BREATHING PATTERNS IN NONLINEAR RELAXATION

JUSTIN HOLMER AND MACIEJ ZWORSKI

Abstract. In numerical experiments involving nonlinear solitary waves propagating through nonhomogeneous media one observes "breathing" in the sense of the amplitude of the wave going up and down on a much faster scale than the motion of the wave – see Fig. 2 below. In this paper we investigate this phenomenon in the simplest case of stationary waves in which the evolution corresponds to relaxation to a nonlinear ground state. The particular model is the popular δ_0 impurity in the cubic nonlinear Schrödinger equation on the line. We give asymptotics of the amplitude on a finite but relevant time interval and show their remarkable agreement with numerical experiments, see Fig. 1. We stress the nonlinear origin of the "breathing patterns" caused by the selection of the ground state depending on the initial data, and by the non-normality of the linearized operator.

1. Introduction

We study a simple model of relaxation to a nonlinear ground state. Our equation is the one dimensional nonlinear cubic Schrödinger equation with a small delta potential:

$$i\partial_t u + \tfrac{1}{2}\partial_x^2 u + q\delta_0(x)u + u|u|^2 = 0\,, \tag{1.1}$$

where $0 < |q| \ll 1$. The nonlinear ground state minimizes the corresponding energy (2.1) for a prescribed L^2 norm, and is explicitly given by

$$v_\lambda(x) = \lambda \operatorname{sech}(\lambda|x| + \tanh^{-1}(q/\lambda))\,, \quad \|v_\lambda\|_2^2 = 2(\lambda - q)\,, \quad \lambda > |q|\,. \tag{1.2}$$

A simple rescaling allows the reduction to the case $\lambda = 1$ and we obtain

Theorem 1. *Suppose that $u(x,t)$ solves* (1.1), *$u(x,0) \in H^1(\mathbb{R})$ is real and even, and that*

$$\|x^k \partial_x^\ell w_0\|_{L^\infty((0,\infty))} \le C_{kl}|q|\,, \quad w_0(x) \stackrel{\text{def}}{=} u(x,0) - v_1(x)\,, \quad k\,,\ell \in \mathbb{N}\,. \tag{1.3}$$

Then for $\lambda = 1 + \int_{\mathbb{R}} w_0(x)v_1(x)dx$ and $0 \le t \ll |q|^{-1/2}$, we have

$$\|u(x,t) - e^{it\lambda^2/2}\Big(v_\lambda(x) + w(\lambda x, \lambda^2 t)\Big)\|_{H^1_x} \le C|q|^{3/2} + Ct^2q^2\,, \tag{1.4}$$

where $w(x,t)$ is given explicitly in (5.6). *In particular, for $1 \ll t \le C|q|^{-2/7}$,*

$$u(0,t) = e^{it\lambda^2/2}\left(\lambda - \sqrt{\frac{2}{\pi t}}e^{i(\lambda^2 t/2 + \pi/4)}\int_{\mathbb{R}} w_0(x)dx\right) + \mathcal{O}\left(\frac{q}{t^{3/2}}\right)\,. \tag{1.5}$$

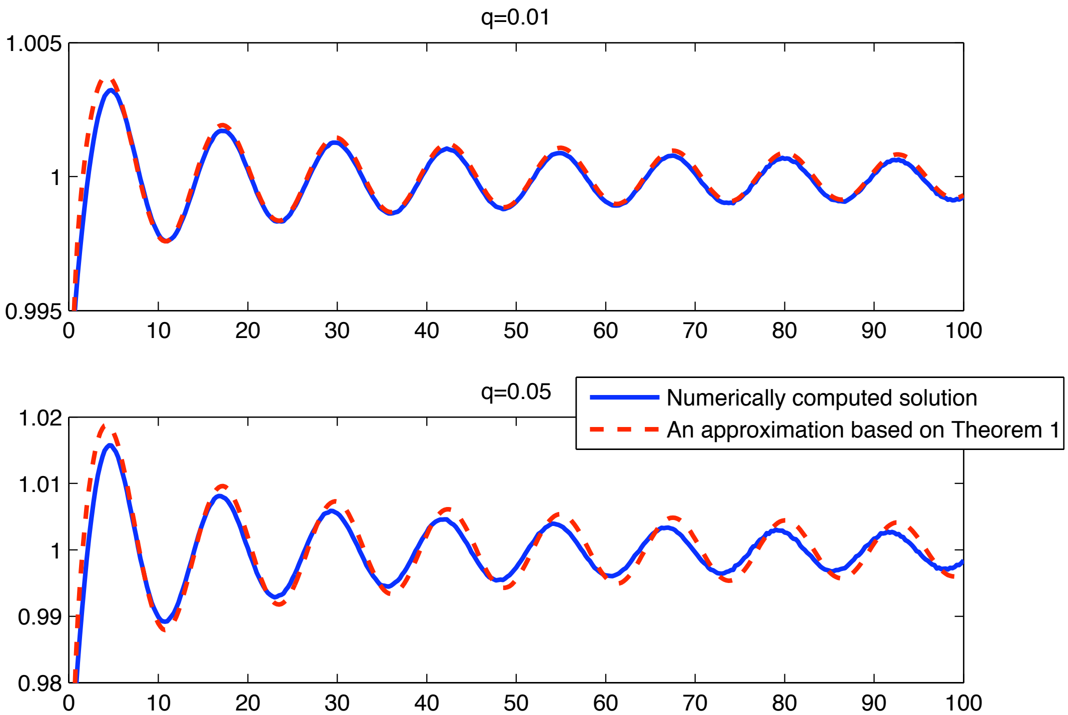

FIGURE 1. Breathing patterns for $u(x,0) = \operatorname{sech}(x/(1+q))/(1+q)$ (the initial data is rescaled so that the ground state to which it relaxes is $v_1(x)$): the plots show $|u(0,t)|$ and the asymptotic prediction (1.5) given in Theorem 1, for $q = 0.05$ and $q = 0.01$. The agreement is remarkably good for times much longer than given in the theoretical result.

The conditions on w_0 in (1.3) can be weakened considerably, and in particular we only need estimates for $k\,,\ell \leq N$ for some N. Theorem 2 below gives a statement which depends only on $\|w_0\|_{H^1}$ being small and the more explicit results in Theorem 1 come from our close analysis of the propagator $\exp(-it\mathcal{L}_{q,\lambda})$ appearing in (3.2). As explained below our motivation comes from the study of solitons and the intial data in which we are most interested is $u(x,0) = \operatorname{sech} x$. For that case, the comparison of the theorem with numerical results is shown in Fig.1.

The reduction to the case $\lambda = 1$, mentioned before the statement of Theorem 1 is straightforward: let $\tilde{u}(x,t) = \lambda^{-1}u(\lambda^{-1}x, \lambda^{-2}t)$. Then $\tilde{u}$ solves (1.1) with q replaced by $\lambda^{-1}q$, $i\partial_t\tilde{u} + \frac{1}{2}\partial_x^2\tilde{u} + q\lambda^{-1}\delta_0(x)\tilde{u} + \tilde{u}|\tilde{u}|^2 = 0$. Now, if we suppose the theorem holds when applied to $\tilde{u}$ replacing u and q/λ replacing q, then we can deduce the theorem in its current form. Thus, it suffices to prove the $\lambda = 1$ case.

In the remainder of the introduction we will discuss our motivation and relations to existing literature, possible approaches to obtaining finer asymptotics, and a simple example of a breathing pattern for nonnormal operators.

1.1. **Motivation.** Mathematical studies of relaxation to ground states for nonlinear Schrödinger equations have been recently conducted in a number of mathematical papers, see Soffer-Weinstein [26], Tsai-Yau [28], Gang-Sigal [11], Gang-Weinstein [12], and references given there. The particular focus is on the behaviour as $t \to \infty$ (genuine relaxation in the sense of pure mathematics) and the allowed non-linearities typically exclude standard examples from the physical literature. For the cubic nonlinear Schrödinger equation (NLS) on the line, that is for (1.1) with $q = 0$, the nonlinear relaxation can be studied in great detail using methods of inverse scattering theory pioneered by Zakharov-Shabat [31] – see Deift-Its-Zhou [7], Deift-Zhou [6], and [14, Appendix B] for recent advances and references. The case of $q \neq 0$ with even initial data is also in principle accessible by these methods as was pointed out by Fokas [9].

The goals of this paper are more modest: we explain a phenomenological fact occuring on shorter time scales for a simple physically relevant model of NLS with small δ impurities. Numerous references for this model in the physics literature can for instance be found in [4] (where it is used to model more realistic narrow traps), [5],[13], and [22]. See also [24] for a recent numerical study and further pointers to the literature.

Our motivation came from observing a common phenomenon illustrated in Fig.2. In [16] we have shown that the solution of (1.1) with $u(x, 0) = \operatorname{sech}(x - a_0)e^{ixv_0}$ satisfies

$$\|u(t, \bullet) - e^{i(\bullet - a(t))v(t)}e^{i\gamma(t)} \operatorname{sech}(\bullet - a(t))\|_{H^1(\mathbb{R})} \leq C|q|^{1-3\delta}\,, \tag{1.6}$$

for $0 < t < \delta(v_0^2 + |q|)^{-1/2} \log(1/|q|)$, $0 < |q| \ll 1$, and where a, v, and γ solve the following system of equations

$$\begin{gathered} \frac{d}{dt}a = v\,, \quad \frac{d}{dt}v = \frac{1}{2}q\partial_x(\operatorname{sech}^2)(a)\,, \\ \frac{d}{dt}\gamma = \frac{1}{2} + \frac{v^2}{2} + q\operatorname{sech}^2(a) + \frac{1}{2}qa\partial_x(\operatorname{sech}^2)(a)\,, \end{gathered} \tag{1.7}$$

with initial data $(a_0, v_0, 0)$ (please note that the sign convention for q has been changed here). As was pointed out there, as seen from explicit constants in coercive estimates, these asymptotics require $q \lesssim 0.01$ to hold accurately. From the semiclassical point of view $q = h^2$, where h is the *effective Planck constant* of the problem, so that means $h \lesssim 0.1$ – see [17] for an explanation of this scaling philosophy.

In Fig. 2 the dashed line shows the motion of the center of the soliton in the case of $q = 0.05$ (which is a borderline case for the applicability of (1.6)). We see oscillations with the period proportional to $q^{-1/2}$ in agreement with (1.7). The continuous line

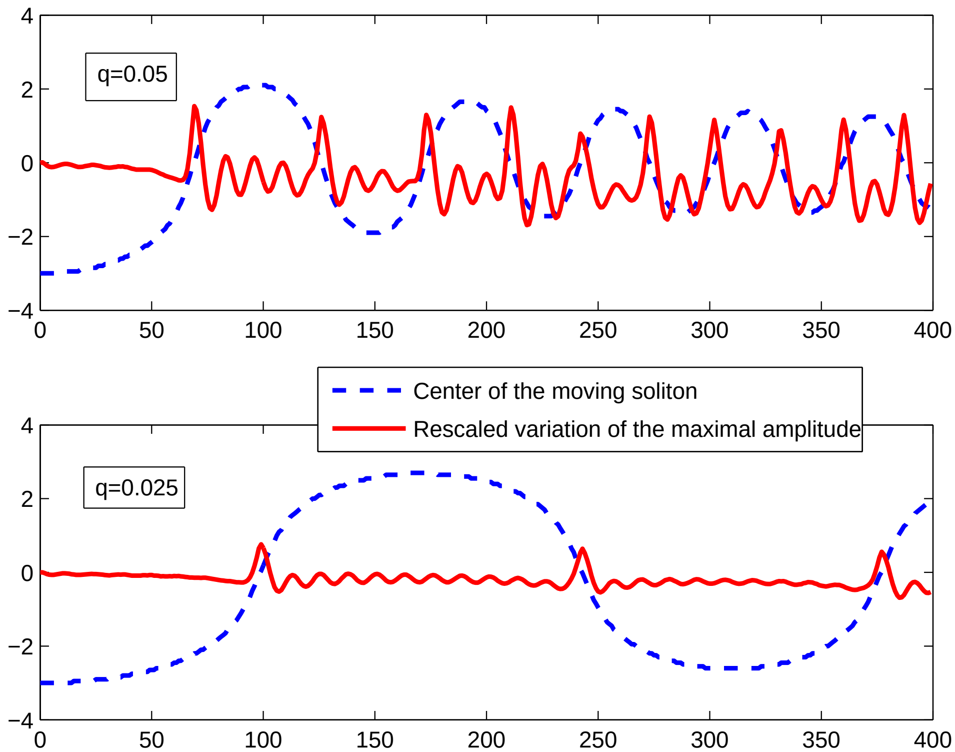

FIGURE 2. Fast and slow oscillations in the motion of the soliton with initial condition $\operatorname{sech}(x+3)$ moving in the field of $-\delta_0(x)/20$ (top figure) and $-\delta_0(x)/40$ (bottom figure). The center of the soliton oscillates around $x = 0$ with a much larger period than the rescaled amplitude $A(t) = 30(|u(a(t), t)| - 1)$, where $a(t)$ is the center of the moving of the moving soliton. The periods of the fast oscillations are close.

shows the oscillation of the amplitude: we look at the deviations of the value of the solution at the maximum of $|u(x, t)|$ in x from 1, the maximal value of the absolute value of the soliton solution. The oscillations are much faster than the oscillations of the center of the soliton and the period is close to being fixed. Numerical observations suggest that the period is almost independent of q.

This "breathing" behaviour is even more striking in movies of numerical solutions (see for instance the last movie in `http://math.berkeley.edu/~zworski/msg.pdf`). The slowing down of the soliton and the shedding of its mass seem closely related to these breathing patterns.

As the first step to understand solitons moving in nonhomogeneous media we study the stationary case, that is (1.1) with initial data given by $u(x, 0) = \operatorname{sech} x$. The results of [16] recalled in (1.6) and (1.7) show that

$$\|u(t,\bullet) - e^{i\gamma(t)}\operatorname{sech}(\bullet)\|_{H^1(\mathbb{R})} \leq C|q|^{1-3\delta},$$

for $0 < t < \delta|q|^{-1/2}\log(1/|q|)$, and $\gamma(t) = (1/2+q)t$. In fact, an application of the method of [16] shows that for some $\tilde{\gamma}(t)$,

$$\|u(t,\bullet) - e^{i\tilde{\gamma}(t)}\operatorname{sech}(\bullet)\|_{H^1(\mathbb{R})} \leq C|q|,$$

for all times. Here we could replace sech with v_λ for any $\lambda = 1 + \mathcal{O}(q)$.

Hence the breathing patterns must involve higher order asymptotics and since $|q| = h^2$, the natural next step is $|q|^{3/2} = h^3$. Theorem 1 provides that next step on a time scale which allows seeing a large number of oscillations. The numerical experiments show a very good agreement with asymptotics provided by (1.5) and suggest that they are valid for times longer than $t \ll |q|^{-1/2}$.

Finer asymptotics might be possible if one adapts some of the methods of [26], [11], and [12], but it is not clear which direction should be taken for the efficient study of moving solitons. We opted for the simplest at this early stage.

1.2. **Nonlinear aspects of "breathing".** We first compare the breathing patterns observed here with amplitude oscillations in the relaxation to the ground state of a linear problem, $iu_t = -u_{xx}/2 - q\delta_0 u$, $0 < q \ll 1$. If the initial data is equal to $u_0(x)$ and is real and even, a heuristic approximation for the solution is

$$u(0,t) \sim e^{itq^2/2} q \int_{\mathbb{R}} u_0(x) e^{-q|x|} dx + \frac{\hat{u}_0(0)}{\sqrt{t}}.$$

Although the zero resonance disappears (see [23] and §4.1), for q small we expect the behaviour $1/\sqrt{t}$ to persist for long times – see §5.4.

This is very different from (1.5). The main difference is that in the nonlinear problem the eigenvalue is not fixed but it is selected depending on the initial condition. The approximate selection is given by the formula for λ in Theorem 1 and a more precise selection method is given in Proposition 3.1 preceding Theorem 2. In particular, the periods oscillation for a fixed initial condition are approximately independent of q. Since the linear eigenvalue is fixed and depends on q this is strikingly different in linear relaxation – see Fig.6.

The origin of the phase in the second term in (1.5) lies in the properties of the non-normal linearized operator for (1.1), and in particular in the coupling responsible for the nonnormality. We do not yet have a fully conceptual explanation for that other than the analysis of (5.6).

