I.Scattering Resonances and Inverse Problems?

Workshop on Inverse Problem
MSRI

Maciej Zworski

UC Berkeley

July 27, 2009

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- General introduction to resonances (scattering poles).

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- Recent mathematical and experimental results in higher dimensions, a survey.

  http://math.berkeley.edu/∼zworski/ipw2.pdf/

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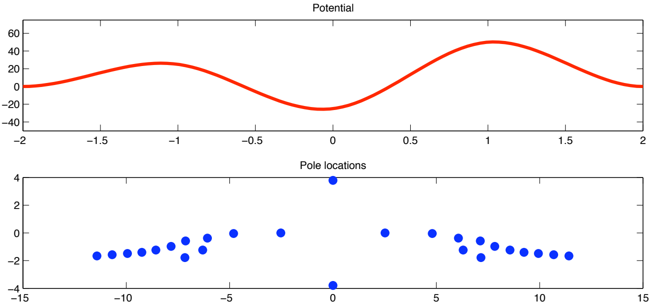

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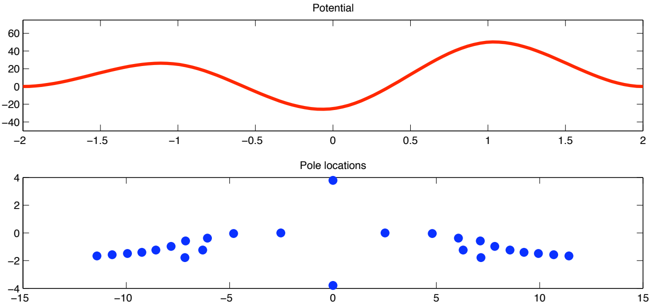

Codes by David Bindel

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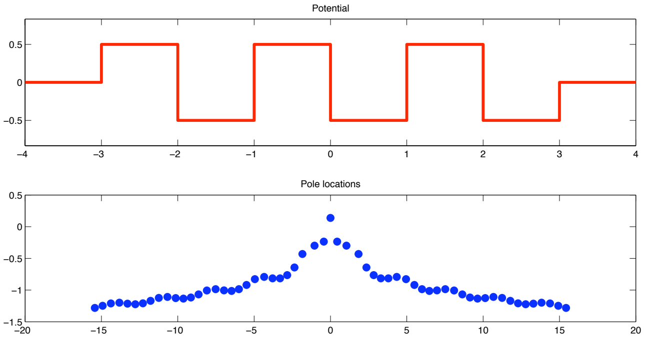

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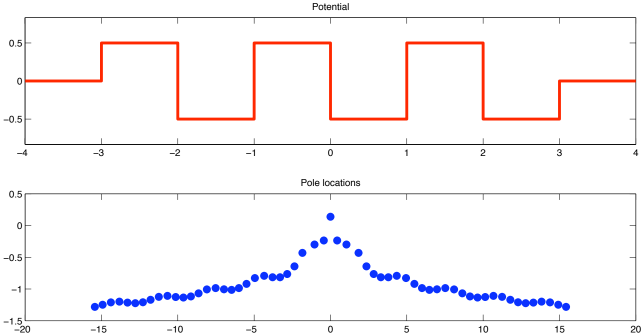

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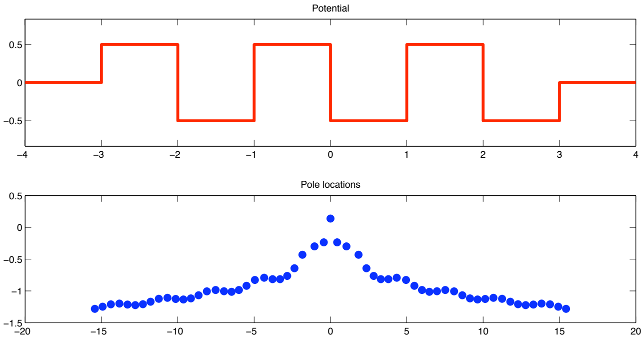

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http://www.math.univ-paris13.fr/~klopp/conf/

All of this is very well, but what do this blue dots really mean?

