Solitons in External Fields

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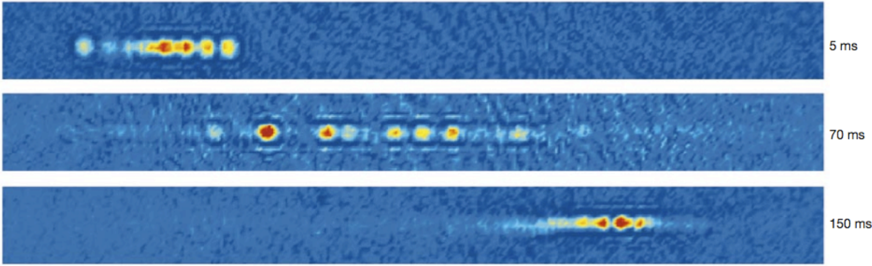
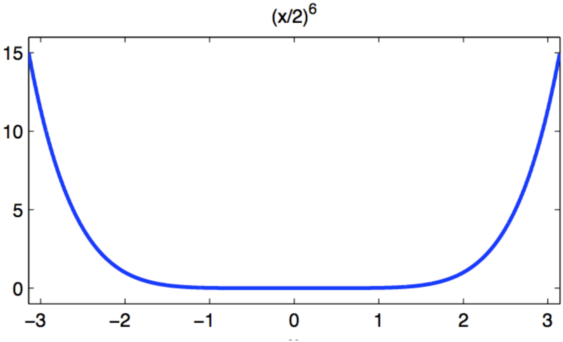
Motivation



Formation and propagation of matter-wave soliton trains,
K.E. Strecker et al **Nature**, May, 2002.

Nonlinear Schrödinger Equation

$$i\partial_t u + \frac{1}{2}\partial_x^2 u + u|u|^2 - V(x)u = 0\,,$$



Exact solution vs. effective dynamics

Numerical results: Potter 2010; the mathematics still not understood but similar results for mKdV Holmer-Perelman-Z 2010.

$$i\partial_t u + \frac{1}{2}\partial_x^2 u + u|u|^2 = 0\,,$$

This equation has traveling wave solutions:

$$u(x,t) = e^{i\gamma(t)}\mu\mathrm{sech}\,(\mu(x-a-vt))\,,$$

$$\mu > 0\,,\quad v, a, \gamma \in \mathbf{R}\quad,$$

$$\gamma(t) = \gamma + vx + (\mu^2 - v^2)t/2\,.$$

One of the amazing features in the stability of solitary waves in interaction. Collision of $\mu = 1$ and $\mu = 0.75$:

Martel-Merle 2009: stable interaction persists for some *non-integrable* gKdV equations!

$$iu_t = -u_{xx}/2 - |u|^2 u\,, \quad u(x,0) = 2\,\mathrm{sech}\,x\,.$$

$$u(x,t) = 2e^{it/2}\,\mathrm{sech}\,x\,\frac{4 + 3\,\mathrm{sech}^2(e^{4it} - 1)}{4 - 3\,\mathrm{sech}^4\,x\,\sin^2 2t}$$

This solution is obtained using the inverse scattering method.

One of the simplest models of an point impurity interacting with nonlinear waves is given by the Gross-Pitaevskii equation with the δ_0-function potential:

$$i\partial_t u + \tfrac{1}{2}\partial_x^2 u + q\delta_0 u + |u|^2 u = 0$$

We are interested in the effective dynamics of a soliton ($q = 0$):

$$u(x,t) = \mu \operatorname{sech}\left(\mu(x - vt - a)\right) \exp(i\gamma + ivx + i(\mu^2 - v^2)t/2)\,,$$

$$\mu > 0\,, \quad v \in \mathbb{R} \;\;,$$

The behaviour of solutions to

$$i\partial_t u + \frac{1}{2}\partial_x^2 u + q\delta_0 u + |u|^2 u = 0$$

depends dramatically on the size of q and on the initial parameters in

$$u(x,0) = \mu \operatorname{sech}(\mu(x-a))e^{ivx}\,.$$

For one thing, $q > 0$ is attractive, and $q < 0$ is replusive.

$$q = 3,\ v = 3,\ x_0 = -3.$$

Holmer-Marzuola-Z 2007 ($q < 0$), Holmer-Datchev 2008 ($q > 0$):

Theorem

Let $u(x,0) = e^{ixv}\,\mathrm{sech}(x - x_0)$ *and fix* δ, $2/3 < \delta < 1$. *If* $x_0 < -v^{1-\delta}$, *then for* $|x_0|/v + 1 \le t \le (1-\delta)\log v$, *we have*

$$u(t,x) = u_R(t,x) + u_T(t,x) + \mathcal{O}_{L^2_x}\left(\frac{1}{v^{1-\frac{3}{2}\delta}}\right) + \mathcal{O}_{L^\infty_x}\left(\frac{1}{\sqrt{t}}\right),$$

where

$$u_T(t,x) = \; A_T e^{i\varphi_T} e^{ivx + i(A_T^2 - v^2)t/2}\,\mathrm{sech}(A_T(x - x_0 - tv))$$