We present a simple example illustrating how non-normality can be responsible for "breathing", that is oscillations in the amplitute, absent for normal operators. Suppose that $\alpha, \beta \in \mathbb{R}$ (note that we allow both positive and negative α, β), $\vec{w} =$

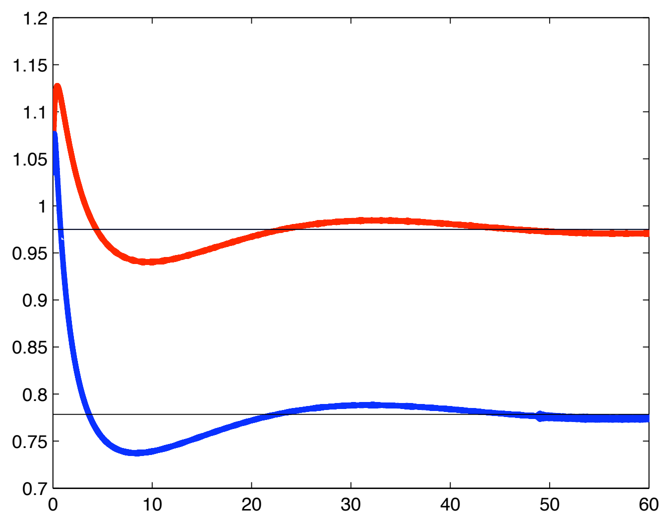

FIGURE 3. Examples of linear relaxation: the initial condition is $u(x,0) = \operatorname{sech} x$ and the potentials are $-q\delta_0(x)$ with$q = 1/2$ for the top graph and $q = \sqrt{2}/4$ for the bottom graph. We expect the periods of oscillations to be $4\pi/q^2$. Consequently, changing $q = 1/2$ to $q/\sqrt{2} = \sqrt{2}/2$ we expect the period to double and to see that we plot $|u(0,t)|$ in the top graph and $|u(0,t/2)|$ in the bottom graph: the agreement of the periods is striking. The horizontal lines correspond to the asymptotic values $v_q(0)\int \operatorname{sech} x v_q(x)dx$, $v_q = \sqrt{q}\exp(-q|x|)$.

$[\operatorname{Re} w \;\; \operatorname{Im} w]^T$,

$$R = \begin{bmatrix} 0 & -\beta - \partial_x^2 \\ \alpha + \partial_x^2 & 0 \end{bmatrix} \tag{1.8}$$

and we consider the evolution

$$\partial_t \vec{w} = R\vec{w} \tag{1.9}$$

We consider R as an operator on $H^2(\mathbb{R};\mathbb{C}) \times H^2(\mathbb{R};\mathbb{C})$ and write out explicit formulas for the complex-valued vector plane-wave solutions associated to (generalized) eigenvalues $\pm i\omega$ of R. From these, we build real-valued plane-wave solutions $\vec{w} = [w_1 \; w_2]^T$ to the matrix equation (1.9). When these are converted to complex numbers as $w = w_1 + iw_2$, we find that if $\alpha = \beta$, then w is unimodular but if $\alpha \neq \beta$, then w is not unimodular and we see oscillations in amplitude.

Let $\omega \geq 0$. We seek complex-valued (vector) plane-wave solutions to (1.9) with generalized eigenvalue $\pm i\omega$. Let $\gamma \geq \max(\sqrt{\max(\alpha,0)}, \sqrt{\max(\beta,0)}) \geq 0$ be the unique solution to

$$\omega^2 = (\gamma^2 - \alpha)(\gamma^2 - \beta)\,.$$

Set

$$\sigma = \sqrt{\frac{\gamma^2 - \alpha}{\gamma^2 - \beta}}\,.$$

Now let

$$v(\gamma) = \begin{bmatrix} 1 \\ i\sigma \end{bmatrix} e^{i\gamma x}, \qquad \tilde{v}(\gamma) = \begin{bmatrix} 1 \\ -i\sigma \end{bmatrix} e^{i\gamma x}\,.$$

Then $v(\gamma)$, $v(-\gamma)$ are two plane-wave solutions to $Rv = i\omega v$ and $\tilde{v}(\gamma)$, $\tilde{v}(-\gamma)$ are two plane-wave solutions to $R\tilde{v} = -i\omega\tilde{v}$. From this we see that

$$e^{-it\omega} v(\pm\gamma), \qquad e^{it\omega}\tilde{v}(\pm\gamma)$$

are four solutions to (1.9). Thus

$$\vec{w} = \begin{bmatrix} \cos(t\omega + \gamma x) \\ \sigma \sin(t\omega + \gamma x) \end{bmatrix} = \tfrac{1}{2} e^{-it\omega} v(-\gamma) + \tfrac{1}{2} e^{it\omega}\tilde{v}(\gamma)$$

is a real solution. Forming a complex number from this vector, we obtain

$$\cos(t\omega + \gamma x) + i\sigma \sin(t\omega + \gamma x)$$

which has constant-in-time modulus if and only if $\sigma = 1$.

The linearization of (1.1) around the solution v_1 gives the non-normal operator $\mathcal{F}_q$ defined below in (2.12). The motivation for studying the operator R above is that $\mathcal{F}_q$ has the form of R with $\alpha = \beta = -1$ for $|x|$ large but with $\alpha = 5$, $\beta = 1$ for $|x|$ near 0.

1.3. **Method of proof and organization of the paper.** The proof Theorem 1 consists of two parts. The first is a nonlinear perturbation theory presented in Theorem 2 and gives an approximation of the solution by the linearized flow. The second part is the precise analysis of that linearized flow on time scales consistent with the approximation given in Theorem 2.

In §2 we present various standard facts about the nonlinear Schrödinger flow with an external delta function potential. The Hamiltonian structure of this flow with respect to the symplectic form $\omega(u, v) = \operatorname{Im}\int u\bar{v}$ is particularly crucial. It plays an important rôle in §3 where it is used to select the nonlinear eigenvalue of the limiting "relaxed state". The other component of the proof of Theorem 2, which is the main result of that section, is the coercivity estimate allowing the control of H^1 norm by the linearized operator. The estimates on the propagator are similar to the estimates in §5 of [17]: instead of the L^2-energy method we estimate $\partial_t \langle \mathcal{L}_q u, u\rangle$, where $\mathcal{L}_q$ is essentially the Hessian of the Hamiltonian (see (2.8)). The initial data is assumed to be even which is particularly important in the case of $q < 0$ (repulsive δ potential) as the ground state is then unstable – see [22].

In §4 we recall Kaup's explicit spectral decomposition of the linearized operator for the focusing cubic NLS on the line. As shown in Appendix A that basis can also be discovered via simple numerical experimentation. We use it to obtain a representation of the propagator and apply it to see the relaxation to solitons in the free case. For initial data close to the solitons this crude approximation is remarkably close to the precise results given by full inverse spectral method – see Fig.5.

The results of §4 lead to an almost explicit spectral decomposition for the operator with $0 < |q| \ll 1$ – that is presented in §5 and Appendices B and C. We follow the general theory of Buslaev-Perelman and Krieger-Schlag in that particular setting. More general non-linearities could also be allowed but since we are ultimately interested in the comparison with numerics the explicit nature of Kaup's basis is very useful. As in §4 this leads to a representation of the propagator, (5.6), used in the statement of Theorem 1. An asymptotic analysis of §5.4 gives the approximation (1.5) illustrated in Fig.1.

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2. Preliminaries

In this section we review various basic aspects of the equation (1.1).

2.1. **Hamiltonian structure.** The nonlinear Schrödinger equation (1.1) describes the Hamiltonian flow on on $H^1(\mathbb{R}, \mathbb{C})$ for the Hamiltonian

$$H_q(v) \stackrel{\text{def}}{=} \frac{1}{4} \int (|\partial_x v|^2 - |v|^4) dx - \frac{1}{2} q |v(0)|^2 \,. \tag{2.1}$$

More precisely, we consider

$$V = H^1(\mathbb{R}, \mathbb{C}) \simeq H^1(\mathbb{R}, \mathbb{R}) \oplus H^1(\mathbb{R}, \mathbb{R}) \,, \quad u \simeq (\operatorname{Re} u, \operatorname{Im} u) \,,$$

as a *real* Hilbert space with the inner product and the symplectic form given by

$$\langle u, v \rangle \stackrel{\text{def}}{=} \operatorname{Re} \int u\bar{v} \,, \quad \omega(u, v) \stackrel{\text{def}}{=} \langle iu, v \rangle = \operatorname{Im} \int u\bar{v} \,, \tag{2.2}$$

Let H_q given by (2.1), or be a more general function, $H : V \to \mathbb{R}$. The associated Hamiltonian vector field is a map $\Xi_H : V \to TV$, which means that for a particular point $u \in V$, we have $(\Xi_H)_u \in T_u V$. The vector field Ξ_H is defined by the relation

$$\omega(v, (\Xi_H)_u) = d_u H(v) \,, \tag{2.3}$$

where $v \in T_u V$, and $d_u H : T_u V \to \mathbb{R}$ is defined by

$$d_u H(v) = \frac{d}{ds}\Big|_{s=0} H(u + sv) \,.$$

In the notation above

$$dH_u(v) = \langle dH_u, v\rangle \,, \quad (\Xi_H)_u = \frac{1}{i} dH_u \,. \tag{2.4}$$

For $H = H_q$ given by (2.1) we compute

$$\begin{aligned} d_u H(v) &= \operatorname{Re} \int ((1/2)\partial_x u \partial_x \bar{v} - |u|^2 u \bar{v}) dx - \operatorname{Re}(q u(0) \bar{v}) \\ &= \operatorname{Re} \int (-(1/2)\partial_x^2 u - |u|^2 u - q\delta_0(x) u)\bar{v} \,. \end{aligned}$$

Thus, in view of (2.4) and (2.3),

$$(\Xi_H)_u = \frac{1}{i}\left(-\frac{1}{2}\partial_x^2 u - |u|^2 u - q\delta_0(x) u\right)$$

The flow associated to this vector field (Hamiltonian flow) is

$$\dot{u} = (\Xi_H)_u = \frac{1}{i}\left(-\frac{1}{2}\partial_x^2 u - |u|^2 u - q\delta_0(x) u\right) \,. \tag{2.5}$$

2.2. **Well posedness in H^1.** The discussion here has been formal but it is well known that the equation (1.1) has global solutions in H^1 for more general nonlinearities, $|u|^{p-1}u$, $1 < p < 5$. For the reader's convenience we recall the standard argument.

We have the following basic estimates:

$$\|u\|_{L^\infty}^2 \leq C\|u\|_{L^2}\|u'\|_{L^2} \,, \tag{2.6}$$

(which follows from the fundamental theorem of calculus: $u(x)^2 = \int_{-\infty}^x 2u(y)u'(y)dy$) and thus

$$\frac{1}{p+1}\int |u|^{p+1} \leq \frac{1}{p+1}\|u\|_{L^\infty}^{p-1}\|u\|_{L^2}^2 \leq C\|u'\|_{L^2}^{\frac{p-1}{2}}\|u\|_{L^2}^{\frac{p+3}{2}} \leq \frac{1}{16}\|u'\|_{L^2}^2 + C'\|u\|_{L^2}^{\frac{2(p+3)}{5-p}} \,,$$

$$\frac{q}{2}|u(0)|^2 \leq \frac{1}{16}\|u'\|_{L^2}^2 + Cq^2\|u\|_{L^2}^2 \,.$$

Hence,

$$\begin{aligned} H_q(u) &= \frac{1}{4}\|u'\|_{L^2}^2 - \frac{1}{p+1}\|u\|_{L^{p+1}}^{p+1} - \frac{q}{2}|u(0)|^2 \\ &\geq \frac{1}{8}\|u'\|_{L^2}^2 - C\|u\|_{L^2}^{\frac{2(p+3)}{5-p}} - Cq^2\|u\|_{L^2}^2 \end{aligned}$$

and consequently,

$$\|u\|_{H^1}^2 \leq 8H_q(u) + C\|u\|_{L^2}^{\frac{2(p+3)}{5-p}} + (Cq^2+1)\|u\|_{L^2}^2 \,.$$

Since the energy, $H_q(u)$, and mass, $\|u\|_{L^2}^2$, are conserved, we see that if the solution exists in H^1, its H^1 norm is uniformly bounded. Thus we only need to show *local* existence in H^1. Let us fix $T>0$ and, for $u=u(x,t)$, define the norm

$$\|u\|_X \stackrel{\text{def}}{=} \sup_{0\leq t\leq T} \|u(\bullet,t)\|_{H^1}\,.$$

Solving (1.1), $u(x,0)=u_0(x)$, is equivalent to finding the fixed point of the operator

$$\Phi \;:\; u(x,t) \longmapsto e^{it(\partial_x^2/2+q\delta_0(x))}u_0(x) - \frac{1}{i}\int_0^t e^{i(t-s)(\partial_x^2/2+q\delta_0(x))}(|u|^{p-1}u)(x,s)ds\,.$$

Here the operator $\exp(it(\partial_x^2/2+q\delta_0(x)))$ is unitary on L^2 (see the discussion of the operator L given in (2.9) below) and preserves

$$\tilde{H}_q(u) = \frac{1}{4}\|u'\|_{L^2}^2 - \frac{q}{2}|u(0)|^2\,.$$

Again using (2.6),

$$\frac{1}{8}\|u'\|_{L^2}^2 - Cq^2\|u\|_{L^2}^2 \leq \tilde{H}_q(u(t)) \leq \frac{1}{2}\|u'\|_{L^2}^2 + Cq^2\|u\|_{L^2}^2\,.$$

Therefore, if $u(t)=\exp(it(\partial_x^2/2+q\delta_0(x)))u_0$,

$$\begin{aligned}\frac{1}{8}\|u'(t)\|_{L^2}^2 &\leq H_q(u(t)) + Cq^2\|u(t)\|_{L^2}^2\\ &= H_q(u_0) + Cq^2\|u_0\|_{L^2}^2\\ &\leq \frac{1}{2}\|u_0'\|_{L^2}^2 + Cq^2\|u_0\|_{L^2}^2\end{aligned}$$

From this, we see that

$$e^{it(\partial_x^2/2+q\delta_0(x))} \;:\; H^1(\mathbb{R}) \longrightarrow H^1\,(\mathbb{R})\,,$$

is bounded with norm independent of t. This and the estimate

$$\begin{aligned}\||u|^{p-1}u - |v|^{p-1}\|_{H^1} &\leq C(\||u|^{p-1}\|_{H^1} + \||v|^{p-1}\|_{H^1})\|u-v\|_{H^1}\\ &\leq C(\|u\|_{H^1}+\|v\|_{H^1})^{p-1}\|u-v\|_{H^1}\,.\end{aligned}$$

give

$$\|\Phi(u)-\Phi(v)\|_X \leq CT(\|u\|_X+\|v\|_X)^{p\ -1}(\|u-v\|_X)$$

so that for T small fixed point arguments can be used to obtain a solution in H^1.

2.3. **Nonlinear ground states.** The minimizers with a prescribed L^2 norm are given by critical points of the Hamiltonian with the constraint added (and $\lambda^2/4$ playing the rôle of the Lagrange multiplier):

$$\mathcal{E}_{q,\lambda}(u) \stackrel{\text{def}}{=} H_q(u) + \frac{\lambda^2}{4}\|u\|_{L^2}^2\,. \tag{2.7}$$

Then for the ground state given by (1.2) we obtain

$$\mathcal{E}_{q,\lambda}'(v_\lambda) = 0\,,\quad \mathcal{E}_{q,\lambda}''(v_\lambda) = \mathcal{L}_q\,,$$

where the Hessian is the following self-adjoint operator on $H^1(\mathbb{R},\mathbb{C}) \simeq H^1(\mathbb{R},\mathbb{R}) \oplus H^1(\mathbb{R},\mathbb{R})$:

$$\mathcal{L}_q \stackrel{\text{def}}{=} \begin{bmatrix} L_{q+} & 0 \\ 0 & L_{q-} \end{bmatrix} , \tag{2.8}$$

where

$$\begin{aligned} L_{q+} &= \frac{1}{2}(\lambda^2 - \partial_x^2 - 6v^2 - 2q\delta_0) , \\ L_{q-} &= \frac{1}{2}(\lambda^2 - \partial_x^2 - 2v^2 - 2q\delta_0) . \end{aligned}$$

In view of the δ functions, the definition of the operators $L_{q\pm}$ is given by choosing the correct domain for the operator. To see what it is let us first examine the basic case of

$$L = -\partial_x^2 + V - q\delta_0 , \tag{2.9}$$

on $\mathbb{R}$, where V is a smooth real-valued potential, rapidly decaying at ∞. Suppose that $u \in L^2(\mathbb{R})$, $Lu = f$, and $f \in L^2(\mathbb{R})$. This implies that away from 0 we have that $\partial_x^2 u \in L^2$, and thus $u \in H^2(\mathbb{R}\backslash\{0\})$. In order that f remain a function across $x = 0$, we must have that $u(x)$ is continuous at $x = 0$ and

$$u'(0+) - u'(0-) = -qu(0) \tag{2.10}$$

Thus a natural domain to consider for L is

$$\mathcal{D} = \{\, u \,|\, u \in H^2(\mathbb{R}\backslash\{0\}) \,, u \text{ is continuous at } x = 0 \text{ and (2.10) holds}\} \tag{2.11}$$

By verifying that the operators $L \pm i$ are both symmetric and surjective on $\mathcal{D}$ we see that L is self-adjoint with domain $\mathcal{D}$.