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Li Bai 750 "In bells of frost I heard the resonance 余响 (餘響) die"

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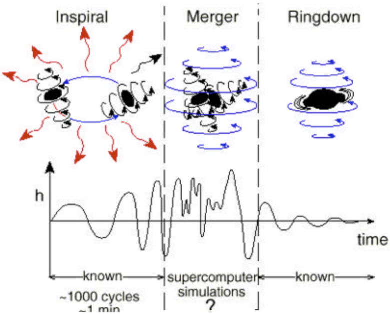

Picture Credit: Kip Thorne

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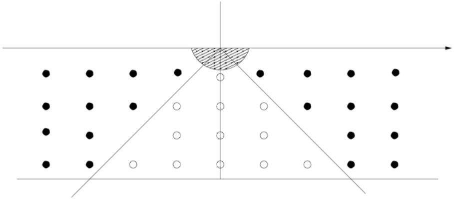

FIGURE 1. The lattice, $3^{-\frac{3}{2}}m(1-9\Lambda m^2)^{\frac{1}{2}}(\pm\mathbb{N}\pm\frac{1}{2}-\frac{i}{2}(\mathbb{N}_0+1/2))$, of pseudo-poles approximating resonances (dark dots) in a conic neighbourhood of the continuous spectrum.

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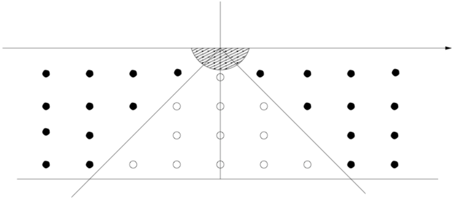

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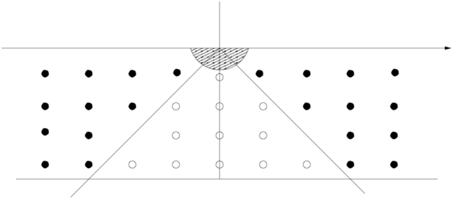

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Sá Barreto-Zworski 1996 (long earlier tradition in the physics literature)

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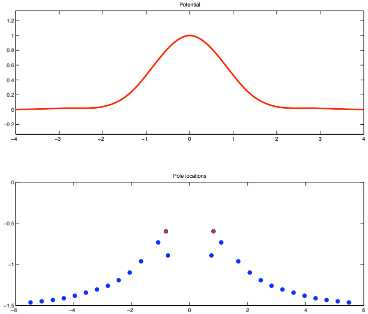

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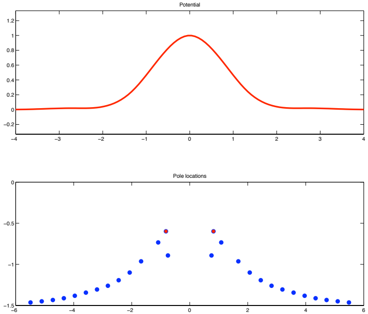

The $*$ resonances are generated by the unstable equilibrium points. The numerical false resonances come from the trancation of the support: the potential is approximated by a C^1 spline.

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And there are more concrete examples (Lecture 5).

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$$\sum_{j=1} c_j z_j^m p(z_j) = 0\,,$$

so if $v_m = [c_1 z_1^m, \cdots, c_n z_n^m]$ span $\mathbb{C}^n$, z_j's are the roots of $p(z)$.

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Wei-Majda-Strauss 1988 used it to compute resonances for 1D potentials by solving the wave equation numerically (Bindel's method, twenty years later, is direct and more efficient).

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The condition for large $|x|$ is called the outgoing condition.

A resonance at $\lambda = 0$ means that we have a bounded solution to

$$(-\partial_x^2 + V(x))u = 0.$$

Note that $V(x) = 0$ has a zero resonance (its only one).