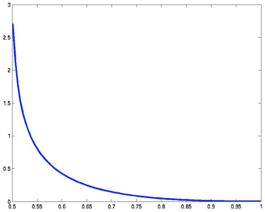
$$\varphi_T = \arg t_q(v) + \varphi_0(|t_q(v)|) + (1 - A_T^2)|x_0|/2v\,,$$

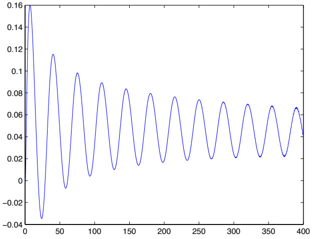
$$A_T(v/q) = (2|t_q(v)| - 1)_+\,,$$

where $t_q(v)$ *is transmission coefficient of the* $q\delta_0$ *potential, and* φ_0 *is an explicit functions. Similar expression involving the reflection coefficient is valid for* u_R.

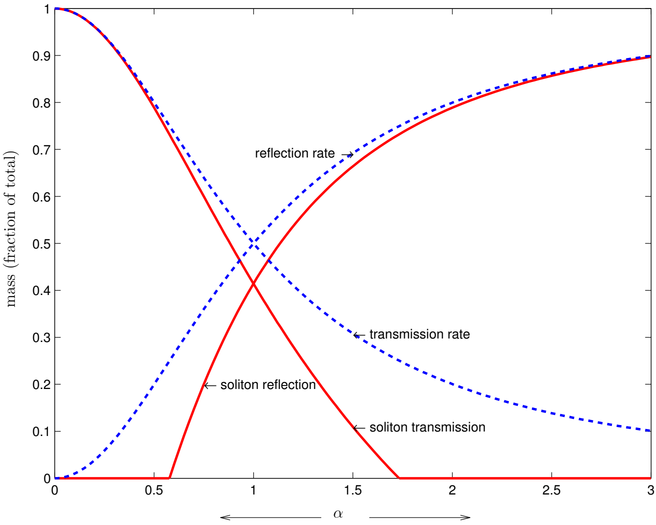
What is $\varphi(\alpha)$?

$$\varphi(\alpha) = \int_0^\infty \log\left(1 + \frac{\sin^2 \pi\alpha}{\cosh^2 \pi\zeta}\right) \frac{\zeta}{\zeta^2 + (2\alpha - 1)^2} d\zeta\,,$$

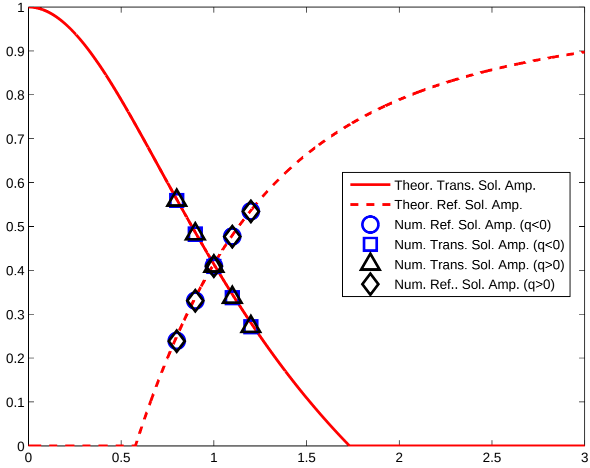




Notice that the plot on the right appears to be slowly converging to $\varphi(0.8) \simeq 0.045$. This plot represents the difference of two numbers of size ~ 100 by the end of the computation, and must therefore be taken with a grain of salt.



Soliton scattering rates compared with quantum scattering rates.



Numerical vefification of the theorem for attractive ($q > 0$) and repulsive potentials ($q < 0$).

Related results:

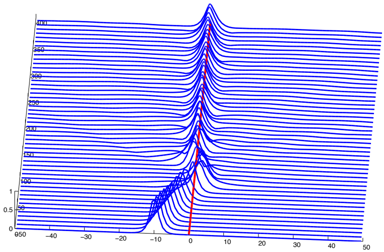
Perelman 2008: for the quintic NLS the soliton disappears (disperses) completely in the fast interaction with the delta function.

Perelman 2009: for the NLS with nonlinearity close to cubic, a fast soliton interacting with a stationary high mass soliton (δ_0-like) splits into two solitons described using the scattering matrix of the "high" soliton.
The results hold on the same time scale as the ones above. The analysis is however more subtle.

Abou Salem-Sulem 2010: resonant scattering of solitons through two delta functions (a more interesting scattering matrix).