2.4. **Linearization and the Hamiltonian map.** For $\mathcal{E} : V \to \mathbb{R}$ satisfying $\mathcal{E}'(u) = 0$ we can invariantly define the *Hamiltonian map*,

$$\mathcal{F} \;:\; T_u V \longrightarrow T_u V \,,$$

using the well defined Hessian of $\mathcal{E}$ at u:

$$\langle \mathcal{E}''(u) X, Y \rangle = \omega(Y, \mathcal{F} X) \,.$$

In other words, the Hamiltonian map is the *linearization* of the Hamilton vector field of $\mathcal{E}$. See for instance [18, Sect.21.5] for a general discussion, and [17, Lemmas 2.1, 2.2] for relevant facts in our context.

For $V = H^1(\mathbb{R},\mathbb{C})$ with the symplectic form (2.2) we have

$$\mathcal{F} = -i\mathcal{E}'' \,,$$

and for $\mathcal{E}$ given by (2.7) we have

$$\mathcal{F}_q = -i\mathcal{L}_q = \begin{bmatrix} 0 & L_{q-} \\ -L_{q+} & 0 \end{bmatrix} . \tag{2.12}$$

The matrix representation is based on the identification

$$H^1(\mathbb{R},\mathbb{C}) \ni u \simeq [\operatorname{Re} u, \operatorname{Im} u]^t \in H^1(\mathbb{R},\mathbb{R})^2\,,$$

It is also convenient to consider the equivalent matrix representation using the identification,

$$H^1(\mathbb{R},\mathbb{C}) \ni u \simeq [u,\bar u]^t \in \Delta \subset H^1(\mathbb{R},\mathbb{C})^2\,,$$

which gives

$$\frac{1}{2}H_q \stackrel{\text{def}}{=} \frac{1}{i} U \mathcal{F}_q U^*\,, \quad U \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}\,,$$

$$H_0 = \begin{bmatrix} -\partial_x^2 + \lambda^2 & 0 \\ 0 & \partial_x^2 - \lambda^2 \end{bmatrix} + \operatorname{sech}^2 x \begin{bmatrix} -4 & -2 \\ 2 & 4 \end{bmatrix}\,, \tag{2.13}$$

$$H_{q,\lambda} = H_q = \begin{bmatrix} -\partial_x^2 + \lambda^2 & 0 \\ 0 & \partial_x^2 - \lambda^2 \end{bmatrix} + v_\lambda^2(x) \begin{bmatrix} -4 & -2 \\ 2 & 4 \end{bmatrix} - 2q\delta_0 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

(when there is no, or little, chance of confusion we supress λ in our notation; most of the time its value is taken to be 1). This representation is convenient when we study the spectral decomposition of F_q. The factor $\frac{1}{2}$ was introduced to make the notation simpler and to have a better agreement with the standard notation of [19], [1], [21].

Since the energy $H_q(u)$ differs from $2\mathcal{E}_{q,\lambda}$ by the additive mass term, $\lambda^2\|u\|^2/2$, these linearizations differ from the linearization of (1.1) by a constant only.

For future reference we also note the symmetries of H_q. Let σ_j be the Pauli matrices,

$$\sigma_1 \stackrel{\text{def}}{=} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\,, \quad \sigma_2 \stackrel{\text{def}}{=} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\,, \quad \sigma_3 \stackrel{\text{def}}{=} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\,. \tag{2.14}$$

We recall that they are characterized by the properties that $\sigma_j^2 = I$ and $\sigma_j^* = \sigma_j$. Using this notation,

$$\sigma_1 H_q \sigma_1 = -H_q, \qquad \sigma_3 H_q \sigma_3 = H_q^*\,. \tag{2.15}$$

General considerations show that $\sigma(H_q) \subset \mathbb{R} \cup i\mathbb{R}$, and in fact $\sigma(H_q) \setminus \sigma_{\rm pp}(H_q) = (-\infty,-1] \cup [1,+\infty)$. The fact that all pure point spectrum is contained in $\mathbb{R} \cup i\mathbb{R}$ follows by examining the squared operator H_q^2 which turns out to be self-adjoint. (see Buslaev-Perelman [1, §2.2.3]). This can be done despite the fact that the operator contains the δ_0 potential. To see that consider again the operator L given in (2.9) and suppose that we want to consider L^2, the squared operator. Away from $x = 0$, we see that we must have $u \in H^4(\mathbb{R}\backslash\{0\})$. If $Lu = f$, then we need $f \in \mathcal{D}$. Since f is continuous at 0, we see from the equation $Lu = f$ that $\lim_{x\to 0-} \partial_x^2 u(x) = \lim_{x\to 0+} \partial_x^2 u(x)$. Moreover, taking

$$u''(0) \stackrel{\text{def}}{=} \lim_{x\to 0-} \partial_x^2 u(x) = \lim_{x\to 0+} \partial_x^2 u(x)$$

implies

$$u''(0) = V(0)u(0) - f(0)$$

Away from $x = 0$, we have $-u''' + V'u + Vu' = f'$ and thus the condition that $f'(0+) - f'(0-) = -qf(0)$ becomes

$$u'''(0+) - u'''(0-) = -qu''(0) \tag{2.16}$$

Define

$$\tilde{\mathcal{D}} = \{\, u \,|\, u \in H^4(\mathbb{R}\backslash\{0\})\,, u, u'' \text{ are continuous at } x = 0 \text{ and (2.10),(2.16) hold}\} \tag{2.17}$$

Provided $u \in \tilde{\mathcal{D}}$, $L^2 u$ is defined and belongs to $L^2(\mathbb{R})$ – so there is no need to worry about the square of the delta function not being defined. Indeed, as soon as we know that $u \in \tilde{\mathcal{D}}$ as defined here, then one need only compute $(-\partial_x^2 + V)^2 u$ away from $x = 0$ to obtain $L^2 u$. It is thus natural to consider the squared operator H_q^2 on $\tilde{\mathcal{D}} \times \tilde{\mathcal{D}}$, and H_q^2 can in fact be shown to be self-adjoint on this domain.

2.5. **Symmetries and the generalized kernel.** As in [16] and [17] it is convenient to introduce a natural group action on H^1:

$$\begin{gathered} H^1 \ni u \longmapsto g \cdot u \in H^1\,, \quad (g \cdot u)(x) \stackrel{\text{def}}{=} e^{i\gamma} e^{iv(x-a)} \mu u(\mu(x-a))\,, \\ g = (a, v, \gamma, \mu) \in \mathbb{R}^3 \times \mathbb{R}_+\,. \end{gathered} \tag{2.18}$$

This action gives a group structure on $\mathbb{R}^3 \times \mathbb{R}_+$ and it is easy to check that this transformation group is a semidirect product of the Heisenberg group H_3 and $\mathbb{R}_+$:

$$G = H_3 \ltimes \mathbb{R}_+\,, \quad \mu \cdot (a, v, \gamma) = (\frac{a}{\mu}, \mu v, \gamma)\,.$$

The Lie algebra of G, denoted by $\mathfrak{g}$, is generated by e_1, e_2, e_3, e_4, which in the infinitesimal representation obtained from (2.18) is given by

$$e_1 = -\partial_x\,, \quad e_2 = ix\,, \quad e_3 = i\,, \quad e_4 = \partial_x \cdot x\,. \tag{2.19}$$

It acts, for instance, on $\mathcal{S}(\mathbb{R}) \subset H^1$, and by $X \in \mathfrak{g}$ we will denote a linear combination of the operators e_j. We note that for $q = 0$ (and hence $v = v_\lambda = \lambda\,\mathrm{sech}(\lambda x)$)

$$\begin{gathered} \omega(e_1 \cdot v, e_2 \cdot v) = 1\,, \quad \omega(e_3 \cdot v, e_4 \cdot v) = 1\,, \\ \omega(e_j \cdot v, e_3 \cdot v) = \omega(e_j \cdot v, e_4 \cdot v) = 0\,, \quad j = 1, 2\,. \end{gathered} \tag{2.20}$$

In the case of $q = 0$ the Hamilton vector fields, Ξ_{H_0}, $\Xi_{\mathcal{E}_{0,\lambda}} = \Xi_{H_0} + \lambda/4$, are tangent to the *manifold of solitons*, $G \cdot v$ – see [10, §3] or [17, §2.2]. Hence $\mathcal{F}_0 = -i\mathcal{L}_0$ preserves $T_v(G \cdot v) \simeq \mathfrak{g} \cdot v$. In fact, $\mathfrak{g} \cdot v$ is the generalized kernel of $-i\mathcal{L}_0$:

$$\begin{aligned} i\mathcal{L}_0(e_1 \cdot v) &= 0\,, \qquad i\mathcal{L}_0(e_2 \cdot v) = e_1 \cdot v\,, \\ i\mathcal{L}_0(e_3 \cdot v) &= 0\,, \qquad i\mathcal{L}_0(e_4 \cdot v) = e_3 \cdot v\,. \end{aligned} \tag{2.21}$$

The first and third equation are an immediate consequence of the invariance of solutions under the circle action $u \mapsto e^{i\theta} u$, and translation in x.

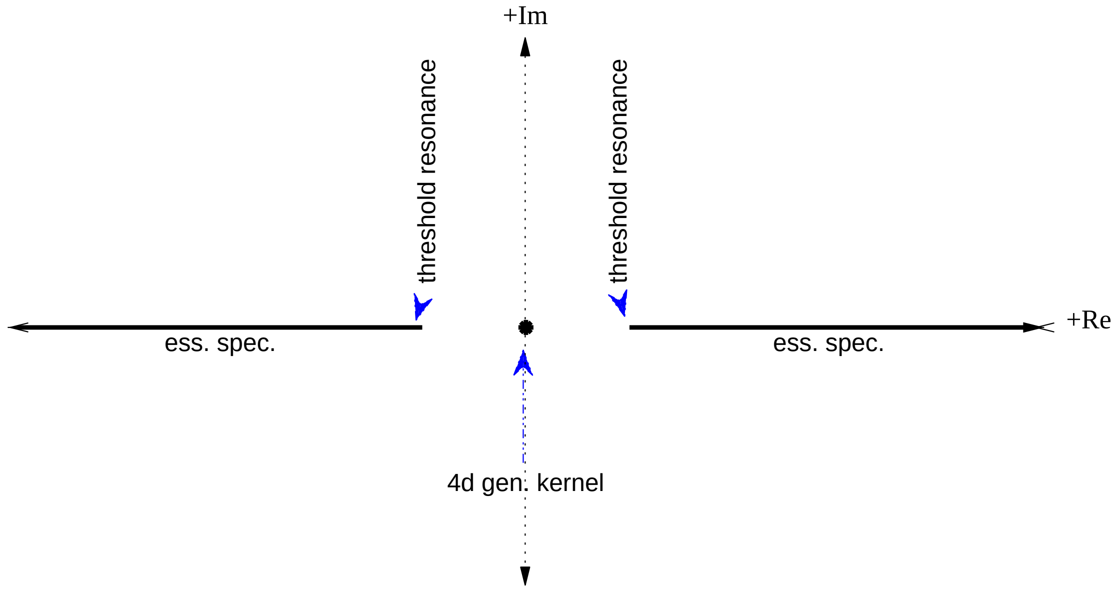

FIGURE 4. The spectrum of the operator H_0. In the notation of §2.5, the generalized eigenspace at 0 is spanned by $e_j \cdot \mathrm{sech}$, $j = 1, \cdots, 4$.

When $q \neq 0$ we lose the translation invariance but we still have $i\mathcal{L}_q(e_1 \cdot v) = 0$ due to the preserved circle action symmetry. As shown in Appendix B the generalized kernel is given by

$$V_0 \stackrel{\text{def}}{=} \mathrm{span}\{v_3, v_4\}\,,$$

where

$$v_3(x) = iv_\lambda(x)\Big|_{\lambda=1}\,, \quad v_4(x) = \partial_\lambda\Big|_{\lambda=1} v_\lambda(x), \quad i\mathcal{L}_q v_3 = 0\,, \quad i\mathcal{L}_q v_4 = v_3\,.$$

Hence v_j, $j = 3, 4$, are the generalizations of $e_j \cdot v$, and in fact,

$$v_3 = e_3 \cdot v\,, \quad v_4 = e_4 \cdot v + \mathcal{O}_{H^1}(q)\,,$$

2.6. **Coercivity estimate.** Finally we recall the crucial coercivity estimate which in a more general form is well known since the work of Weinstein [29]. For the special case at hand an elementary presentation can be found in [16, §4].

For $q = 0$ we have the following estimate: Let $w \in H^1(\mathbb{R}, \mathbb{C})$ and suppose that for any $X \in \mathfrak{g}$, $\omega(w, X \cdot \eta) = 0$. Then,

$$\langle \mathcal{L}_0 w, w \rangle \geq \frac{2\rho_0}{7 + 2\rho_0} \|w\|^2_{H^1} \simeq 0.0555 \|w\|^2_{H^1}\,, \quad \rho_0 = \frac{9}{2(12 + \pi^2\ )}\,. \tag{2.22}$$

3. Nonlinear perturbation theory

In this section we prove a result describing eigenstate selection and nonlinear flow approximation for a time depending on the initial data and on the size of q. Although we restrict our attention to the physical (and completely integrable) case of the cubic NLS the arguments apply to nonlinearities for which the Weinstein coercivity conditions are satisfied (see [29] and Lemma 3.4 below).

Recall

$$v_{\lambda,q}(x) = \lambda \operatorname{sech}(\lambda|x| + \tanh^{-1}(q/\lambda)) \,, \quad \|v_{\lambda,q}(x)\|_{L^2}^2 = 2(\lambda - q) \,.$$

Define the projection

$$P_{\lambda,q}\varphi \overset{\text{def}}{=} \omega(\varphi, \partial_\lambda v_{\lambda,q}) i v_{\lambda,q} - \omega(\varphi, i v_{\lambda,q}) \partial_\lambda v_{\lambda,q} \tag{3.1}$$

onto the generalized kernel

$$V_{\lambda,q} \overset{\text{def}}{=} \operatorname{span}_{\mathbb{R}}\{ i v_{\lambda,q}, \partial_\lambda v_{\lambda,q} \}$$

of $\mathcal{L}_{\lambda,q}$. We will only use the q subscript when it is needed for clarity (e.g. in the scaling argument below). We will also drop the λ-subscript when $\lambda = 1$. Recall that $\langle u, v \rangle = \operatorname{Re} \int u\bar{v}$.

