_j^\# + \mathcal{O}_{L^2\to\mathcal{C}^\infty}(h^\infty) = \Pi_j^\# + \mathcal{O}_{L^2\to\mathcal{C}^\infty}(h^\infty)\,.
$$

From Lemma 6.5 we easily get the bounds

$$
(6.51) \quad (\mathcal{M}_g)_{kj}(z)\,(I - \Pi_j^\#) = \mathcal{O}_{L^2\to\mathcal{C}^\infty}(h^\infty)\,, \qquad (I - \Pi_k^\#)\,(\mathcal{M}_g)_{kj}(z) = \mathcal{O}_{L^2\to\mathcal{C}^\infty}(h^\infty)\,.
$$

The operator

$$
(6.52) \quad P_j \overset{\mathrm{def}}{=} \widetilde{\Pi}_{j,A} + \widetilde{\Pi}_{j,D} + I - \Pi_j^\#\,.
$$

is not a projection but it can be easily modified to yield a projection with desired properties:

Lemma 6.9. *Let γ be positively oriented curve around $\zeta = 1$, $\gamma : t \mapsto 1 + \epsilon\exp(2\pi it)$, ϵ small and fixed. For P_j given by (6.52) define*

$$
\widetilde{\Pi}_j \overset{\mathrm{def}}{=} \frac{1}{2\pi i} \int_\gamma (\zeta - P_j)^{-1} d\zeta\,.
$$

Then $\widetilde{\Pi}_j$ is an orthogonal projection satisfying

$$
(6.53) \quad \widetilde{\Pi}_j = \widetilde{\Pi}_{j,A} + \widetilde{\Pi}_{j,D} + I - \Pi_j^\# + \mathcal{O}_{L^2\to\mathcal{C}^\infty}(h^\infty)\,,
$$

where $\widetilde{\Pi}_{j,\bullet}$, $\Pi_j^\sharp$ are the projections in (6.49) and (6.50).

Proof. The operator P_j is not a projection but it is self-adjoint and satisfies

$$
(6.54) \quad P_j^2 = P_j + \mathcal{O}_{L^2\to\mathcal{C}^\infty}(h^\infty)\,,
$$

which we check using (6.49), (6.50) and the properties of the functions $\chi_{j,\bullet}$, χ_j. Hence its spectrum is contained in $[0, \mathcal{O}(h^\infty)] \cup [1 - \mathcal{O}(h^\infty), 1 + \mathcal{O}(h^\infty)]$. For ϵ small enough and fixed

we can take γ including all spectrum near 1 (the statements implicitly assume that h is small enough). For $\zeta \in \gamma$ we write

$$(\zeta - P_j)^{-1} = \zeta^{-1}(I + (\zeta - 1)^{-1}P_j)\left(I - \zeta^{-1}(\zeta - 1)^{-1}(P_j^2 - P_j)\right)^{-1}.$$

The inverse on the right hand side exists in view of (6.54) and satisfies

$$\left(I - \zeta^{-1}(\zeta - 1)^{-1}(P_j^2 - P_j)\right)^{-1} = I + \mathcal{O}_{L^2 \to \mathcal{C}^\infty}(h^\infty)$$

uniformly on γ. Inserting these two formulæ into the integral defining Π_j gives (6.53). $\qquad\square$

The next lemma provides the property crucial in the construction of the Grushin problem:

Lemma 6.10. *For $\widetilde{\Pi}_j$ defined above and any $k \neq j \neq i$ we have*

$$(\mathcal{M}_g)_{kj}(z)\,\widetilde{\Pi}_j\,(\mathcal{M}_g)_{ji}(z) = \mathcal{O}(h^\infty) \; : \; L^2(\partial\mathcal{O}_i) \longrightarrow \mathcal{C}^\infty(\partial\mathcal{O}_k) \,,$$

uniformly for $z \in \Omega_0$.

Proof. To simplify the notation we write $\mathcal{M}_{ij}$ instead of $(\mathcal{M}_g)_{ij}(z)$.