(Some) Open Problems:

- Long time behaviour: remove the time limitation $t \leq C \log v$ for the soliton with speed v; a well known issue for cubic NLS.
- Trapped soliton: for a range of q's and v's Cao-Malomed 1995 observed soliton trapping:



How to explain this mathematically?

The situation changes dramatically when q is small:

$$q = -0.02\,, \quad v_0 = 0\,, \quad a_0 = -3\,.$$

Theorem (Holmer-Z 2007)

Suppose $u_0(x) = e^{ixv_0}\operatorname{sech}(x - a_0)$. *Then for*

$$|t| \leq \delta(v_0^2 + q)^{-1/2}\log(1/|q|)\,,$$

$$\|u(\bullet, t) - e^{i(\bullet - a(t))v(t)}e^{i\gamma(t)}\operatorname{sech}(\bullet - a(t))\|_{H^1} \lesssim |q|^{1-3\delta}$$

where a, v, γ *solve the following system*

$$\begin{cases} \dot{a} = v \\ \dot{v} = \frac{1}{2}q\partial_x(\operatorname{sech}^2)(a) \\ \dot{\gamma} = \frac{1}{2} + \frac{1}{2}v^2 + q\operatorname{sech}^2(a) + \frac{1}{2}qa\partial_x(\operatorname{sech}^2)(a) \end{cases}$$

with initial data $(a_0, v_0, 0)$.

This theorem was inspired by the works of on the motion of solitons in slowly varying external fields:

Qualitative results
Bronski-Jerrard(2000),
Keraani(2002),
D.M.A. Stuart(2007)

Quantitative results:

Fröhlich-Tsai-Yau (2002): NL Hartree equation
Fröhlich-Gustafson-Jonsson-Sigal (2004),(2006): NLS, NLH, $\cdots$
Fröhlich–Jonsson-Lenzmann (2007): dynamics of boson stars (as solitons)
W.K. Abou Salem (2007): time dependent slowly varying potentials.
Muñoz (2010): long time behaviour for special slowly varying potentials.
Pocovnicu (2010): effective dynamics small Toeplitz perturbations for nonlinear Szegö equation.

...

Here is a typical result:

Theorem (Fröhlich et al 2004)

Let $V(x) = W(hx)$, $W \in C^2$.
Then, for $0 \le t \le \delta/h$,

$$\|u(t,\bullet) - e^{i\bullet v(t)} e^{i\gamma(t)} \operatorname{sech}((\bullet - a(t)))\|_{H^1} \le Ch\,,$$

where a, v, *and* γ *solve*

$$\dot{a} = v + O(h^2)\,, \quad \dot{v} = -V'(a) + O(h^2)\,,$$

$$\dot{\gamma} = \frac{1}{2} + \frac{v^2}{2} - V(a) + O(h^2)\,.$$

The δ_0-theorem is in fact a special case of a more general

Theorem (Holmer-Z 2007)

Let $V(x) = W(hx)$, $W \in C^3$, *or* $V = h^2 W$, $W \in H^{-1}$. *Then, for* $0 \leq t \leq C\delta \log(1/h)/h$

$$\|u(t, \bullet) - e^{i\bullet v(t)} e^{i\gamma(t)} \operatorname{sech}((\bullet - a(t)))\|_{H^1} \leq C h^{2-\delta} ,$$

where a, v, *and* γ *solve*

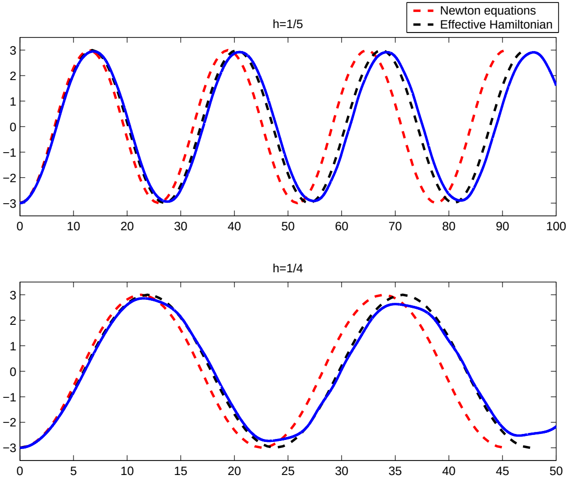
$$\dot{a} = v \,, \quad \dot{v} = -\frac{1}{2}(V * \operatorname{sech}^2)'(a) \,,$$

$$\dot{\gamma} = \frac{1}{2} + \frac{v^2}{2} - \frac{1}{2} V * \operatorname{sech}^2(a) + V * (x \tanh(x) \operatorname{sech}^2(x))(a) \,.$$

The main points are the more precise effective dynamics and better, $h \to h^2$, error bounds.

In fact, we build a more general framework in terms of Hamiltonian mechanics that in addition provides an explanation for the phase equation.