Proposition 3.1 (Symplectic orthogonality)**.** *There exists $\delta > 0$ such that the following holds. If $\varphi \in H^1$ and there exists $\lambda_0 > 0$, $\theta_0 \in \mathbb{R}$ such that $\|\varphi - e^{i\theta_0} v_{\lambda_0}\|_{H^1} \leq \delta$, then there exists $\lambda \in (0, +\infty)$, $\theta \in \mathbb{R}$ such that $P_\lambda(e^{-i\theta}\varphi - v_\lambda) = 0$.*

Proof. Let $F : H^1 \times (0, +\infty) \times \mathbb{R} \to \mathbb{R}^2$ be given by

$$F(u_0, \lambda, \theta) = \begin{bmatrix} \omega(u_0 - e^{i\theta} v_\lambda, \ i e^{i\theta} v_\lambda) \\ \omega(u_0 - e^{i\theta} v_\lambda, \ e^{i\theta} \partial_\lambda v_\lambda) \end{bmatrix}$$

Fix θ_0, λ_0. Note that $F(e^{i\theta_0} v_{\lambda_0}, \lambda_0, \theta_0) = 0$ and the matrix

$$[\partial_\lambda F \quad \partial_\theta F] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

is (uniformly in λ, θ) nondegenerate at $u_0 = e^{i\theta_0} v_{\lambda_0}$, $\lambda = \lambda_0$, $\theta = \theta_0$. The implicit function theorem completes the proof. $\square$

Theorem 2 (Nonlinear perturbation theory)**.** *Let $I \Subset (0, +\infty)$ and $|q| \ll 1$. Suppose that $u(x,t)$ is an even solution to (1.1) and $w_0(x) \overset{\text{def}}{=} u(x,0) - e^{i\theta} v_\lambda(x)$ satisfies $\|w_0\|_{H^1} \leq h \ll 1$ and $P_\lambda(e^{-i\theta} w_0) = 0$ for some $\lambda \in I$, $\theta \in \mathbb{R}$. Then*

$$\|u(t) - e^{it\lambda^2/2}\Big(v_\lambda + e^{-it\mathcal{L}_{\lambda,q}} w_0 \Big)\|_{H^1_x} \leq Ct(1+t)h^2 \,, \tag{3.2}$$

for all $0 \leq t \ll h^{-1/2}$.

Remark 3.2. The constant C depends on I (the range of values in which λ lies), and the restrictions $|q| \ll 1$, $h \ll 1$ and $t \ll h^{-1/2}$ all indicate an implicit (small) constant depending on I.

Remark 3.3. We will ultimately take $h = Cq$ to prove Theorem 1 in §5.5. Note that our use of h here is different from the connection to the semiclassical problem (discussed in the introduction) where $q = h^2$.

Lemma 3.4 (Coercivity). *There exists $c_0 > 0$ (independent of q) with the following property: If $|q| \ll 1$, $P_q f = 0$ and f is even, then*

$$\|f\|_{H_x^1}^2 \leq c_0 \langle \mathcal{L}_q f, f\rangle$$

Proof. We have, with $f = f_1 + f_2$

$$\langle \mathcal{L}_q f, f\rangle = \langle L_{q+} f_1, f_1\rangle + \langle L_{q-} f_2, f_2\rangle\,,$$

where L_{q+} and L_{q-} are the self-adjoint operators defined in (2.8). It suffices to prove that if f is even and real-valued, then

$$\langle f, v\rangle = 0 \implies \langle L_+ f, f\rangle \geq c\|f\|_{L^2}^2, \tag{3.3}$$

and

$$\langle f, \partial_\lambda v_\lambda|_{\lambda=1}\rangle = 0 \implies \langle L_- f, f\rangle \geq c\|f\|_{L^2}^2\,. \tag{3.4}$$

The operators $L_\pm$ (defined as $L_{q\pm}$ with $q = 0$) were analysed in [16, §4], and it was proved there that $\sigma(L_+) = \{-\frac{3}{2}, 0\} \cup [\frac{1}{2}, +\infty)$ and $\sigma(L_-) = \{0\} \cup [\frac{1}{2}, +\infty)$. Moreover, the eigenvalues and L^2 normalized eigenfunctions are given explicitly:

$$L_+(\tfrac{\sqrt{3}}{2}\operatorname{sech}^2) = -\tfrac{3}{2}(\tfrac{\sqrt{3}}{2}\operatorname{sech}^2), \qquad L_+(\sqrt{\tfrac{3}{2}}\operatorname{sech}') = 0, \qquad L_-(\tfrac{1}{\sqrt{2}}\operatorname{sech}) = 0\,.$$

By perturbation theory (this is standard perturbation theory for 2nd-order scalar self-adjoint operators, as opposed to the perturbation theory of Appendix B), $\sigma(L_{q+}) = \{\lambda_1, \lambda_0\} \cup [\frac{1}{2}, +\infty)$, where $\lambda_1 = -\frac{3}{2} + \mathcal{O}(q)$ and $\lambda_0 = \mathcal{O}(q)$. Moreover, the L^2 normalized associated eigenfunctions, g_1 and g_0,

$$L_{q+} g_1 = \lambda_1 g_1, \qquad L_{q+} g_0 = \lambda_0 g_0\,,$$

satisfy

$$g_1(x) = \tfrac{\sqrt{3}}{2}\operatorname{sech}^2 x + \mathcal{O}(q), \qquad g_0(x) = \sqrt{\tfrac{3}{2}}\operatorname{sech}' x + \mathcal{O}(q)\,.$$

In particular, $g_0(x)$ is "nearly" odd. However, we also have that $L_{q+} g_0(-x) = \lambda_0 g_0(-x)$, and hence $g_0(-x) = cg(x)$ for some constant c. If $g_0(0) \neq 0$, then $c = 1$ and hence $g_0(x)$ is even which contradicts the fact that it is nearly odd. From this we conclude $g_0(0) = 0$, and since $g_0(x)$ is not identically zero and solves a second order ODE, we must have $g_0'(0) \neq 0$. Taking the derivative of the identity $g_0(-x) = cg_0(x)$ and evaluating at $x = 0$ gives that $c = -1$. Hence g_0 is exactly odd.

Now we prove (3.3). Since f is assumed even, we have by the Spectral Theorem that if $\langle f, g_1\rangle = 0$, then $\langle L_{q+} f, f\rangle \geq \frac{1}{2}\|f\|_{L^2}^2$. By [16, Lemma 4.2] (with, in the notation of that Lemma, $v_0 = g_1$, $v_1 = v$, $c_0 = -\lambda_1$), we have that if $\langle f, v\rangle = 0$, then

$$\langle L_{q+} f, f\rangle \geq \left(\lambda_1 + (\tfrac{1}{2} - \lambda_1)\frac{\langle g_1, v\rangle^2}{\|v\|_{L^2}^2}\right)\|f\|_{L^2}^2 .$$

By the perturbation theory, the coefficient evaluates to $\frac{3\pi^2}{16} - \frac{3}{2} + \mathcal{O}(q) > 0$.

Now we carry out the analysis of L_{q-}. By perturbation theory, we know that $\sigma(L_{q-}) = \{\lambda_2\} \cup [\frac{1}{2}, +\infty)$, where $\lambda_2 = \mathcal{O}(q)$. However, direct calculation shows that $L_{q-} v = 0$ (where $v(x) = \operatorname{sech}(|x| + \tanh^{-1} q)$), and thus $\lambda_2 = 0$. By the Spectral Theorem, if $\langle f, v\rangle = 0$, then $\langle L_{q-} f, f\rangle \geq \frac{1}{2}\|f\|_{L^2}^2$. By [16, Lemma 4.2], we have that if $\langle f, \partial_\lambda v_\lambda|_{\lambda=1}\rangle = 0$,

$$\langle L_{q-} f, f\rangle \geq \left(\tfrac{1}{2}\frac{\langle \partial_\lambda v_\lambda|_{\lambda=1}, v\rangle^2}{\|\partial_\lambda v_\lambda|_{\lambda=1}\|_{L^2}^2 \|v\|_{L^2}^2}\right)\|f\|_{L^2}^2 .$$

Elliptic regularity completes the argument. □

Proof of Theorem 2. Let $\tilde{u}(x,t) = e^{-i\theta}\lambda^{-1}u(\lambda^{-1}x, \lambda^{-2}t)$. Then $\tilde{u}$ solves

$$i\partial_t \tilde{u} + \tfrac{1}{2}\partial_x^2 \tilde{u} + \frac{q}{\lambda}\delta_0 \tilde{u} + |\tilde{u}|^2\tilde{u} = 0$$

((1.1) with q replaced by q/λ) with initial data $\tilde{u}_0(x) = e^{-i\theta}\lambda^{-1}u_0(\lambda^{-1}x)$. Moreover,

$$\|u_0 - e^{i\theta}v_{\lambda,q}\|_{H^1} \leq h \implies \|\tilde{u}_0 - v_{1,q/\lambda}\|_{H^1} \leq C(\lambda)h$$

and

$$P_{\lambda,\theta,q}(u_0 - e^{i\theta}v_\lambda) = 0 \implies P_{1,0,q/\lambda}(\tilde{u}_0 - v_{1,q/\lambda}) = 0$$

Hence, it suffices to prove the theorem in the case $\lambda = 1$, $\theta = 0$. Let

$$v_3(x) = iv_\lambda(x)\Big|_{\lambda=1}, \quad v_4(x) = \partial_\lambda\Big|_{\lambda=1} v_\lambda(x)$$

Note (by direct computation) that $i\mathcal{L}_q v_3 = 0$ and $i\mathcal{L}_q v_4 = v_3$ and

$$Pw = \omega(w, v_4)v_3 - \omega(w, v_3)v_4$$

is the symplectic orthogonal projection onto the generalized kernel $V_0 = \operatorname{span}\{v_3, v_4\}$.

Define $v(x) \stackrel{\text{def}}{=} v_\lambda(x)\big|_{\lambda=1}$ and $w(t)$ by the relation $u(t) = e^{it/2}(v + w(t))$, and then note that w solves

$$\begin{cases} \partial_t w = -i\mathcal{L}_q w + iF \\ w\big|_{t=0} = u_0 - v \end{cases},$$

where

$$F = 2v|w|^2 + vw^2 + |w|^2 w .$$

Also define

$$w_1 \stackrel{\text{def}}{=} e^{-\frac{1}{2}it\mathcal{L}_q}(u_0 - v) \quad \text{and} \quad \tilde{w} \stackrel{\text{def}}{=} w - w_1 ,$$

so that $\tilde{w}$ satisfies

$$\begin{cases} \partial_t \tilde{w} = -i\mathcal{L}_q \tilde{w} + iF \\ w\big|_{t=0} = 0 \end{cases},$$

where now we write F as

$$F = 2v|w_1 + \tilde{w}|^2 + v(w_1 + \tilde{w})^2 + |w_1 + \tilde{w}|^2(w_1 + \tilde{w})\,.$$

Since $\mathcal{L}_q$ is self-adjoint with respect to $\langle\cdot,\cdot\rangle$,

$$\partial_t \langle \mathcal{L}_q w_1, w_1\rangle = 2\langle \mathcal{L}_q w_1, \partial_t w_1\rangle = 2\langle \mathcal{L}_q w_1, i\mathcal{L}_q w_1\rangle = 0\,.$$

By Lemma 3.4,

$$\|w_1(t)\|_{H^1}^2 \lesssim \langle \mathcal{L}_q w_1(t), w_1(t)\rangle = \langle \mathcal{L}_q w_0, w_0\rangle \lesssim \|w_0\|_{H^1}^2\,.$$

Hence, there exists a constant $c_1 > 0$ such that

$$\|w_1(t)\|_{H^1} \le c_1 h \quad \text{for all } t\,. \tag{3.5}$$

It can be checked by direct computation using $\mathcal{L}_q v_4 = iv_3$ that

$$P \circ \mathcal{L}_q = \mathcal{L}_q \circ P = \omega(w, v_3)v_3\,. \tag{3.6}$$

(An abstract argument using the fact that $\mathcal{L}_q$ preserves V_0 can be given to justify the first equality, which is, in fact, all we use for now). Let

$$\hat{w} = \tilde{w} - P\tilde{w}$$

so that $P\hat{w}(t) = 0$ for all t (and hence Lemma 3.4 will be applicable to $f = \hat{w}(t)$). Using (3.6), we find that

$$\partial_t \hat{w} = -\tfrac{1}{2} i\mathcal{L}_q \hat{w} + (I - P)iF\,.$$

Using the self-adjointness of $\mathcal{L}_q$ with respect to $\langle\cdot,\cdot\rangle$ and the above equation for $\partial_t\hat{w}$, we find that

$$\begin{aligned}\partial_t \langle \mathcal{L}_q \hat{w}, \hat{w}\rangle &= \langle \mathcal{L}_q \hat{w}, \partial_t \hat{w}\rangle = 2\langle \mathcal{L}_q \hat{w}, -\tfrac{1}{2} i\mathcal{L}_q \hat{w} + (1-P)F\rangle \\ &= 2\langle \mathcal{L}_q \hat{w}, F\rangle\,.\end{aligned}$$

Let $[0,T]$ be a time interval over which $\tilde{w}$ remains

$$\|\tilde{w}\|_{L^\infty_{[0,T]}H^1_x} \le c_1 h\,, \tag{3.7}$$

where c_1 is given in (3.5) (so that we know *a priori* that $\tilde{w}$ is at least no worse that w_1, although of course we want to show that it is better). [All future instances of $\lesssim$ mean "less than a constant times", where the constant depends upon c_0 (in Lemma 3.4) and c_1.] This gives an estimate for F: $\|F\|_{L^\infty_{[0,T]}H^1_x} \lesssim h^2$. By integrating, for $0 \le t \le T$, we have

$$|\langle \mathcal{L}_q \hat{w}(t), \hat{w}(t)\rangle| \lesssim T\, \|\hat{w}\|_{L^\infty_{[0,T]}H^1_x} h^2\,.$$

Taking the sup over $t \in [0,T]$ and employing Lemma 3.4,

$$\|\hat{w}\|^2_{L^\infty_{[0,T]}H^1_x} \lesssim Th^2 \|\hat{w}\|_{L^\infty_{[0,T]}H^1_x} \,,$$

and hence

$$\|\hat{w}\|_{L^\infty_{[0,T]}H^1_x} \lesssim Th^2 \,. \tag{3.8}$$

From this, we need to infer a bound on $\tilde{w}$. We compute

$$\begin{aligned} \partial_t \, \omega(\tilde{w}, v_3) &= \omega(-\tfrac{1}{2} i \mathcal{L}_q \tilde{w} + iF, v_3) = -\tfrac{1}{2}\langle \mathcal{L}_q \tilde{w}, v_3\rangle + \langle F, v_3\rangle \\ &= \langle F, v_3\rangle \,, \end{aligned}$$

and thus, on $[0,T]$, we have the bound

$$|\partial_t \, \omega(\tilde{w}, v_3)| \lesssim h^2 \,.$$

Integrating in time, we find that for all $t \in [0,T]$,

$$|\omega(\tilde{w}(t), v_3)| \lesssim h^2 \, t \,. \tag{3.9}$$

Now we perform a similar computation for $\omega(\tilde{w}, v_4)$.

$$\begin{aligned} \partial_t \, \omega(\tilde{w}, v_4) &= \omega(-\tfrac{1}{2} i \mathcal{L}_q \tilde{w} + iF, v_4) = -\tfrac{1}{2}\langle \mathcal{L}_q \tilde{w}, v_4\rangle + \langle F, v_4\rangle \\ &= -\tfrac{1}{2}\langle \tilde{w}, iv_3\rangle + \langle F, v_4\rangle = \tfrac{1}{2}\omega(\tilde{w}, v_3) + \langle F, v_4\rangle \,. \end{aligned}$$

Appealing to (3.9), we obtain the bound

$$|\partial_t \, \omega(\tilde{w}, v_4)| \lesssim h^2 \, t + h^2 \,,$$

which, integrated in time, yields

$$|\omega(\tilde{w}(t), v_4)| \lesssim t^2 h^2 + t h^2 \,. \tag{3.10}$$

The estimates (3.9) and (3.10) give

$$\|P\tilde{w}\|_{L^\infty_{[0,T]}H^1_x} \lesssim h^2 \, T^2 + h^2 \, T \,.$$

Now, provided $T \lesssim h^{-1/2}$, we obtain

$$\|P\tilde{w}\|_{L^\infty_{[0,T]}H^1_x} \le \tfrac{1}{4} c_1 h \,,$$

and thus

$$\|\tilde{w}\|_{L^\infty_{[0,T]}H^1_x} = \|\hat{w}\|_{L^\infty_{[0,T]}H^1_x} + \|P\tilde{w}\|_{L^\infty_{[0,T]}H^1_x} \le \tfrac{1}{2} c_1 h \,,$$

for h suitably small (in terms of the constants c_0 and c_1). Thus we have that the bootstrap assumption (3.7) indeed remains valid over $[0,T]$, and moreover, that

$$\|\tilde{w}(t)\|_{H^1} \lesssim t(1+t)h^2$$

holds over the whole interval $[0,T]$. □

The following corollary is useful in streamlining Theorem 1:

Corollary 3.5. *Suppose the hypothesis of Theorem 2 holds, and that $\tilde\lambda$ satisfies $|\lambda - \tilde\lambda| \lesssim h^2$. Then, in place of (3.2), we have*

$$\|u(t) - e^{it\tilde\lambda^2/2}\big(v_{\tilde\lambda} + e^{-it\mathcal{L}_{\tilde\lambda,q}}w_0\big)\|_{H^1_x} \leq C(1+t)^2h^2\,.$$

Proof. It suffices to show that for any f such that $P_{\lambda,q}f = 0$, we have

$$\|e^{-it\mathcal{L}_{\lambda,q}}f - e^{-it\mathcal{L}_{\tilde\lambda,q}}f\|_{H^1_x} \lesssim h^2(1+t)\|f\|_{H^1_x}\,.$$

(In fact, this is stronger than necessary since $\|w_0\|_{H^1_x} \leq h$. The dominant error term arises from the fact that $\|e^{it\lambda^2/2}v_\lambda - e^{it\tilde\lambda^2/2}v_{\tilde\lambda}\|_{H^1_x} \lesssim h^2$.) Let $u(t) = e^{-it\mathcal{L}_{\lambda,q}}f$ and $\tilde u(t) = e^{-it\mathcal{L}_{\tilde\lambda,q}}f$. Henceforth we will drop the q subscript. Then,

$$\partial_t(u-\tilde u) = -i\mathcal{L}_\lambda(u-\tilde u) + i(\mathcal{L}_\lambda - \mathcal{L}_{\tilde\lambda})\tilde u\,.$$

Note that

$$(\mathcal{L}_\lambda - \mathcal{L}_{\tilde\lambda})\tilde u = \tfrac12(\lambda^2-\tilde\lambda^2)\tilde u - 2(v_\lambda^2 - v_{\tilde\lambda}^2)\tilde u - (v_\lambda^2 - v_{\tilde\lambda}^2)\bar{\tilde u}\,.$$

As in the proof of Theorem 2, we compute

$$\partial_t\langle \mathcal{L}_\lambda(u-\tilde u), u-\tilde u\rangle = 2\langle \mathcal{L}_\lambda(u-\tilde u),\ \partial_t(u-\tilde u)\rangle\,.$$

Substituting the above formulas and using that $\langle \mathcal{L}g, i\mathcal{L}g\rangle = 0$, we obtain

$$\partial_t\langle \mathcal{L}_\lambda(u-\tilde u), u-\tilde u\rangle = 2\langle \mathcal{L}_\lambda(u-\tilde u),\ \big(\tfrac12(\lambda^2-\tilde\lambda^2)\tilde u - 2(v_\lambda^2 - v_{\tilde\lambda}^2)\tilde u - (v_\lambda^2 - v_{\tilde\lambda}^2)\bar{\tilde u}\big)\rangle\,.$$

The next step is to write out the operator $\mathcal{L}_\lambda$, and address the above expression term by term. For the ∂_x^2 term, integrate by parts once, and then apply the Cauchy-Schwarz inequality. For the δ_0 term, use that $|g(0)| \leq \|g\|_{H^1_x}$. For all other terms, directly apply the Cauchy-Schwarz inequality. The resulting bound is

$$|\partial_t\langle \mathcal{L}_\lambda(u-\tilde u), u-\tilde u\rangle| \lesssim |\lambda-\tilde\lambda|\|u-\tilde u\|_{H^1_x}\|\tilde u\|_{L^\infty_{[0,t]}H^1_x}\,.$$

Following the argument used to obtain the bound (3.5) in the proof of Theorem 2, we conclude that

$$\|u(t)\|_{H^1_x} \lesssim \|f\|_{H^1_x}\,,$$

uniformly for all t. Using that $\|\tilde u\|_{H^1_x} \leq \|u-\tilde u\|_{H^1_x} + \|\tilde u\|_{H^1_x}$, we obtain

$$|\partial_t\langle \mathcal{L}_\lambda(u-\tilde u), u-\tilde u\rangle| \lesssim h^2\|u-\tilde u\|_{H^1_x}^2 + h^2\|u-\tilde u\|_{H^1_x}\|f\|_{H^1_x}\,.$$

Integrating over $[0,t]$ and using that $u(0) = \tilde u(0)$, we obtain

$$|\langle \mathcal{L}_\lambda(u(t)-\tilde u(t)), u(t)-\tilde u(t)\rangle| \lesssim th^2\|u-\tilde u\|^2_{L^\infty_{[0,t]}H^1_x} + th^2\|u-\tilde u\|_{L^\infty_{[0,t]}H^1_x}\|f\|_{H^1_x}\,. \tag{3.11}$$

By Lemma 3.4,

$$\|(u-\tilde u) - P_\lambda(u-\tilde u)\|^2_{H^1_x} \lesssim \langle \mathcal{L}_\lambda\big((u-\tilde u) - P_\lambda(u-\tilde u)\big),\ \big((u-\tilde u) - P_\lambda(u-\tilde u)\big)\rangle\,.$$

By Cauchy-Schwarz, we deduce the bound

$$\|u-\tilde u\|^2_{H^1_x} \lesssim \langle \mathcal{L}_\lambda(u-\tilde u), u-\tilde u\rangle + (\|u-\tilde u\|_{H^1_x} + \|P_\lambda(u-\tilde u)\|_{H^1_x})\|P_\lambda(u-\tilde u)\|_{H^1_x}\,. \tag{3.12}$$

Note that

$$P_\lambda(u-\tilde u) = -P_\lambda \tilde u = -(P_\lambda - P_{\tilde\lambda})\tilde u - P_{\tilde\lambda}\tilde u\,, \tag{3.13}$$

Since $P_{\tilde\lambda}\circ e^{-it\mathcal{L}_{\tilde\lambda}} = e^{-it\mathcal{L}_{\tilde\lambda}}\circ P_{\tilde\lambda}$,

$$P_{\tilde\lambda}\tilde u = e^{-it\mathcal{L}_{\tilde\lambda}} P_{\tilde\lambda} f.$$

This is just the evolution of the (generalized) kernel, and hence

$$\|P_{\tilde\lambda}\tilde u\|_{H^1_x} \le t\|P_{\tilde\lambda} f\|_{H^1_x} = t\|(P_{\tilde\lambda}-P_\lambda)f\|_{H^1_x} \lesssim th^2\|f\|_{H^1_x}\,.$$

Similarly,

$$\|(P_\lambda - P_{\tilde\lambda})\tilde u\|_{H^1_x} \lesssim h^2\|\tilde u\|_{H^1_x} \lesssim h^2(\|u-\tilde u\|_{H^1_x} + \|f\|_{H^1_x})\,,$$

which, together with the previous bound, gives (see (3.13))

$$\|P_\lambda(u-\tilde u)\|_{H^1_x} \lesssim h^2\|u-\tilde u\|_{H^1_x} + (1+t)h^2\|f\|^2_{H^1_x}\,.$$

Combining this with (3.12), we obtain the bound (absorbing $h^2\|u-\tilde u\|^2_{H^1_x}$ on the left side)

$$\|u-\tilde u\|^2_{H^1_x} \lesssim \langle \mathcal{L}_\lambda(u-\tilde u), u-\tilde u\rangle + (1+t)h^2\|u-\tilde u\|_{H^1_x}\|f\|_{H^1_x} + (1+t)^2h^4\|f\|^2_{H^1_x}\,.$$

Combining this with (3.11), provided $th^2 \ll 1$, we obtain the bound

$$\|u-\tilde u\|^2_{L^\infty_{[0,t]}H^1_x} \lesssim (1+t)h^2\|u-\tilde u\|_{H^1_x}\|f\|_{H^1_x} + (1+t)^2h^4\|f\|^2_{H^1_x}\,,$$

from which it follows that

$$\|u(t)-\tilde u(t)\|_{H^1_x} \lesssim (1+t)h^2\|f\|_{H^1_x}\,,$$

which implies the estimate in the corollary. □

4. The free case

We will discuss the case of $q=0$ and the initial data close to a stationary soliton $\operatorname{sech} x$. Very precise information can in principle be obtained in this case using the inverse scattering method [31] – see also [6], [7],[8]. However we are not aware of any reference containing that information – see [14, Appendix B] for a discussion.

4.1. **Spectral theory of the linearized operator.** The explicit spectral decomposition of the operator H_0 was discovered by Kaup [19] (see also [30] for a recent discussion and generalizations). We now present it in a way which will make the spectral decomposition of H_q natural.

Spectral theory of operators of the form

$$H = \frac{1}{2}\begin{bmatrix} -\partial_x^2+1 & 0 \\ 0 & \partial_x^2 - 1\end{bmatrix} + \begin{bmatrix} V_1 & V_2 \\ -V_2 & -V_1\end{bmatrix}\,, \tag{4.1}$$

was studied systematically by Buslaev-Perelman [1] and Krieger-Schlag [21]. Despite the non-normality of H (if $V_2 \neq 0$) a spectral decomposition is available once the existence and properties of the four dimensional set of solutions to

$$H\psi = (k^2+1)\psi$$

is established. They are characterized by their behaviour as $x \to \infty$:

$$\psi(x) \sim e^{\pm ikx}\begin{bmatrix}1\\0\end{bmatrix}\,, \quad e^{\pm\mu x}\begin{bmatrix}0\\1\end{bmatrix}\,, \quad \mu^2 = k^2+2\,, \quad k \neq 0\,, \quad \mu \neq 0\,,$$

see [21, Section 5] for a careful discussion.

For the linearization of the cubic NLS the set of four solutions can be found *explicitly*[1] and it is given by

$$\{\,\Psi_+(\cdot,k), \quad \Psi_+(\cdot,-k), \quad \Psi_-(\cdot,i\mu), \quad \Psi_-(\cdot,-i\mu)\,\}\,, \quad k\neq 0\,, \quad \mu \neq 0\,, \tag{4.2}$$

where

$$\Psi_+(x,k) = \begin{bmatrix}(\tanh x - ik)^2\\ -\operatorname{sech}^2 x\end{bmatrix} e^{ikx}\,, \quad \Psi_- = \sigma_1\Psi_+\,, \quad \mu = (k^2+2)^{1/2}\,. \tag{4.3}$$

Since $\sigma_1 H_0 \sigma_1 = -H_0$, we have that

$$\{\,\Psi_-(\cdot,k), \quad \Psi_-(\cdot,-k), \quad \Psi_+(\cdot,i\mu), \quad \Psi_+(\cdot,-i\mu)\,\}$$

is a basis for the solution space to $H_0\psi = -(k^2+1)\psi$.

Now let

$$\varphi_+(x,k) = \sigma_3\Psi_+(x,k)\,, \quad \varphi_-(x,k) = \sigma_1\varphi_-(x,k)\,,$$

Then since $\sigma_3 H_0 \sigma_3 = H_0^*$, we have that

$$\{\,\varphi_+(\cdot,k), \quad \varphi_+(\cdot,-k), \quad \varphi_-(\cdot,i\mu), \quad \varphi_+(\cdot,-i\mu)\,\}$$

is a basis of the solution space to $H_0^*\psi = (k^2+1)\psi$. Note that

$$\varphi_- = \sigma_1\varphi_+ = \sigma_1\sigma_3\Psi_+ = -\sigma_3\sigma_1\Psi_+ = -\sigma_3\Psi_-$$

Finally, using that $\sigma_1 H_0^* \sigma_1 = -H_0^*$, we obtain that

$$\{\,\varphi_-(\cdot,k), \quad \varphi_-(\cdot,-k), \quad \varphi_+(\cdot,i\mu), \quad \varphi_+(\cdot,-i\mu)\,\}$$

is a basis of the solution space to $H_0^*\psi = -(k^2+1)\psi$ for $k \neq 0$.

The eigenvalue 0 corresponds to a generalized eigenspace $\operatorname{span}\{\Psi_1,\Psi_2,\Psi_3,\Psi_4\}$, to be described now. Let $\eta(x) = \operatorname{sech} x$ and

$$e_1 = -\partial \qquad e_2 = ix \qquad e_3 = i \qquad e_4 = \partial x$$

[1]In Appendix A we show how this solution can be guessed by performing a simple numerical experiment even if, as we were, one is ignorant of the inverse scattering developments. We are grateful to Galina Perelman for explaining to us the structure of solutions to linearized operators in the completely integrable case.

Then e_1 (translation) and e_2 (Galilean) are symplectically dual and we have (2.21). Let

$$\Psi_j = i^{j-1} U(e_j \cdot \eta)\,,$$

where U is given in (2.13). Then

$$H_0\Psi_1 = H_0\Psi_3 = 0, \qquad H_0\Psi_2 = \Psi_1, \quad H_0\Psi_4 = \Psi_3$$

The generalized kernel of H_0^* is spanned by $\varphi_j \stackrel{\text{def}}{=} \sigma_3\overline{\Psi}_j$,

$$H_0^*\varphi_1 = H_0^*\varphi_3 = 0\,,\ H_0^*\varphi_2 = \varphi_1\,,\ H_0^*\varphi_4 = \varphi_3\,.$$

Since $U^*\sigma_3 U = \sigma_2$ we also see that

$$\int_{\mathbb{R}} \varphi_2(x)^*\Psi_1(x)dx = \int_{\mathbb{R}} \varphi_1^*(x)\Psi_2(x) = 1\,,$$
$$\int_{\mathbb{R}} \varphi_4(x)^*\Psi_3(x)dx = \int_{\mathbb{R}} \varphi_3^*(x)\Psi_4(x) = -1\,.$$

Finally we recall that H_0 has a simple threshold resonance given by the explicit formula (see Chang-Gustafson-Nakanishi-Tsai [3, §3.7])

$$\begin{bmatrix} \tanh^2 x \\ -\operatorname{sech}^2 x \end{bmatrix}, \tag{4.4}$$

corresponding to $k = 0$. Following [1] and [21, Definition 5.18] (note a slight change of convention between this paper and [21]) we say that H has a *resonance* at 1 if there exists $u \in L^\infty$ such that $Hu = u$. The multiplicity of a resonance is the number of independent solutions with these properties. As we will recall below the maximum multiplicity is 2. Here we include eigenvalues as resonances: "true" resonances satisfy $u \in L^\infty \setminus L^2$.

This definition is equivalent to the more general definition based on the meromorphic continuation of the resolvent. The potential $\operatorname{sech}^2 x$ is exponentially decaying and the resolvent of H_0 (the same operator without the potential term), $R_0(z) = (H_0 - z)^{-1}$, has a global meromorphic continuation to a three sheeted Riemann surface, with poles at ± 1. The resolvent $R(z) = (H-z)^{-1}$ can then be continued from the *physical plane*, $\Sigma \stackrel{\text{def}}{=} \mathbb{C} \setminus ((-\infty,-1)\cup\{0\}\cup(1,\infty))$, to a neighbourhood of Σ on that three sheeted Riemann surface. Near ± 1 the resolvent is meromorphic in λ, $z = \pm(1+\lambda^2)$. The analysis outlined in [1] (see [21, Lemma 5.2] and [21, Lemma 6.5] for detailed presentation) can be used to show that the definition of resonances as poles of the resolvent coincides with the definition above given in terms of solutions.

Let P_c denote the symplectic orthogonal projection onto the essential spectrum, which we define as $I - P_d$ where P_d is the symplectic orthogonal projection onto the discrete spectral subspace E_0. The 2×2 matrix kernel $P_c(x,y)$ of P_c is given by

Kaup's formula [19]

$$P_c(x,y) = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{1}{(1+k^2)^2}(\Psi_+(x,k)\varphi_+(y,k)^* + \Psi_-(x,k)\varphi_-(y,k)^*)\,dk$$

Once we know (4.2), this formula can also be derived by contour deformation and the fact that

$$\frac{1}{2\pi i}\int_\Gamma (H_0 - z)^{-1}\,dz = \mathrm{Id}$$

where Γ is any contour that encloses the spectrum of H_0 – see [21, Lemma 6.8]. As claimed in [20] and [30] it can also be checked by an explicit calculations of the integral.

We now put this into a form that is more consistent with one-dimensional scattering theory (see for instance [27]) and connects the basis with the basis of *scattering solutions* of [1, §2.5.1] and [21, §6].