Using (6.53) and (6.51), we can write

$$\mathcal{M}_{kj}\,\widetilde{\Pi}_j\,\mathcal{M}_{ji} = \mathcal{M}_{kj}(\widetilde{\Pi}_{j,A} + \widetilde{\Pi}_{j,D})\mathcal{M}_{ji} + \mathcal{O}_{L^2 \to \mathcal{C}^\infty}(h^\infty)\,.$$

From (6.49) we use

$$\widetilde{\Pi}_{j,A} = \chi_{j,A}^w\,\widetilde{\Pi}_{j,A} + \mathcal{O}(h^\infty)\,, \qquad \widetilde{\Pi}_{j,D} = \widetilde{\Pi}_{j,D}\,\chi_{j,D}^w + \mathcal{O}(h^\infty)\,,$$

and hence, using (6.48), we complete the proof. $\qquad\square$

Now define the following orthogonal projection:

(6.55) $$\Pi_h \overset{\text{def}}{=} \operatorname{diag}(I - \widetilde{\Pi}_j) \; : \; L^2(\partial\mathcal{O}) \longrightarrow L^2(\partial\mathcal{O})\,.$$

Since each $I - \widetilde{\Pi}_j$ is microlocalized on a compact neighbourhood of $B^*\partial\mathcal{O}_j$, it has a rank comparable with h^{1-n}, and so does Π_h.

Using this projection we obtain the main result of this section:

Theorem 5. *Let Π_h be given by (6.55) and (6.49), and $\mathcal{M}_g(z,h)$ be defined by (6.4) and (6.42).*

If $\mathcal{O}_j$ are strictly convex and satisfy the Ikawa condition (1.1) then the following Grushin problem is well posed for $z \in D(0,C)$:

$$\begin{pmatrix} I - \mathcal{M}_g(z,h) & \Pi_h^t \\ \Pi_h & 0 \end{pmatrix} \; : \; L^2(\partial\mathcal{O}) \oplus W_h \longrightarrow L^2(\partial\mathcal{O}) \oplus W_h \,, \quad W_h \overset{\text{def}}{=} \Pi_h L^2(\partial\mathcal{O})\,,$$

where $\Pi_h^t : W_h \hookrightarrow L^2(\partial\mathcal{O})$.

The effective Hamiltonian is given by

$$E_{-+}(z,h) = -\Big(I_{W_h} - \Pi_h\big(M_g(z,h) + R(z,h)\big)\Pi_h\Big)\,,$$

where

$$(6.56) \qquad M_g(z,h) = \varphi^w \mathcal{M}_g(z,h)\varphi^w \in I^0_{0+}(\partial\mathcal{O} \times \partial\mathcal{O}, F'), \quad \varphi^w = \mathrm{diag}(1 - \tilde\chi^w_{j,A} - \tilde\chi^w_{j,D}),$$

with the relation F given by (1.6), $\tilde\chi_{j,\bullet}$ *satisfying* (6.47), *and*

$$R(z,h) = \mathcal{O}_{L^2(\partial\mathcal{O})\to\mathcal{C}^\infty(\partial\mathcal{O})}(h^\infty).$$

Proof. We first observe that Lemma 6.10 gives

$$(6.57) \qquad \mathcal{M}_g(z,h)\Pi_h\mathcal{M}_g(z,h) = \mathcal{M}_g(z,h)^2 + \mathcal{O}_{L^2(\partial\mathcal{O})\to\mathcal{C}^\infty(\partial\mathcal{O})}(h^\infty).$$

The theorem follows from this. Indeed, (6.57) implies that, in abbreviated notation,

$$\begin{pmatrix} I - \mathcal{M}_g & \Pi^t \\ \Pi & 0 \end{pmatrix} \begin{pmatrix} (I + (I - \Pi)\mathcal{M}_g)(I - \Pi) & \Pi^t + (I - \Pi)\mathcal{M}_g\Pi^t \\ \Pi(I + \mathcal{M}_g(I - \Pi)) & -(I_W - \Pi\mathcal{M}_g\Pi) \end{pmatrix}$$
$$= I_{L^2(\partial\mathcal{O})\oplus W_h} + \mathcal{O}_{L^2(\partial\mathcal{O})\oplus W_h\to\mathcal{C}^\infty(\partial\mathcal{O})\oplus W_h}(h^\infty).$$

The exact inverse in then obtained by a Neumann series inversion. In view of (6.49) and (6.38) we obtain (6.56). $\qquad\square$

Proof of Theorem 1. To apply Theorem 4 from §5.4 we need to show that we can choose g so that $M_g(z,h)$ satisfies the conditions in Definition 2.1. The only assumption that needs verification is (2.4). For that we take g_0 in (6.40) given in Lemma 4.5 (proved in the appendix), with the map being the billiard ball relation, F.