The improvement is numerically striking: $V(x) = -\operatorname{sech}^2(hx)$



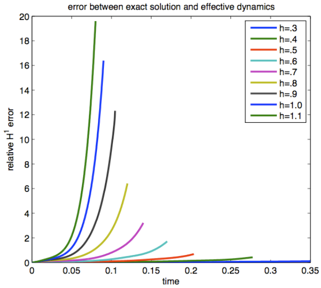
The potential and the initial data were

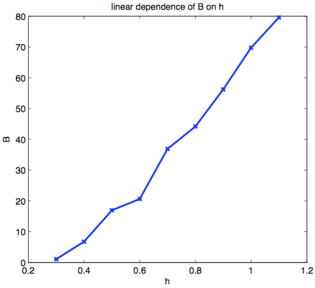
$$V(x) = -\operatorname{sech}^2((x+5)/4) - \operatorname{sech}^2((x-5)/4) - 0.1\operatorname{sech}^2(x/4)\,,$$

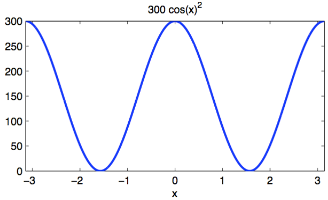
$$u_0(x) = e^{ix/10}\operatorname{sech}(x+8)\,.$$

This means that for $h = 1/4$ we see the limitations of the theorem!
And we also see heavy breathing!
The semiclassical results break down when h is not that small and the time is long.

The error terms in the theorem $h^2 exp(Bt)$, $B = Ch$, and hence the time of validity, $\log(1/h)/h$, are optimal:







Potter 2010

Our point of view is in terms of Hamiltonian mechanics and the group structure of the symmetries.

Let M be a manifold.

- Let $\omega : TM \times TM \to \mathbb{R}$ be a symplectic form (a closed nondegenerate 2-form).
- Let $H : M \to \mathbb{R}$ be a given function, the Hamiltonian

The Hamilton vector field $\Xi_H : M \to TM$ is defined by

$$\forall\, \varphi \in T_u M\,, \qquad \omega(\underbrace{\varphi}_{\substack{\uparrow\\ \in T_u M}}, \underbrace{\Xi_H(u)}_{\substack{\uparrow\\ \in T_u M}}) = d_u H(\varphi)$$

The Hamiltonian flow is

$$\dot{u} = \Xi_H(u)$$

Take $M = H^1(\mathbb{R})$, identify M with TM and define

$$\omega(u, v) = \mathrm{Im} \int u\bar{v}.$$

Let

$$H(u) = \frac{1}{4}\int |\partial_x u|^2 + \frac{1}{2}\int V|u|^2 - \frac{1}{4}\int |u|^4$$

Then

$$d_u H(\varphi) \equiv \frac{d}{ds} H(u + s\varphi)\Big|_{s=0} = \mathrm{Re} \int \left(-\frac{1}{2}\partial_x^2 u + Vu + |u|^2 u\right) \bar{\varphi}$$

and thus

$$\Xi_H(u) = \frac{1}{2} i\partial_x^2 u - iVu + i|u|^2 u$$

and the flow is NLS:

$$\partial_t u = \frac{1}{2} i\partial_x^2 u - iVu + i|u|^2 u$$

Let M be the four dimensional submanifold of H^1

$M = \{\ e^{i\gamma}e^{i(x-a)v}\mu\,\mathrm{sech}(\mu(x-a)) \mid \gamma\in\mathbb{R},\ v\in\mathbb{R},\ a\in\mathbb{R},\ \mu>0\ \}$

Now restrict H to M and ω to $TM \times TM$.

$$H\Big|_M = \frac{\mu v^2}{2} - \frac{\mu^3}{6} + \frac{\mu^2}{2} V * (\mathrm{sech}^2(\mu\bullet))(a)$$

and

$$\omega\Big|_M = \mu dv \wedge da + v d\mu \wedge da + d\gamma \wedge d\mu$$

We see that $\omega|_M$ is nondegenerate, and hence $(M, \omega\Big|_M)$ is a 4-dimensional symplectic submanifold of (H^1, ω).

Suppose we can guarantee that the solution $u(x,t)$ remains close to M for all times, i.e.

$$u(x,t) \approx e^{i\gamma(t)} e^{i(x-a(t))v(t)} \mu(t)\,\mathrm{sech}(\mu(t)(x-a(t)))$$

for some parameters $\gamma(t)$, $v(t)$ $\mu(t)$, $a(t)$.

The simple geometric insight is: The propagation of $(\gamma(t), v(t), \mu(t), a(t))$ in time should be the Hamiltonian flow of $H|_M$ with respect to $\omega|_M$.