Let

$$v_+(x,k) = \frac{1}{(1+i|k|)^2}\Psi_+(x,k)\,. \tag{4.5}$$

Then $H_0 v_+ = (1+k^2)v_+$ and

$$v_+(x,k) \sim \begin{bmatrix}1\\0\end{bmatrix}\begin{cases} e^{ikx} + R_+(k)e^{-ikx} & \text{as } x\to-\infty \\ T_+(k)e^{ikx} & \text{as } x\to+\infty\end{cases} \qquad \text{for } k>0 \tag{4.6}$$

$$v_+(x,k) \sim \begin{bmatrix}1\\0\end{bmatrix}\begin{cases} T_-(k)e^{ixk} & \text{as } x\to-\infty \\ e^{ikx} + R_-(k)e^{-ikx} & \text{as } x\to+\infty\end{cases} \qquad \text{for } k<0 \tag{4.7}$$

with

$$R_+(k) = 0 \quad \text{and} \quad T_+(k) = \frac{(1-ik)^2}{(1+ik)^2}$$

$$R_-(k) = 0 \quad \text{and} \quad T_-(k) = \frac{(1+ik)^2}{(1-ik)^2}$$

Now let

$$v_- \stackrel{\text{def}}{=} \sigma_1 v_+ \implies H_0 v_- = -(1+k^2)v_+$$

$$\tilde v_+ \stackrel{\text{def}}{=} \sigma_3 v_+ \implies H_0^*\tilde v_+ = (1+k^2)\tilde v_+$$

$$\tilde v_- \stackrel{\text{def}}{=} \sigma_1 \tilde v_+ \implies H_0^*\tilde v_- = -(1+k^2)\tilde v_-$$

Then

$$P_c(x,y) = \frac{1}{2\pi}\int_{\mathbb{R}} (v_+(x,k)\tilde v_+(y,k)^* + v_-(x,k)\tilde v_-(y,k)^*)\,dk\,, \tag{4.8}$$

and

$$\frac{1}{2\pi}\int_{\mathbb{R}} \tilde v_\pm(x,k)^* v_\pm(x,k')\,dx = \delta(k-k'), \quad \frac{1}{2\pi}\int_{\mathbb{R}} \tilde v_\pm(x,k)^* v_\mp(x,k')\,dx = 0 \tag{4.9}$$

4.2. **The free linearized propagator.** It follows from (4.8) that the propagator $e^{-\frac{1}{2}itH_0}P_c$ on the essential spectrum is represented by the Schwartz kernel

$$e^{-\frac{1}{2}itH_0}P_c(x,y) = \frac{1}{2\pi}\int_{\mathbb{R}}(e^{-\frac{1}{2}it(1+k^2)}v_+(x,k)\tilde{v}_+(y,k)^* + e^{\frac{1}{2}it(1+k^2)}v_-(x,k)\tilde{v}_-(y,k)^*)dk\,.$$

We will now study $e^{-\frac{1}{2}itH_0}P_cw_0$ for w_0 appearing in Theorem 2.

Proposition 4.1. *Suppose that $w_0 \in \mathcal{S}(\mathbb{R})$ is real valued. Then*

$$\begin{gathered}
e^{-\frac{1}{2}itH_0}P_cw_0(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}(a(x,k)e^{-\frac{1}{2}i(k^2+1)t} + b(x,k)e^{\frac{1}{2}i(k^2+1)t})f(k)dk\,, \\
f(k) = \frac{1}{\sqrt{2\pi}(1-i|k|)^2}\int_{-\infty}^{+\infty}(1+2ikt(x)-k^2t(x)^2)w_0(x)e^{-ikx}\,dx \\
a(x,k) = \frac{(t(x)-ik)^2}{(1+i|k|)^2}e^{ikx}\,, \quad b(x,k) = \frac{-s(x)^2}{(1+i|k|)^2}e^{-ikx}\,,
\end{gathered} \tag{4.10}$$

where we used the notation $t(x) = \tanh x$ and $s(x) = \operatorname{sech} x$. Consequently,

$$e^{-\frac{1}{2}itH_0}P_cw_0(0,t) = -\frac{1}{\sqrt{2\pi t}}e^{\frac{1}{2}it}e^{i\frac{\pi}{4}}\int_{\mathbb{R}}w_0(x)dx + \mathcal{O}(t^{-3/2})\,. \tag{4.11}$$

Proof. Let

$$V_+g(x) = \frac{1}{\sqrt{2\pi}}\int_k v_+(x,k)g(k)\,dk\,,$$

be the "inverse distorted Fourier transform," which gives

$$V_+^*f(k) = \frac{1}{\sqrt{2\pi}}\int_x v_+^*(x,k)f(x)\,dx\,,$$

the "distorted Fourier transform," associated to the operator H_0. With this notation, we have

$$P_ce^{-\frac{1}{2}itH} = V_+M(t)V_+^*\sigma_3 + \sigma_1V_+M(-t)V_+^*\sigma_3\sigma_1, \quad M(t)f(k) \stackrel{\text{def}}{=} e^{-\frac{1}{2}i(k^2+1)t}f(k)$$

Consequently, for w_0 real,

$$\begin{aligned}
e^{-\frac{1}{2}itH_0}P_cw_0 &= \begin{bmatrix}1 & 0\end{bmatrix}\ P_ce^{-\frac{1}{2}itH}\begin{bmatrix}1\\1\end{bmatrix}w_0 \\
&= \begin{bmatrix}1 & 0\end{bmatrix}\ (V_+M(t)V_+^*\sigma_3 + \sigma_1V_+M(-t)V_+^*\sigma_3\sigma_1)\begin{bmatrix}1\\1\end{bmatrix}w_0 \\
&= \begin{bmatrix}1 & 0\end{bmatrix}\ (V_+M(t) + \sigma_1V_+M(-t))V_+^*\begin{bmatrix}1\\-1\end{bmatrix}w_0
\end{aligned}$$

We write the function $v_+(x,k)$ given by (4.5) as

$$v_+(x,k) = \begin{bmatrix} a(x,k) \\ b(x,k) \end{bmatrix} e^{ikx}\,, \qquad \begin{aligned} a(x,k) &= \frac{(t(x)-ik)^2}{(1+i|k|)^2} e^{ikx} \\ b(x,k) &= \frac{-s(x)^2}{(1+i|k|)^2} e^{-ikx} \end{aligned}$$

and define

$$f(k) \stackrel{\text{def}}{=} \left(V_+^* \begin{bmatrix} 1 \\ -1 \end{bmatrix} w_0 \right)(k) = \frac{1}{\sqrt{2\pi}} \int_x (\bar{a}(x,k) - \bar{b}(x,k)) w_0(x)\, dx\,. \tag{4.12}$$

Then,

$$\begin{aligned} e^{-\frac{1}{2}itH} P_c w(0,t) &= \frac{1}{\sqrt{2\pi}} \int (a(0,k) e^{-\frac{1}{2}i(k^2+1)t} + b(0,k) e^{\frac{1}{2}i(k^2+1)t}) f(k)\, dk \\ &= \frac{1}{\sqrt{2\pi}} \int \frac{-k^2 e^{-\frac{1}{2}i(k^2+1)} - e^{\frac{1}{2}i(k^2+1)t}}{(1+i|k|)^2} f(k)\, dk \\ &= -\frac{1}{\sqrt{t}} e^{\frac{1}{2}it} e^{i\frac{\pi}{4}} f(0) + \mathcal{O}(t^{-3/2})\,, \end{aligned}$$

by the method of stationary phase. □

4.3. **Nonlinear perturbation theory in the free case.** Let us take a particular example:

$$w_0(x) = \frac{1+h}{1+2h} \operatorname{sech}\left(\frac{x}{1+2h} \right) - \operatorname{sech} x$$

(the choice of scaling was made so that $\operatorname{sech} x$ is selected as the nonlinear ground state by Theorem 2). In this case we compute $f(0) = h\sqrt{\pi/2}$ which gives

$$e^{-\frac{1}{2}itH_0} P_c w_0(0,t) = -\sqrt{\frac{\pi}{2}}\, e^{\frac{1}{4}i\pi} \, \frac{1}{\sqrt{t}} \, e^{\frac{1}{2}it} h + \mathcal{O}\left(\frac{h}{t^{3/2}} \right)$$

We also have $P_3 w_0 = \omega(w_0, v_4) v_3 = 0$ and $P_4 w_0 = \omega(w_0, v_3) v_4 \approx 0.4 \cdot h^2 v_4$. Thus, for $t \gg 1$, we have

$$e^{-\frac{1}{2}itH_0} P_c w_0(0,t) = 0.2 \cdot ih^2 t - \, e^{\frac{1}{4}i\pi} \sqrt{\frac{\pi}{2t}} \, e^{\frac{1}{2}it} h + \mathcal{O}(h^2) + \mathcal{O}\left(\frac{h}{t^{3/2}} \right)$$

We can now apply Theorem 2 to see that the solution of

$$iu_t = -u_{xx}/2 - |u|^2 u\,, \quad u(x,0) = \frac{1+h}{1+2h} \operatorname{sech}\left(\frac{x}{1+2h} \right)\,,$$

satisfies

$$e^{-it/2} u(0,t) = 1 - e^{\frac{1}{4}i\pi} \sqrt{\frac{\pi}{2t}} \, e^{\frac{1}{2}it} h + \mathcal{O}(t^2 h^2) + \mathcal{O}\left(\frac{h}{t^{3/2}} \right)\,, \quad 1 \ll t \ll h^{-1/2}\,. \tag{4.13}$$

Figure 5 compares this asymptotic expression with the numerical solution.

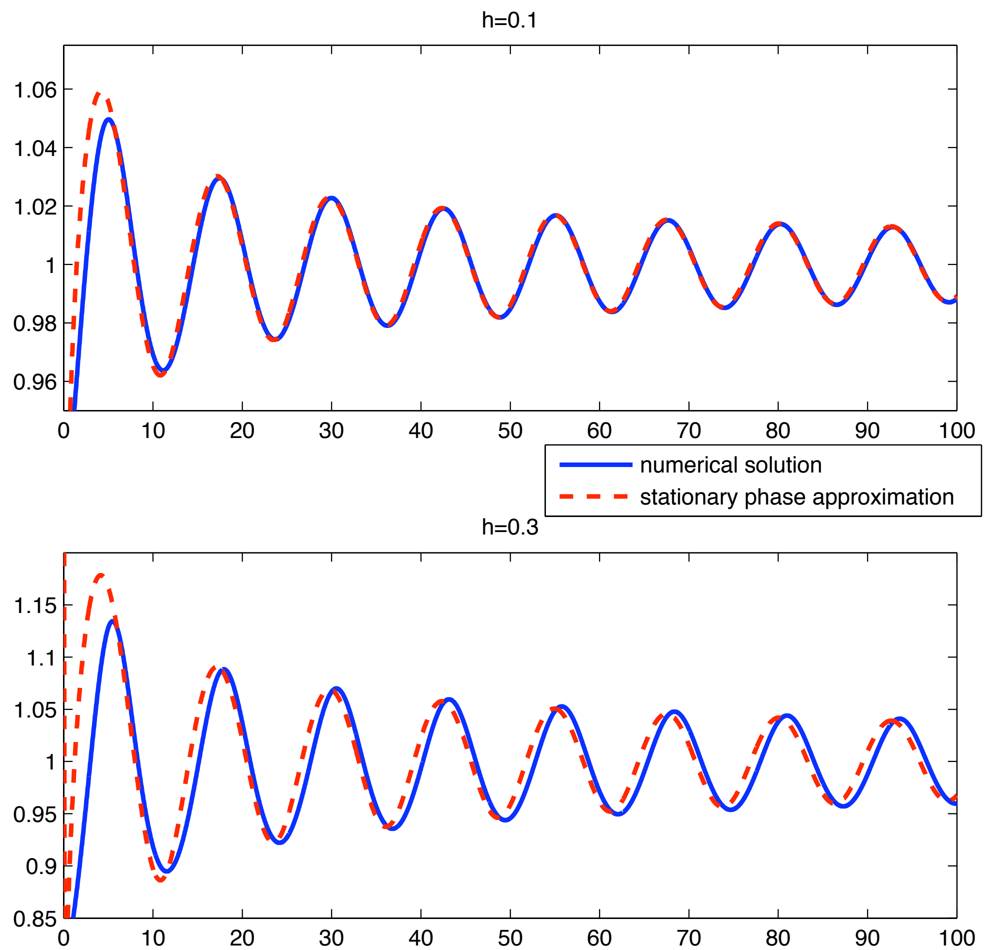

FIGURE 5. Breathing patterns for $q = 0$ and $u(x, 0) = (1 + h)/(1 + 2h)\,\mathrm{sech}(x/(1+2h))$ (the rescaling is rigged so that the resulting ground state is simply $\mathrm{sech}\, x$): we show $|u(0, t)|$ and the asymptotic prediction for $h = 0.1$, $h = 0.3$. The agreement is remarkably good for times much longer than given in the theoretical result. A more precise statement can be obtained using the inverse scattering method.

Remark. We should stress that a more precise result valid for all values of h can in principle be obtained using the inverse scattering method – see [14, Appendix B] and references given there. It would be very interesting to compare those exact expressions with our rough asymptotics. The results of [14, Appendix B] show already that (4.13) can be corrected since we know that

$$u(x,t) = e^{i\varphi(h)} \operatorname{sech} x + \mathcal{O}_{L^\infty}\left(\frac{1}{\sqrt{t}}\right),$$

where

$$\begin{aligned}\varphi(h) &= \int_0^\infty \log\left(1 + \frac{\sin^2 \pi h}{\cosh^2 \pi\zeta}\right) \frac{\zeta}{\zeta^2 + (1+2h)^2}\, d\zeta \\ &\simeq \pi^2 h^2 \int_0^\infty \frac{\zeta \operatorname{sech}^2 \pi\zeta}{1+\zeta^2} d\zeta \simeq 0.6\, h^2\,, \quad h \longrightarrow 0\end{aligned}$$

Hence, in the application of Theorem 2 the error terms $\mathcal{O}(h^2)$ in (4.13) are optimal.

5. Small external delta potential

In this section we will use Theorem 2 to prove Theorem 1 stated in the introduction. For that we will follow the same path as in §4 and provide a spectral decomposition of the linearized operator with the δ_0 potential. The scattering coefficients, $R_\pm$ and T, appearing in (4.6) and (4.7) are now more singular which makes the asymptotic analysis more complicated.

5.1. **Basis of solutions to $H_q\psi = \pm(k^2+1)\psi$.** Using the Kaup basis (4.2) for the free problem we find a complete set of solutions ψ to the equation $H_q\psi = (k^2+1)\psi$, where

$$H_q = \begin{bmatrix} -\partial_x^2 + 1 & 0 \\ 0 & \partial_x^2 - 1 \end{bmatrix} + 2\operatorname{sech}^2(x + \operatorname{sgn}(x)\theta) \begin{bmatrix} -2 & -1 \\ 1 & 2 \end{bmatrix} - 2q \begin{bmatrix} \delta_0 & 0 \\ 0 & -\delta_0 \end{bmatrix},$$

with $\theta = \tanh^{-1} q$, see (2.13).

Let $s = \operatorname{sech}(x + \operatorname{sgn}(x)\theta)$, $t = \tanh(x + \operatorname{sgn}(x)\theta)$, and $\mu = (k^2+2)^{1/2} > 0$. With unknown coefficients $A(k)$, $B(k)$, $C(k)$, and $D(k)$, we look for $\psi(x,k)$ of the form

(5.1)

$$\begin{aligned}\psi = \;& \left(\begin{bmatrix} (t-ik)^2 \\ -s^2 \end{bmatrix} e^{ik(x-\theta)} + A \begin{bmatrix} -s^2 \\ (t-\mu)^2 \end{bmatrix} e^{\mu(x-\theta)} \right) x_-^0 \\ &+ \left(B \begin{bmatrix} (t-ik)^2 \\ -s^2 \end{bmatrix} e^{ik(x+\theta)} + C \begin{bmatrix} (t+ik)^2 \\ -s^2 \end{bmatrix} e^{-ik(x+\theta)} + D \begin{bmatrix} -s^2 \\ (t+\mu)^2 \end{bmatrix} e^{-\mu(x+\theta)} \right) x_+^0\end{aligned}$$

For the unknowns $A(k)$, $B(k)$, $C(k)$, and $D(k)$, two equations are obtained by requiring continuity at $x = 0$ and two more equations are obtained by requiring the appropriate jump condition in the derivatives at $x = 0$. This gives rise to the 4×4 system analysed in detail in Appendix C.

By comparing $\psi(x,-k)$ and $\overline{\psi(x,k)}$ asymptotically as $x \to -\infty$, and noting that both solve $H_q\psi = (1+k^2)\psi$, we find that $\psi(x,-k) = \overline{\psi(x,k)}$ and hence $A(-k) = \overline{A(k)}$, and similarly for B, C, and D.