Egorov's theorem (Proposition 3.10) then shows that (2.4) holds and we can make $\mathcal{W}$ as close to the trapped set as we wish. $\qquad\square$

Appendix: Construction of an escape function for an open map F

In this appendix we prove Lemma 4.5 by explicitly constructing an escape function g_0. The only assumption we need on F is the fact that the trapped set $\mathcal{T} \Subset \widetilde{D}$, $\mathcal{T} \Subset \widetilde{A}$, where $\widetilde{D}$, resp. $\widetilde{A}$ are the (closed) departure and arrival sets of F. Our construction is inspired by similar constructions in [5, Lemma 4.3] and [51]. Here we will independently construct functions $g_\pm$ with good escape properties away from the incoming and outgoing tails $\mathcal{T}_\mp$, respectively.

Let us start with the function g_+. We can take $V_\pm \Subset \mathcal{U}$, open neighbourhoods of the tails $\mathcal{T}_\pm$, such that

$$V_+ \cap V_- \Subset \mathcal{W}_2,$$

where $\mathcal{W}_2$ is the neighbourhood of $\mathcal{T}$ in the statement of Lemma 4.5. We will first construct a function $g_+ \in \mathcal{C}^\infty_c(T^*Y)$ such that

$$(A.1) \qquad \begin{aligned} \forall\rho \in \mathcal{W}_3, \quad & g_+(F(\rho)) - g_+(\rho) \geq 0, \\ \forall\rho \in \mathcal{W}_3 \setminus V_-, \quad & g_+(F(\rho)) - g_+(\rho) \geq 1. \end{aligned}$$
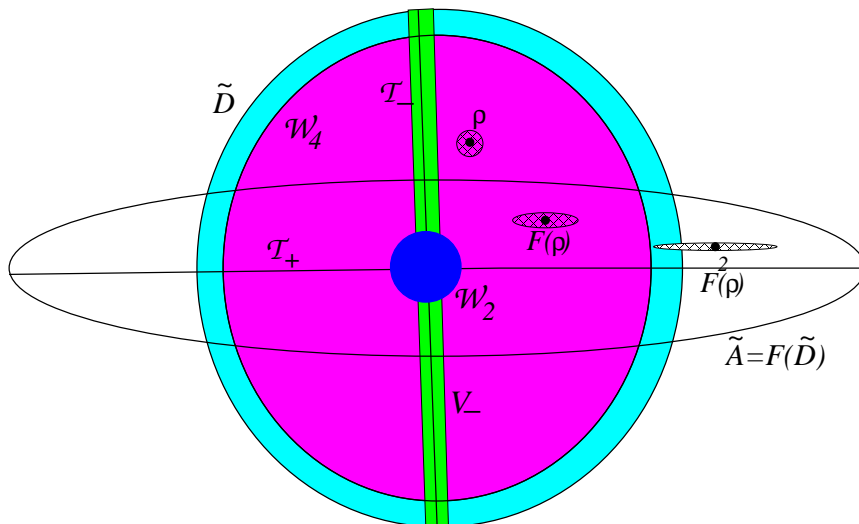
FIGURE 7. Sketch of the construction of an escape function $g_{\rho,+}$. The larger inner disk (respectively the small inner disk) is the neighbourhood $\mathcal{W}_2$ (respectively $\mathcal{W}_4$) of the trapped set inside $\widetilde{D}$, the vertical rectangle is the neighbourhood V_- of the incoming tail $\mathcal{T}_-$. The three ellipses with patterns indicate the sets $F^k(V_\rho)$ surrounding $F^k(\rho)$, $0 \leq k \leq 2$. Here the escape time $n_+(\rho) = 2$.

We will perform a local construction, based on a finite set of points $\rho \in \widetilde{D} \setminus V_-$. Consider a compact set $\mathcal{W}_4$ such that $\mathcal{W}_3 \Subset \mathcal{W}_4 \Subset \widetilde{D}$. Take any point $\rho \in \mathcal{W}_4 \setminus V_-$. We define its forward escape time by

$$n_+ = n_+(\rho) \overset{\text{def}}{=} \min\{k \geq 1, \ F^k(\rho) \notin \mathcal{W}_4\}.$$

Since $\rho \notin V_-$, this time is uniformly bounded from above. Besides, the forward trajectory $\{F^k(\rho), \ 0 \leq k \leq n_+\}$ is a set of mutually different points, with $F^{n_+}(\rho) \in \widetilde{A} \setminus \mathcal{W}_4$. Since $\widetilde{A} \setminus \mathcal{W}_4$ is relatively open, there exists a small neighbourhood V_ρ of ρ, such that the neighbourhoods $F^k(V_\rho)$, $0 \leq k \leq n_+ - 1$, are all inside $\widetilde{D} \setminus \mathcal{T}_-$, while $F^{n_+}(V_\rho) \in \widetilde{A} \setminus \mathcal{W}_4$ (see Fig. 7).