Recall the restricted Hamiltonian and symplectic form

$$H\Big|_M = \frac{\mu v^2}{2} - \frac{\mu^3}{6} + \frac{\mu^2}{2} V * (\operatorname{sech}^2(\mu\bullet))(a)$$

$$\omega\Big|_M = \mu dv \wedge da + v d\mu \wedge da + d\gamma \wedge d\mu$$

The corresponding flow is:

$$\begin{aligned} \dot{a} &= v\,, & \dot{v} &= -\frac{\mu^2}{2} V' * (\operatorname{sech}^2(\mu\bullet))(a)\,, \\ \dot{\mu} &= 0\,, & \dot{\gamma} &= \frac{1}{2}v^2 + \frac{1}{2}\mu^2 - \mu V * (\operatorname{sech}^2(\mu\bullet))(a) \\ & & &\quad + \mu V * (x \operatorname{sech}^2(x)\tanh(x)\Big|_{x=\mu\bullet})(a)\,. \end{aligned}$$

The evolution of a and v is simply the Hamiltonian evolution of

$$(v^2 + \mu^2 V * \operatorname{sech}^2(\mu\bullet)(a))/2, \quad \mu = \text{const}$$

The more mysterious evolution of the phase γ is now explained.

Recently, Datchev-Ventura have applied our method to the Hartree equation

$$\begin{cases} i\partial_t u = -\frac{1}{2}\Delta u + V(x)u - \left(|u|^2 * |x|^{-1}\right)u \\ u(x,0) = u_0(x) \end{cases}$$

where $u : \mathbb{R}^{3+1} \to \mathbb{C}$, improving Fröhlich-Tsai-H.T. Yau (2002). Motivated by earlier works they allowed more flexibility as far as the closeness of the initial data to the soliton is concerned (and got the same improvement for the case we considered). The crucial new component was the careful spectral analysis of the linearized operator by Lenzmann 2009.

Theorem (Datchev-Ventura 2009)

Fix constants $0 < c_1$, $0 \le 2\delta \le \delta_0 < 3/4$ *and* $(v_0, a_0) \in \mathbb{R}^3 \times \mathbb{R}^3$. *Suppose that*

$$\|u_0 - e^{iv_0\cdot(x-a_0)}\eta(x-a_0)\| = \epsilon < h^{1/2+\delta_0} .$$

Then for

$$0 \le t \le \frac{c_1}{h} + \frac{\delta|\log h|}{c_2 h} ,$$

we have

$$\|u(x,t) - e^{iv(t)\cdot(x-a(t))}e^{i\gamma(t)}\eta(x-a(t))\|_{H^1} \le h^{-\delta}\tilde{\epsilon} ,$$

where

$$\tilde{\epsilon} = \epsilon + h^2 ,$$

and where $a(t) \in \mathbb{R}^3$, $v(t) \in \mathbb{R}^3$, $\gamma(t) \in \mathbb{R}$, *evolve according to the ODEs obtained by symplectic projection and perturbed by* $\mathcal{O}(\tilde{\epsilon}^2)$.

The simplest setting for the study of multiple solitons is provided by the modified Korteveg-de Vries (mKdV) equation:

$$\partial_t u = -\partial_x(\partial_x^2 u + 2u^3)\,,$$

and its Hamiltonian perturbations:

$$\partial_t u = -\partial_x(\partial_x^2 u + u^3 - b(x,t)u)$$

with $u : \mathbb{R}^{1+1} \to \mathbb{R}$.

If

$$b, \partial_x b, \partial_t b \ \in L_t^\infty(L_x^\infty \cap L_x^2)$$

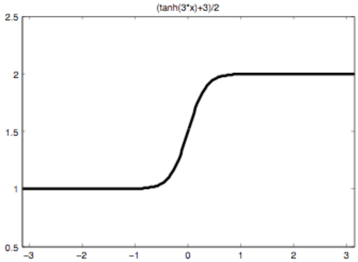
then the equation is locally well-posed in H^1 by a contraction argument using the local smoothing estimates of Kenig-Ponce-Vega (1993).

For b slowly varying and small, mKdV (and the much nastier KdV) the effective dynamics of single solitons was previously studied by Dejak-Jonsson(2006) and Dejak-Sigal (2006).

Stronger results for mKdV follow the same strategy, while more subtle stronger results for KdV have been obtained by Holmer and by Muñoz.