Here is a typical consequence of the formulas from Appendix C. An eigenvalue at $1 + k^2$ comes from finding a solution to

$$B(k, q) = 0\,,$$

with $\operatorname{Im} k < 0$. More generally, a solution will give a resonance or a pole of the resolvent. The following lemma is derived from the computations in Appendix C.

Lemma 5.1. *For $q < 0$, $0 < |q| \ll 1$ the operator H_q has one eigenvalue, $\mu_q^\pm$, near ± 1,*

$$\mu_q^\pm = 1 - q^2\,. \tag{5.2}$$

The corresponding eigenfuctions $u_q^\pm$ can be chosen to be real and satisfy $\sigma_1 u_q^\pm = u_q^\mp$, where

$$u_q^+(x) = |q|^{1/2} \begin{bmatrix} (\tanh(|x| + \theta) - q)^2 \\ -\operatorname{sech}^2(|x| + \theta) \end{bmatrix} e^{q(|x|+\theta)}\,, \quad \theta = \tanh^{-1} q\,.$$

Consequently,

$$|u_q^\pm(x)| \leq C|q|^{\frac{1}{2}} e^{-|qx|}\,, \quad \int u_q^\pm(x)^*\, \sigma_3 u_q^\pm(x)\, dx = 1\,. \tag{5.3}$$

For $0 < q \ll 1$ the operator H_q has no eigenfunctions near ± 1 and the thresholds ± 1 are not *resonances.*

Remark. The normalization of $u_q^\pm$ is consistent with the spectral decomposition of H_q – see §5.5.

We next analyse what happens when $k = -i$ and $\mu = 1$. Since explicit formulæ in that case do not play a rôle in our analysis, the spectrum of H_q (and equivalently of $F_q = -i\mathcal{L}_q$) near zero is analyzed by more general methods in Appendix B. There we proof the following lemma:

Lemma 5.2. *For $0 < |q| \ll 1$ the generalized kernel of F_q is given by $\{iv_1, \partial_\lambda v_\lambda|_{\lambda=1}\}$,*

$$F_q(iv_1) = 0\,, \quad F_q(\partial_\lambda v_\lambda|_{\lambda=1}) = iv_1\,.$$

In a neighbourhood of 0, F_q has two eigenvalues

$$\lambda_q^\pm = \begin{cases} \pm q^{\frac{1}{2}} + \mathcal{O}(q^{3/2}) & q > 0\,, \\ \pm i|q|^{\frac{1}{2}} + \mathcal{O}(|q|^{3/2}) & q < 0\,. \end{cases}$$

The two eigenfuctions, $w_q^\pm$, are odd, *and satisfy $\sigma_3 w_q^\pm = w_q^\mp$.*

Remark 5.3. Note that $H_q = 2F_q$, and thus the eigenvalues of H_q occur at $2\lambda_q^\pm$.

We also see that there are *no* embedded eigenvalues in the continuous spectrum: they would correspond to *real* poles in A and D. Hence for $k \in \mathbb{R} \setminus \{0\}$ the solutions

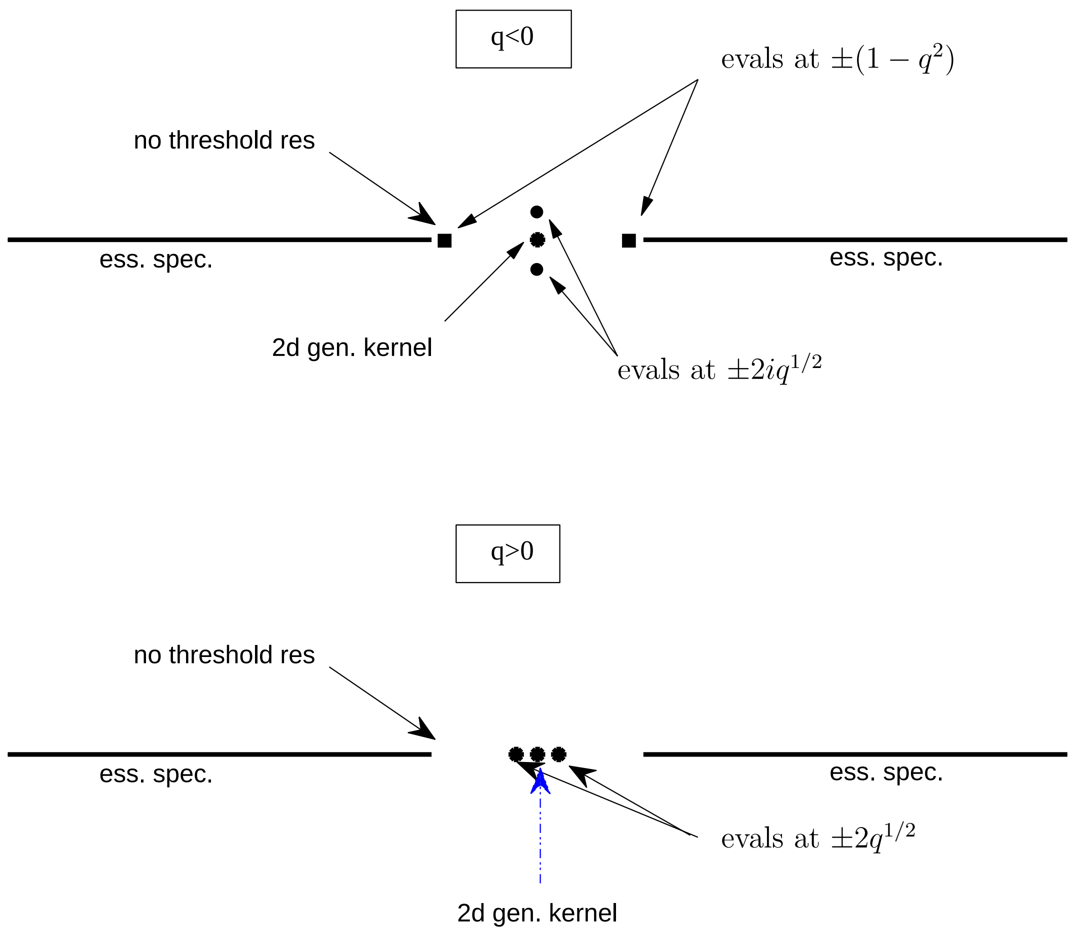

FIGURE 6. The spectrum of the operator H_q for $q < 0$ (repulsive δ potential) and $q > 0$ (attractive δ potential). The threshold resonances of the free problem become eigenvalues for the repulsive potential which is counterintuitive.

$\psi(x,k)$ and $\psi(x,-k)$, or the solutions $\psi(x,k)$ and $\psi(-x,k)$, form a basis of *tempered solutions* to

$$H_q u = (k^2+1)u\,.$$

Our operator H_q is the Hamiltonian matrix for the quadratic form given by

$$L = JH_q = \begin{bmatrix} 0 & -\partial_x^2+1 \\ -\partial_x^2+1 & 0 \end{bmatrix} + 2\,\mathrm{sech}^2(x+\mathrm{sgn}(x)\theta)\begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix} - 2q\begin{bmatrix} 0 & \delta_0 \\ \delta_0 & 0 \end{bmatrix},$$

but all we need are the general structural properties.

5.2. **Spectral decomposition of H_q.** Let P_q^c be the symplectic projection on the symplectic orthogonal of the discrete spectrum of H_q – which we know consists of 4 eigenvalues for $q > 0$ and 6 eigenvalues for $q < 0$.

As in the case of $q = 0$, we want to write the Schwartz kernel of P_q^c as

$$P_q^c(x,y) = \frac{1}{2\pi}\int_{\mathbb{R}} (v_+(x,k)\tilde{v}_+(y,k)^* + v_-(x,k)\tilde{v}_-(y,k)^*)dk\,,$$

where

$$H_q v_\pm = \pm(k^2+1)v_\pm\,, \quad H_q^* \tilde{v}_\pm = \pm(k^2+1)\tilde{v}_\pm\,,$$

and

$$\frac{1}{2\pi}\int_{\mathbb{R}} \tilde{v}_\pm(x,k)^* v_\pm(x,k')dx = \delta(k-k')\,, \quad \frac{1}{2\pi}\int_{\mathbb{R}} \tilde{v}_\pm(x,k)^* v_\mp(x,k')dx = 0\,. \tag{5.4}$$

Now the generalized eigenfunctions are in fact double with $\pm k$ corresponding to the single generalized eigenvalue $k^2 + 1$.

A comparison with standard one dimensional scattering – see [27, (1.26),(1.30)] – shows that the states $v_+(x,k)$ should be chosen so that they satisfy[2] (4.6) and (4.7) – see [1, §2.2.2] and [21, Proposition 6.9] for a full justification of this in the case of the system (4.1).

We compare these asymptotic formulæ to the properties of $\psi(x,k)$:

$$\psi(x,k) \sim \begin{bmatrix}1\\0\end{bmatrix} \begin{cases} B(k)(1-ik)^2 e^{ikx} + C(k)(1+ik)^2 e^{-ikx}\,, & x \to +\infty \\ (1+ik)^2 e^{ixk}\,, & x \to -\infty \end{cases}\,,$$

and

$$\psi(-x,-k) \sim \begin{bmatrix}1\\0\end{bmatrix} \begin{cases} (1-ik)^2 e^{ixk}\,, & x \to +\infty \\ B(-k)(1+ik)^2 e^{ikx} + C(-k)(1-ik)^2 e^{-ikx}\,, & x \to -\infty \end{cases}\,.$$

This shows that

$$v_+(x,k) \stackrel{\text{def}}{=} \begin{cases} a_+(k)\psi(-x,-k) & k > 0 \\ a_-(k)\psi(x,k) & k < 0\,, \end{cases} \tag{5.5}$$

where

$$a_\pm(k) = \frac{1}{(1 \pm ik)^2 B(\mp k)}\,.$$

Note that $v_+(-x,-k) = v_+(x,k)$. Define

$$v_-(x,k) \stackrel{\text{def}}{=} \sigma_1 v_+(x,k)\,,$$

and

$$\tilde{v}_\pm(x,k) \stackrel{\text{def}}{=} \sigma_3 v_\pm(x,k)\,.$$

[2]The states with $\pm k > 0$ correspond to $e_\pm$ in the notation of [27].

5.3. **Propagator** $\exp(-itH_q/2)$**.** The continuous spectrum part of the propagator appearing in Theorems 1 and 2 can now be written as

$$e^{-\frac{1}{2}itH_q}P_c(x,y) = \frac{1}{2\pi}\int_{\mathbb{R}}(e^{-\frac{1}{2}it(1+k^2)}v_+(x,k)\tilde{v}_+(y,k)^* + e^{\frac{1}{2}it(1+k^2)}v_-(x,k)\tilde{v}_-(y,k)^*)dk\,,$$

where $v_\pm$ are given in § 5.2. We have the analogue of the first part of Proposition 4.1. Since the proof is exactly the same, it is omitted.

Proposition 5.4. *Suppose that* $w_0 \in \mathcal{S}(\mathbb{R}\setminus\{0\})\cap L^\infty(\mathbb{R})$ *is real valued and even. Then*

$$\begin{aligned} e^{-\frac{1}{2}itH_q}P_c w_0(x,t) &= \frac{2}{\sqrt{2\pi}}\int_0^\infty (a_{\mathrm{ev}}(x,k)e^{-\frac{1}{2}i(k^2+1)t} + b_{\mathrm{ev}}(x,k)e^{\frac{1}{2}i(k^2+1)t})f(k)dk\,, \\ f(k) &= \frac{2}{\sqrt{2\pi}}\int_0^\infty (\overline{a_{\mathrm{ev}}(x,k)} - \overline{b_{\mathrm{ev}}(x,k)})w_0(x)\,dx\,, \end{aligned} \tag{5.6}$$

where a and b are defined by

$$v_+(x,k) = \begin{bmatrix} a(x,k) \\ b(x,k)\end{bmatrix}\,, \tag{5.7}$$

and $a_{\mathrm{ev}}(x,k) = (a(x,k)+a(-x,k))/2$, $b_{\mathrm{ev}}(x,k) = (b(x,k)+b(-x,k))/2$ *.*

In the above proposition, we reexpressed integrals over $\mathbb{R}$ as integrals over $(0,+\infty)$ in (5.6) using that $a(x,k) = a(-x,-k)$ and $b(x,k) = b(-x,-k)$. This implies that $f(k)$ is even, and that $a_{\mathrm{ev}}(x,k)$ and $b_{\mathrm{ev}}(x,k)$ are even in both x and k.

5.4. **Asymptotic analysis of the breathing patterns.** We will now prove (1.5) by describing the asymptotics of

$$w(x,t) \stackrel{\mathrm{def}}{=} e^{-\frac{1}{2}itH_q}P_q^c w_0(x,t)\,, \tag{5.8}$$

at $x=0$. Here w_0 is assumed to satisfy (1.3). In particular,

$$w(0,t) = \sqrt{\frac{2}{\pi}}\int_0^\infty (a(0,k)e^{-\frac{1}{2}it(1+k^2)} + b(0,k)e^{\frac{1}{2}it(1+k^2)})f(k)\,dk\,, \tag{5.9}$$

where a and b are defined by (5.7) and f is given in (5.6).

We now focus on the form of the expression for $f(k)$ and derive some of its smoothness and decay properties. The behaviour of f for large values of k can be deduced directly from the definition of $f(k)$ as the pairing of w_0 with a solution ψ to $H_q\psi = (1+k^2)\psi$.

Lemma 5.5. *For real and even w_0 satisfying* (1.3)*, and for f defined by* (5.6) *we have* $f|_{\mathbb{R}_\pm}\in C^\infty(\mathbb{R}_\pm)$*. For* $|k|>\epsilon$ *we have*

$$|f^{(p)}(k)| \leq \frac{C_{\epsilon,p}q}{1+k^2}\,,$$

uniformly in q.

Proof. We recall from (4.12) (the formal structure of $f(k)$ is the same as in the free case) that

$$f(k) = \left(V_+^* \begin{bmatrix} 1 \\ -1 \end{bmatrix} w_0\right)(k) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} v_+(x,k)^* \begin{bmatrix} w_0(x) \\ -w_0(x) \end{bmatrix} dx\,.$$

The formulæ for $a(x,k)$ and $b(x,k)$ above, and the formulæ in Appendix C, show that the $v_+(x,k)$ are uniformly bounded in x and in k, for $|k| > \epsilon$. Since $(k^2+1)v_+(x,k) = H_q v_+(x,k)$, integration by parts (see the formula for H_q in (2.13)) shows that

$$\begin{aligned}(1+k^2)f(k) = {} & \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left(\begin{bmatrix} -\partial_x^2 + 1 - 4v^2 & -2v^2 \\ -2v^2 & -\partial_x^2 + 1 - 4v^2 \end{bmatrix} v_+(x,k) \right)^* \begin{bmatrix} w_0(x) \\ w_0(x) \end{bmatrix} dx \\ & - \frac{2q}{\sqrt{2\pi}} v_+(0,k)^* \begin{bmatrix} w_0(0) \\ w_0(0) \end{bmatrix} \\ = {} & \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}\backslash 0} v_+(x,k)^* \begin{bmatrix} -\partial_x^2 + 1 - 4v^2 & -2v^2 \\ -2v^2 & -\partial_x^2 + 1 - 4v^2 \end{bmatrix} \begin{bmatrix} w_0(x) \\ w_0(x) \end{bmatrix} dx \\ & + \frac{1}{\sqrt{2\pi}} v_+(0,k)^* \left(2q \begin{bmatrix} w_0(0) \\ w_0(0) \end{bmatrix} + \begin{bmatrix} w_0'(0-) - w_0'(0+) \\ w_0'(0-) - w_0'(0+) \end{bmatrix} \right)\,,\end{aligned}$$

where the last term came from the fact that $w_0(x) = u(x,0) - v_1(x)$, $u(x,0) \in H^1$, so that w_0 is continuous at $x = 0$, and the $w_0'(0\pm)$ terms come from integation by parts.

The right hand side is uniformly bounded for $|k| > \epsilon$ which proves the lemma for $p = 0$.