Resonant state for the Eckhart barrier: $V(x) = sech^2(x)$

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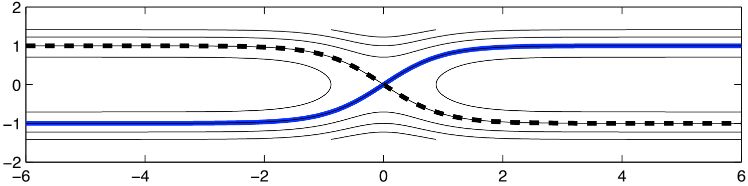
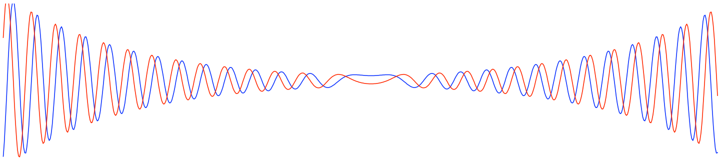
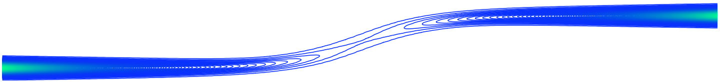
Phase space picture of the Eckart barrier



Real and imaginary parts of the first resonant state



Density plot of the FBI transform of the first resonant state



Significance of zero resonances: long time behaviour.

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A much improved version is due to Krieger-Schlag 2005.

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splinepot((j/4)*[0,2,-4,4,0],[-2,-1,0,1,2])
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The leading asymptotics of the number of resonances depend only on the support of V: Regge 1958, Zworski 1987, Froese 1997, Simon 2000.

But the dynamics is far from understood even in dimension one.

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Magnified view near the imaginary axis: the coupling constant gets larger so the well in the middle gets deeper generating more resonances.

Summary

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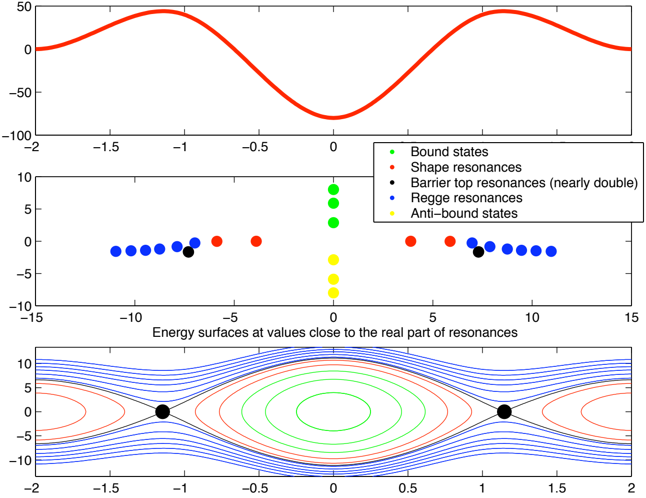

Bound states
Shape resonances
Barrier top resonances (nearly double)
Regge resonances
Anti–bound states
Energy surfaces at values close to the real part of resonances

A curious behaviour of zero resonances in nonlinear problems
Holmer-Zworski 2008:

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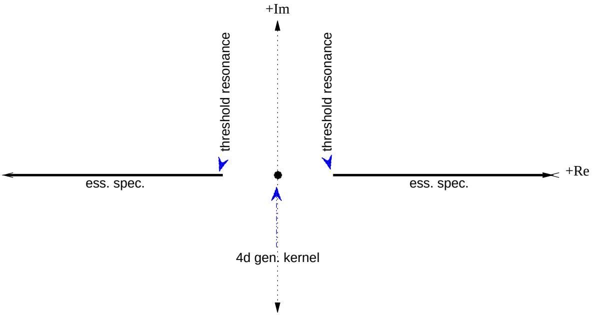

Now consider the case $q > 0$. Two of the elements of the kernel split away to $\pm q^{1/2}$, and the two threshold resonances dissolve into the continuum (become resonances on the nonphysical sheet).

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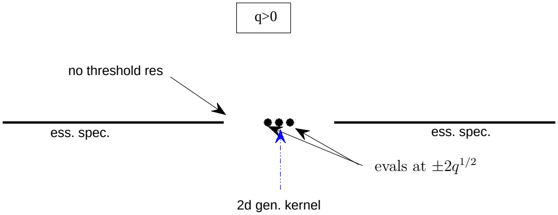

Surprisingly for $q < 0$, that is in the repulsive case, the threshold resonances become eigenvalues, approximately given by $\pm(1 - q^2)$, with eigenfuntions bounded by

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