Muñoz 2010:

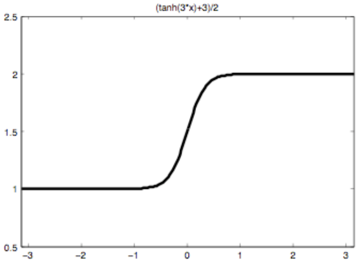
$$u_t + (a(hx)u^m + u_{xx})_x = 0$$



$$m = 1\,, \quad a(x) = \frac{1}{2}(\tanh(3x) + 3)\,, \quad h_{\text{eff}} \sim \frac{1}{30}$$

Muñoz 2010:

$$u_t + (a(hx)u^m + u_{xx})_x = 0$$



$$m = 1\,, \quad a(x) = \frac{1}{2}(\tanh(3x) + 3)\,, \quad h_{\text{eff}} \sim \frac{1}{10}$$

$$u(x,t) = \frac{3}{2} c\, \mathrm{sech}^2 \left(\frac{\sqrt{c}}{2}(x - x_0 - c) \right) + O(\sqrt{h})\,.$$

c determined by energy, x_0 not specified but $t \longrightarrow \infty$.

Hamiltonian structure:

Let $J = \partial_x$ and

$$J^{-1}f(x) = \left(\frac{1}{2}\int_{-\infty}^{x} - \frac{1}{2}\int_{x}^{+\infty}\right) f(y)\,.dy$$

Define the symplectic form

$$\omega(u,v) = \langle u, J^{-1}v\rangle$$

and the Hamiltonian:

$$H(u) = \frac{1}{2}\int |\partial_x u|^2 - |u|^4 + bu^{\ 2}\,.$$

Then the Hamilton flow for perturbed mKdV is

$$\partial_t u = JH'(u) = -\partial_x(\partial_x^2 u + 2u^3 - bu)$$

The family of 2-soliton solutions is parametrized by position constants are $a = (a_1, a_2)$ and scale constants are $c = (c_1, c_2)$.

$$q(x, a, c) = \frac{\det M_1}{\det M}$$

where

$$M = \begin{bmatrix} \frac{1+\gamma_1^2}{2c_1} & \frac{1+\gamma_1\gamma_2}{c_1+c_2} \\ \frac{1+\gamma_1\gamma_2}{c_1+c_2} & \frac{1+\gamma_2^2}{2c_2} \end{bmatrix}, \qquad M_1 = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{M} & \begin{matrix}\gamma_1 \\ \gamma_2\end{matrix} \\ \hline 1 & 1 & 0 \end{array}\right]$$

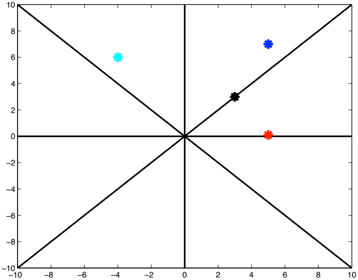
and

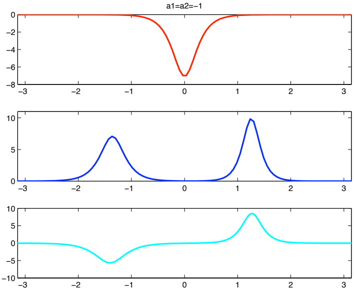
$$\gamma_1 = e^{-c_1(x-a_1)}, \quad \gamma_2 = -e^{-c_2(x-a_2)}.$$

Then remarkably the following solves mKdV:

$$u(x, t) = q(x, a_1 + c_1^2 t, a_2 + c_2^2 t, c_1, c_2)$$

Singular behaviour at $c_1 = \pm c_2$.





In particular, at $c_1 = 0$, $c_2 = c > 0$ we recover the 1-soliton:

$$\eta(x, a, c) = c\,\mathrm{sech}(c(x - a))\,.$$

Theorem (Holmer-Perelman-Z (2009) 2-soliton case)

Suppose that $a(t)$, $c(t)$ satisfy

$$\dot{a}_j = c_j^2 - \frac{1}{2}\partial_{c_j} B(a,c,t)\,, \quad \dot{c}_j = \frac{1}{2}\partial_{a_j} B(a,c,t)\,, \qquad j = 1,2.$$

with initial data $a(0) = a_0$, $c(0) = c_0$, where

$$B(a,c,t) = \int b(x,t) q^2(x,a,c)\, dx\,.$$

If $0 < \delta < |c_1(t) \pm c_2(t)| < \delta^{-1}$,
then for $t \leq \delta h^{-1} \log(1/h)$, the solution $u(t)$ to mKdV with initial data

$$u(\cdot, 0) = q(\cdot, a_0, c_0)$$

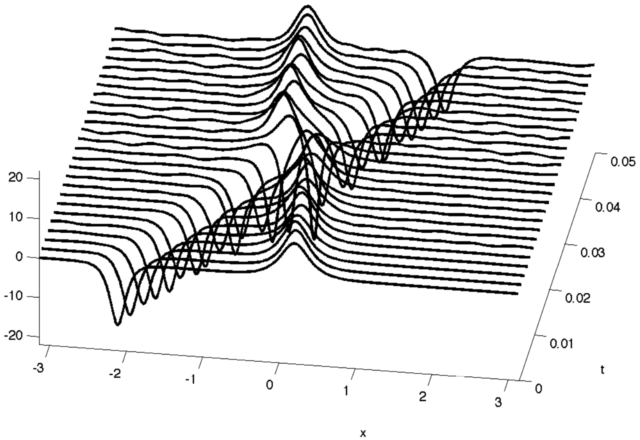
satisfies

$$\|u(\cdot) - q(\cdot, a(t), c(t))\|_{H^2} \leq Ch^{2-\delta}\,.$$

Here is an example of soliton motion in an external field:

$$b = 100\cos^2(x + 1 - 10^3 t) + 50\sin(2x + 2 + 10^3 t)\,,$$

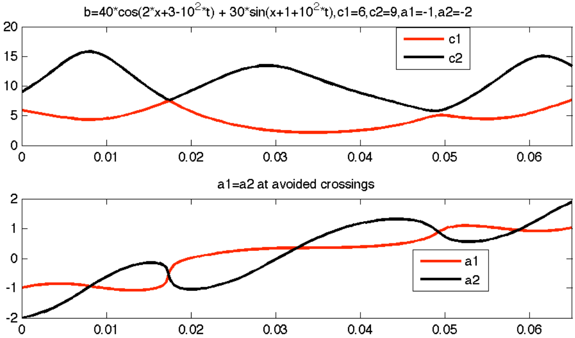
$$c_1 = 6, \quad c_2 = -11, \quad a_1 = 0, \quad a_2 = -2\,.$$



Comparison with the effective dynamics:

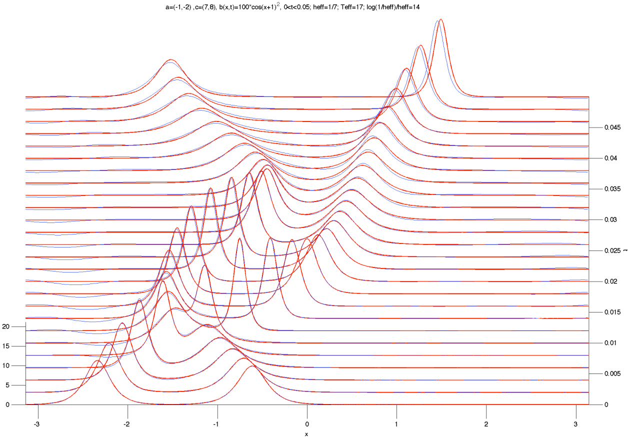
$$h_{\mathrm{eff}} \approx 1\,,\ \ t_{\mathrm{eff}} \approx 50 \gg \log(1/h)/h$$

The case to which the the theorem does not quite apply:



Understanding of the avoided crossing should perhaps be possible by some simple conceptual method...

$$a = (-1, -2)\,, \quad c = (7, 8)\,,$$
$$b(x, t) = 100\cos^2 x\,, \quad t \leq 0.05$$

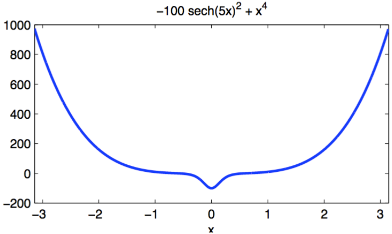


Although dynamics for (a_1, c_1) and (a_2, c_2) looks interesting it should be simplified as the "interesting" features are on irrelevant scales of $\exp(-1/h)$:

Although dynamics for (a_1, c_1) and (a_2, c_2) looks interesting it should be simplified as the “interesting” features are on irrelevant scales of $\exp(-1/h)$:

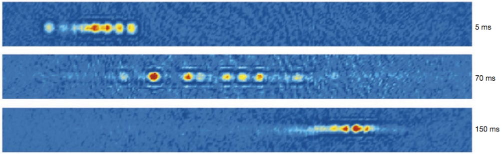
Numerical results of Potter 2010 show that the same results apply to multiple solitons and to the case of NLS.

In fact, one has to work to see the failure of effective dynamics and to check that the errors are optimal:



Outlook

- N-solitons for mKdv
- N-solitons ($N = 2$ and then more) for cubic NLS
- Explanation (or elimination) of avoided crossing.
- Longer times and the analysis of radiation.
- Long time behaviour for traps ($V(x) = x^2$).
- ...