Take a smooth cutoff $\chi_\rho \in \mathcal{C}_c^\infty(V_\rho, [0,1])$, with $\chi_\rho = 1$ in a smaller neighbourhood $V_\rho' \Subset V_\rho$ of ρ, and consider its push-forwards

$$\chi_{\rho,k} \overset{\text{def}}{=} \begin{cases} \chi_\rho \circ F^{-k} & \text{on } F^k(V_\rho), \\ 0 & \text{outside } F^k(V_\rho), \end{cases} \qquad 0 \leq k \leq n_+.$$

The supports of the $\chi_{\rho,k}$, $0 \le k \le n_+ - 1$, are all contained in $\widetilde{D} \setminus \mathcal{T}_-$, while $\operatorname{supp} \chi_{\rho,n_+} \subset \widetilde{A} \setminus \mathcal{W}_4$. From there, we define

$$\text{the set} \quad W'_{\rho,+} \overset{\text{def}}{=} \bigcup_{k=0}^{n_+-1} F^k(V'_\rho) \subset \widetilde{D},$$

$$\text{the function} \quad g_{\rho,+} \overset{\text{def}}{=} \sum_{k=0}^{n_+} (k+1)\chi_{\rho,k}.$$

The function $g_{\rho,+}$ is smooth, and on $\widetilde{D}$ it satisfies

$$g_{\rho,+} \circ F - g_{\rho,+} = \chi_\rho \circ F + \sum_{k=0}^{n_+-1} \chi_{\rho,k} - (n_+ + 1)\chi_{\rho,n_+}.$$

The properties of the supports of the $\chi_{\rho,k}$ imply that

$$g_{\rho,+} \circ F - g_{\rho,+} \ge 0 \ \text{ in } \mathcal{W}_4, \qquad g_{\rho,+} \circ F - g_{\rho,+} \ge 1 \ \text{ in } W'_{\rho,+} \cap \mathcal{W}_4.$$

Since $\mathcal{W}_4 \setminus V_-$ is a compact set, we may extract a finite set of points $\{\rho_j \in \mathcal{W}_4 \setminus V_-\}_{j=1,\dots,J}$ such that $\bigcup_{j=1}^J W'_{\rho_j,+}$ is an open cover of $\mathcal{W}_4 \setminus V_-$. The sum

$$g_+ \overset{\text{def}}{=} \sum_{j=1}^J g_{\rho_j,+}$$

is smooth in $\mathcal{U}$, and satisfies the properties (A.1). Furthermore, for each ρ_j the function $g_{\rho_j,+}$ vanishes near $\mathcal{T}_-$, so there exists $V'_- \Subset V_-$ a neighbourhood of $\mathcal{T}_-$ such that all $g_{\rho_j,+}$, and also g_+, vanish on V'_-.

   Applying the same construction in the past direction, we construct a smooth function $\tilde{g}_-$ and a neighbourhood $V'_+ \Subset V_+$ of $\mathcal{T}_+$, such that

$$\forall \rho \in F(V'_+), \quad \tilde{g}_-(\rho) \equiv 0,$$
$$\forall \rho \in F(\mathcal{W}_3), \quad \tilde{g}_-(F^{-1}(\rho)) - \tilde{g}_-(\rho) \ge 0,$$
$$\forall \rho \in F(\mathcal{W}_3 \setminus V_+), \quad \tilde{g}_-(F^{-1}(\rho)) - \tilde{g}_-(\rho) \ge 1.$$

(notice that the sets $F(V'_+)$, $F(V_+)$ and $\mathcal{W}_3$ have the appropriate properties with respect to $F^{-1} : \widetilde{A} \mapsto \widetilde{D}$). The function $g_- \overset{\text{def}}{=} -\tilde{g}_- \circ F$ then satisfies

(A.2)
$$\forall \rho \in V'_+, \quad g_-(\rho) \equiv 0,$$
$$\forall \rho \in \mathcal{W}_3, \quad g_-(F(\rho)) - g_-(\rho) \ge 0,$$
$$\forall \rho \in \mathcal{W}_3 \setminus V_+, \quad g_-(F(\rho)) - g_-(\rho) \ge 1.$$

The function $g_0 \overset{\text{def}}{=} g_+ + g_-$ satisfies conditions in Lemma 4.5, and vanish on $\mathcal{W}_1 \overset{\text{def}}{=} V'_+ \cap V'_-$.

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