We now sketch the proof of Theorem above:

Let $V(x) = W(hx)$, $W \in C^3$. Then, for $0 \leq t \leq C\delta \log(1/h)/h$

$$\|u(t,\bullet) - e^{i\bullet v(t)} e^{i\gamma(t)} \operatorname{sech}((\bullet - a(t)))\|_{H^1} \leq C h^{2-\delta} \,,$$

where a, v, and γ solve

$$\dot{a} = v \,, \quad \dot{v} = -\frac{1}{2}(V * \operatorname{sech}^2)'(a) \,,$$

$$\dot{\gamma} = \frac{1}{2} + \frac{v^2}{2} - \frac{1}{2} V * \operatorname{sech}^2(a) + V * (x \tanh(x) \operatorname{sech}^2(x))(a) \,.$$

The manifold of solitons $M \subset H^1$ is the orbit of $\operatorname{sech} x$ under the group $G = \{g\}$, where $g = (a, v, \gamma, \mu) \in \mathbb{R}^3 \times \mathbb{R}_+$

$$g \cdot \varphi(x) = e^{i\gamma} e^{i(x-a)v} \mu \varphi(\mu(x-a)) .$$

We now reparameterize the evolution $u(t)$ as follows. Given the solution $u(t)$, define $\tilde{u}(t)$ as:

$$u(x,t) = e^{i\gamma} e^{i(x-a)v} \mu \tilde{u}(\mu(x-a), t)$$

That is, we pull-back the solution $u(t)$ by a group element g to obtain a function $\tilde{u}$ that we compare with $\operatorname{sech} x$.

Set

$$w(x,t) = \tilde{u}(x,t) - \operatorname{sech}\, x$$

By an implicit function theorem argument, the parameters μ, γ, v, a can be chosen so that w satisfies the symplectic orthogonality conditions

$$\mathrm{Im} \int w \left\{ \begin{array}{c} i\eta \\ \partial_x \eta \\ ix\eta \\ \partial_x (x\eta) \end{array} \right\} dx = 0\,, \quad \eta(x) = \operatorname{sech} x\,.$$

Here, $\mathfrak{g} = \operatorname{span}\{-\partial_x,\ ix,\ i,\ \partial_x \cdot x\}$ is the Lie algebra associated with G.

Since the action of G is conformally symplectic ($g^*\omega = \mu(g)\omega$),

$$e^{i\gamma} e^{i(x-a)v} \mu \operatorname{sech}(\mu(x-a))$$

is the symplectic orthogonal projection of $u(t)$ onto the manifold of solitons M, and we expect the parameters a, γ, v, μ, to evolve according to the effective Hamiltonian.

Using the equation for u, definition of $\tilde{u}$ as a pull-back of u, and $w = \tilde{u} - \eta$, we find the equation for w: (recall $\eta = \text{sech}$)

$$\partial_t w = X\eta + iF\eta + Xw + iFw - \mu^2 \mathcal{L}w + i\mu^2 \mathcal{N}w$$

where

$$X = (\cdots)(-\partial_x) + (\cdots)ix + (\cdots)i - (\cdots)\partial_x \cdot x$$

with coefficients involving a, v, γ, μ and $\dot{a}, \dot{v}, \dot{\gamma}, \dot{\mu}$

$F =$ a specific function of x, μ, a involving V

$$\mathcal{L}w = -\frac{1}{2}\partial_x^2 w - 2\eta^2 w - \eta^2 \bar{w} + \frac{1}{2}w \text{ (linearized op)}$$

$$\mathcal{N}w = 2|w|^2\eta + \eta w^2 + |w|^2 w \text{ (nonlin terms)}$$

Properties:

- $X = 0 \Leftrightarrow$ our equations of motion hold
- $iF\eta$ is symplectically orthogonal to M.

$$\partial_t w = X\eta + iF\eta + Xw + iFw - \mu^2 \mathcal{L} w + i\mu^2 \mathcal{N} w \qquad (*)$$

By pairing the equation with each element of $\mathfrak{g} \cdot \eta$ and using the symplectic orthogonality conditions

$$\omega(\partial_t w, \mathfrak{g} \cdot \eta) = 0, \qquad \omega(iF\eta, \mathfrak{g} \cdot \eta) = 0,$$

we get

$$|X| \lesssim h^2 \|w\|_{H^1} + \|w\|_{H^1}^2$$

which, if $\|w\|_{H^1} \lesssim h^2$, says that are ODEs are satisfied with error $\sim h^4$.

To prove $\|w\|_{H^1} \lesssim h^2$, we note that the largest forcing term in (*) is $F\eta \sim h^2$. We introduce a correction term w_1 so that

$$\partial_t w_1 + \mu^2 \mathcal{L} w_1 = iF\eta + \mathcal{O}(h^3)$$

and then seek to control $w - w_1$.

For this, we use a quadratic approximation to the Lyapunov functional employed by M. Weinstein (1986) in his study of orbital stability of NLS.

$$\operatorname{Re}\langle \mathcal{L}(w - w_1), w - w_1 \rangle$$

By spectral estimates we have a lower bound in terms of $\|w - w_1\|_{H^1}^2$, and the upper bound is computed from

$$\partial_t \operatorname{Re}\langle \mathcal{L}(w - w_1), w - w_1 \rangle$$

by plugging in the equation for w and carrying out integration by parts manipulations.

The bounds on the ODEs and the bounds on $\|w\|_{H^1}^2$ are bootstrapped.

End proof of Theorem.