We can now proceed by induction noting that

$$\begin{aligned} H_q \partial_k^p v_+(x,k) &= \partial_k^p H_q v_+(x,k) \\ &= (k^2+1)\partial_k^p v_+(x,k) + 2pk\partial_k^{p-1} v_+(x,k) + p(p-1)\partial_k^{p-2} v_+(x,k)\,, \end{aligned}$$

and that for $|k| > \epsilon$, $|\partial_k^p v_+(x,k) w_0(x)| \le C_\epsilon$, $x \in \mathbb{R}$. □

We now derive a workable expression for $f(k)$. The formulæ (5.1) and (5.5) show that for $k > 0$, we have

$$\begin{aligned} a(x,k) = {} & \left(\frac{(t-ik)^2 e^{ik(x+\theta)}}{B(-k)(1+ik)^2} - \frac{A(-k)s^2 e^{-\mu(x+\theta)}}{B(-k)(1+ik)^2} \right) x_+^0 + \\ & + \left(\frac{(t-ik)^2 e^{ik(x-\theta)}}{(1+ik)^2} + \frac{C(-k)(t+ik)^2 e^{-ik(x-\theta)}}{B(-k)(1+ik)^2} - \frac{D(-k)s^2 e^{\mu(x-\theta)}}{B(-k)(1+ik)^2} \right) x_-^0\,, \end{aligned}$$

and

$$\begin{aligned} b(x,k) = {} & \left(\frac{-s^2 e^{ik(x+\theta)}}{B(-k)(1+ik)^2} + \frac{A(-k)(t+\mu)^2 e^{-\mu(x+\theta)}}{B(-k)(1+ik)^2} \right) x_+^0 + \\ & + \left(-\frac{s^2 e^{ik(x-\theta)}}{(1+ik)^2} - \frac{C(-k)s^2 e^{-ik(x-\theta)}}{B(-k)(1+ik)^2} + \frac{D(-k)(t-\mu)^2 e^{\mu(x-\theta)}}{B(-k)(1+ik)^2} \right) x_-^0\,, \end{aligned}$$

where $s = \operatorname{sech}(x+\operatorname{sgn}(x)\theta)$, $t = \tanh(x+\operatorname{sgn}(x)\theta)$, $\theta = \tanh^{-1}(q)$, and $\mu = (k^2+2)^{\frac{1}{2}}$. From these expressions, we deduce that for $x > 0$, $k > 0$,

$$\begin{aligned} a_{\text{ev}}(x,k) = \; & \frac{(1+C(-k))(t-ik)^2 e^{ik(x+\theta)}}{2B(-k)(1+ik)^2} + \frac{(t+ik)^2 e^{-ik(x+\theta)}}{2(1+ik)^2} \\ & - \frac{(A(-k)+D(-k))s^2 e^{-\mu(x+\theta)}}{2B(-k)(1+ik)^2} \,, \end{aligned}$$

$$b_{\text{ev}}(x,k) = -\frac{(1+C(-k))s^2 e^{ik(x+\theta)}}{2B(-k)(1+ik)^2} - \frac{s^2 e^{-ik(x+\theta)}}{2(1+ik)^2} + \frac{(A(-k)+D(-k))(t+\mu)^2 e^{-\mu(x+\theta)}}{2B(-k)(1+ik)^2} \,,$$

and thus (using that $A(-k) = \overline{A(k)}$, etc.) for $k > 0$, we have

$$f(k) = \frac{1+C(k)}{2B(k)(1-ik)^2} f_1(k) + \frac{1}{2(1-ik)^2} f_1(-k) - \frac{A(k)+D(k)}{2B(k)(1-ik)^2} f_2(k) \,, \tag{5.10}$$

where

$$\begin{aligned} f_1(k) &= \frac{1}{\sqrt{2\pi}} \int_0^\infty ((t+ik)^2 + s^2) e^{-ik(x+\theta)} w_0(x)\, dx \,, \\ f_2(k) &= \frac{1}{\sqrt{2\pi}} \int_0^\infty ((t+\mu)^2 + s^2) e^{-\mu(x+\theta)} w_0(x)\, dx \,. \end{aligned} \tag{5.11}$$

By differentiation under the integral sign, integration by parts, and Taylor's theorem, we have

Lemma 5.6. *Let f_j be defined by* (5.11) *and suppose w_0 satisfies* (1.3)*. We have, for each $\ell = 0, 1, 2 \ldots$,*

$$\|f_j^{(\ell)}\|_{L^\infty} \lesssim q(1+|k|) \,,$$

with the implicit constants depending only upon ℓ and w_0 (specifically, the "q" on the right side could be replaced with a finite sum of seminorms of w_0). Moreover,

$$f_1(k) = f_1(0) + k f_1'(0) + k^2 g(k)$$

where $g(k)$ is a smooth function satisfying for each $\ell = 0, 1, 2 \ldots$,

$$\|g^{(\ell)}\|_{L^\infty} \lesssim q(1+|k|) \,.$$

Now we return to the computation of $w(0,t)$ given by (5.9). Because of continuity at $x = 0$ we conclude that

$$a(0,k) = \frac{1}{(1+ik)^2} \Big(\frac{(q-ik)^2 e^{ik\theta}}{B(-k)} + \frac{A(-k)(1-q^2)e^{-\mu\theta}}{B(-k)} \Big) \,, \tag{5.12}$$

and $b(0,k) = b_1(k) + b_2(k)$, where

$$b_1(k) = -\frac{e^{ik\theta}}{(1+ik)^2 B(-k)}, \quad b_2(k) = \frac{q^2 e^{ik\theta} + A(-k)(q+\mu)^2 e^{-\mu\theta}}{(1+ik)^2 B(-k)} \tag{5.13}$$

Upon substituting (5.12), (5.13) and (5.10) into (5.9), we obtain an expression with many terms. We first observe in the following lemma that, fortunately, many of these terms are of lower order.

Lemma 5.7. *For w_0 satisfying* (1.3)*, and $f_1(k)$, $f_2(k)$ defined in* (5.11)*, we have that each of the following*

$$\int_0^\infty e^{-\frac{1}{2}itk^2} a(0,k) f(k)\, dk, \qquad \int_0^\infty e^{\frac{1}{2}itk^2} b_2(k) f(k)\, dk,$$

$$\int_0^\infty e^{\frac{1}{2}itk^2} \frac{b(0,k)(A(k)+D(k))}{2B(k)(1-ik)^2} f_2(k)\, dk\,,$$

is of size

$$\mathcal{O}\Big(\frac{q^2}{t^{1/2}}\Big) + \mathcal{O}\Big(\frac{q}{t^{3/2}}\Big)\,.$$

We will prove this lemma later. In the next lemma, we deduce the asymptotic form of the dominant terms in the expression for $w(0,t)$.

Lemma 5.8. *For w_0 satisfying* (1.3)*, and $f_1(k)$ defined in* (5.11)*, we have*

$$\begin{aligned} &\int_0^\infty \frac{e^{\frac{1}{2}ik^2 t}}{2(1+k^2)^2 B(-k)} \Big(\frac{1+C(k)}{B(k)} f_1(k) + f_1(-k)\Big)\, dk \\ &\quad = \frac{1}{2} t^{-1/2} e^{i\pi/4} \int_0^\infty w_0(x)\, dx + \mathcal{O}(q^2) + \mathcal{O}\left(\frac{q}{t^{3/2}}\right) \end{aligned} \tag{5.14}$$

Combining Lemmas 5.7, 5.8, we obtain the following proposition.

Proposition 5.9. *For $w(0,t)$ given by* (5.9) *we have for $t \gg 1$,*

$$w(0,t) = -\sqrt{\frac{2}{\pi t}} e^{it/2 + i\pi/4} \int_{\mathbb{R}} w_0(x) dx + \mathcal{O}\left(\frac{q}{t^{3/2}}\right) + \mathcal{O}(q^2)\,.$$

Remark. The leading expression in Proposition 5.9 is formally the same as the expression in the case $q = 0$ in §4.3. For the case described in Fig. 1,

$$w_0(x) = \frac{1}{1+q} \operatorname{sech}\left(\frac{x}{1+q}\right) - \operatorname{sech}(|x| + \tanh^{-1} q)\,,$$

and we have

$$\int_{-\infty}^\infty w_0(x) = 2q.$$

Now we develop some preliminaries in order to prove Lemmas 5.7 and 5.8. To streamline the presentation, we introduce a definition:

Definition 3. *A function $h(k,q)$ is* conormal *(at $k=0$) uniformly in q,*

$$h \in \mathcal{A}\,,$$

if for $0 \le \ell \le 4$,

$$
\begin{aligned}
&|\partial_k^l h(k,q)| \le C_\ell \qquad \textit{for} \quad |k| \ge 1\,, \\
&|(k\partial_k)^\ell h(k,q)| \le C_\ell \quad \textit{for} \quad |k| \le 1\,,
\end{aligned} \tag{5.15}
$$

with the constants independent of q.

We note that the sum and product of conormal functions is conormal. The two main types of lower order terms that we encounter arise from either q times a conormal function or k^2 times a conormal function. The former will give an error of size $q^2/t^{1/2}$ and the latter an error of size $q/t^{3/2}$. This will follow (as we will see in more detail in the proof of Lemmas 5.7, 5.8 below) from Lemma 5.6 and the following lemma applied with $f = f_j$, $j = 1, 2$ defined in (5.11).

Lemma 5.10. *Suppose that* $h(k,q)$ *is conormal in the sense of Definition 3. Then*

$$
\left| \int_0^\infty e^{\pm \frac{1}{2} itk^2} h(k,q) f(k)\, dk \right| \lesssim \frac{1}{\sqrt{t}} \sum_{j=0}^{2} \|f^{(j)}\|_{L^\infty}\,, \tag{5.16}
$$

$$
\left| \int_0^\infty e^{\pm \frac{1}{2} itk^2} k^2 h(k,q) f(k)\, dk \right| \lesssim \frac{1}{t^{3/2}} \sum_{j=0}^{4} \|f^{(j)}\|_{L^\infty}\,, \tag{5.17}
$$

with the implicit constants independent of q.

Proof. We begin with (5.16). Let $s = k\sqrt{t}$. Then the integral to be estimated takes the form

$$
\frac{1}{\sqrt{t}} \int_0^\infty e^{\frac{1}{2} is^2} h\Big(\frac{s}{\sqrt{t}}\Big) f\Big(\frac{s}{\sqrt{t}}\Big)\, ds
$$

Let $\chi(s)$ satisfy

$$
\chi \in C_{\rm c}^\infty((-1,1))\,, \quad \chi \text{ is equal to 1 in a neighbourhood of } s = 0. \tag{5.18}
$$

Clearly,

$$
\left| \frac{1}{\sqrt{t}} \int_0^\infty \chi(s) e^{\frac{1}{2} its^2} h\Big(\frac{s}{\sqrt{t}}\Big) f\Big(\frac{s}{\sqrt{t}}\Big)\, ds \right| \lesssim \frac{\|h\|_{L^\infty} \|f\|_{L^\infty}}{\sqrt{t}} \lesssim \frac{\|f\|_{L^\infty}}{\sqrt{t}}
$$

and therefore we just need to estimate

$$
\frac{1}{\sqrt{t}} \int_0^\infty (1-\chi(s)) e^{\frac{1}{2} is^2} h\Big(\frac{s}{\sqrt{t}}\Big) f\Big(\frac{s}{\sqrt{t}}\Big)\, ds \tag{5.19}
$$

Using that $(-is^{-1}\partial_s)^2 e^{\frac{1}{2} is^2} = e^{\frac{1}{2} is^2}$ and two applications of integration by parts gives

$$
\frac{1}{\sqrt{t}} \int_0^\infty e^{\frac{1}{2} is^2} (-i\partial_s\, s^{-1})^2 \Big[(1-\chi(s)) h\Big(\frac{s}{\sqrt{t}}\Big) f\Big(\frac{s}{\sqrt{t}}\Big)\Big]\, ds
$$

Distributing the derivatives and estimating (using the s^{-2} factor to carry out the integration), we obtain the bound

$$\left(\sum_{0\le\ell\le 2}\frac{\|h^{(\ell)}\|_{L^\infty(|k|\ge t^{-1/2})}}{t^{\ell/2}}\right)\left(\sum_{0\le\ell\le 2}\frac{\|f^{(\ell)}\|_{L^\infty(|k|\ge t^{-1/2})}}{t^{\ell/2}}\right).$$

Now we just apply (5.15) to obtain the bound (5.16).

Now we establish (5.17). Let $s = k\sqrt{t}$ to obtain

$$\frac{1}{t^{3/2}}\int_0^\infty e^{\frac12 is^2}s^2 h\Big(\frac{s}{\sqrt{t}}\Big)f_j\Big(\frac{s}{\sqrt{t}}\Big)\,ds$$

The remainder of the proof is similar to that above, except that we need to use $(-is^{-1}\partial_s)^4 e^{\frac12 is^2} = e^{\frac12 is^2}$ and four applications of integration by parts. □

We shall need the following properties of the scattering coefficients A, B, C, and D, obtained from the more precise asymptotics in Appendix C.

Lemma 5.11 (Properties of A, B, C, D)**.** *$1/B(k)$ and $C(k)/B(k)$ are conormal, and in fact*

$$\frac{1}{B(k)} = \frac{k}{k-iq} + q\alpha_1(k,q) + k^2\alpha_2(k,q)\,,$$
$$\frac{C(k)}{B(k)} = \frac{iq}{k-iq} + q\alpha_3(k,q) + k^2\alpha_4(k,q)\,,$$

where $\alpha_j \in \mathcal{A}$ are conormal in the sense of Definition 3. Also,

$$\frac{A(k)}{B(k)} = q\beta_1(k,q)\,,\quad \frac{D(k)}{B(k)} = q\beta_2(k,q)\,,\quad \beta_j\in\mathcal{A}\,.$$

With these preliminaries out of the way, we can now prove Lemmas 5.7 and 5.8.

Proof of Lemma 5.7. We shall give the proof for

$$\int_0^\infty e^{-\frac12 itk^2}a(0,k)\,f(k)\,dk\,. \tag{5.20}$$

The other integrals in the statement of the lemma are treated similarly. By Lemma 5.11, we see that in the expression (5.10), all coefficients of f_1, f_2 are conormal. Also by Lemma 5.11 and (5.12), we see that

$$a(0,k) = qa_1(k,q) + k^2 a_2(k,q)\,,\quad a_j\in\mathcal{A}\,.$$

By the alegbra property of the conormal class, (5.16),(5.17), and Lemma 5.6, we obtain that (5.20) is of size $\mathcal{O}(q^2/t^{1/2}) + \mathcal{O}(q/t^{3/2})$. □

Proof of Lemma 5.8. We write $\approx$ to mean that the two quantities are equal with an error of the form q times conormal or k^2 times conormal. By Lemma 5.11,

$$\frac{1}{B(-k)} \approx \frac{k}{k+iq}\,,\quad \frac{1+C(k)}{B(k)} \approx \frac{k+iq}{k-iq}\,,\quad \frac{1}{(1+k^2)^2}\approx 1\,.$$

We also take the expansion in Lemma 5.6:

$$f_1(k) = f_1(0) + kf'(0) + k^2 g(k)\,,$$
$$f_1(-k) = f_1(0) - kf'(0) + k^2 g(-k)\,.$$

Substituting the above into (5.14) and appealing to (5.16),(5.17) and Lemma 5.6 for the error terms, we see that (5.14) is equal to

$$\int_0^\infty e^{\frac{1}{2}itk^2} \frac{k}{2(k-iq)} \left(\frac{k+iq}{k-iq}(f_1(0)+kf'(0)) + (f_1(0)-kf_1'(0)) \right) dk \\ +\mathcal{O}\left(\frac{q^2}{t^{1/2}}\right) + \mathcal{O}\left(\frac{q}{t^{3/2}}\right).$$

This simplifies to

$$f_1(0)\int_0^\infty e^{\frac{1}{2}itk^2} \frac{k^2}{k^2+q^2}\,dk + iqf_1'(0)\int_0^\infty e^{\frac{1}{2}itk^2} \frac{k^2}{k^2+q^2}\,dk \\ +\mathcal{O}\left(\frac{q^2}{t^{1/2}}\right) + \mathcal{O}\left(\frac{q}{t^{3/2}}\right). \tag{5.21}$$

Note that

$$\int_0^\infty e^{\frac{1}{2}is^2}\frac{s^2}{s^2+\delta^2}\,ds = \int_0^\infty e^{\frac{1}{2}is^2}\,ds - \delta\int_0^\infty e^{\frac{1}{2}i\delta^2 s^2}\frac{1}{s^2+1}\,ds\,,$$

where, in the second term, we made the substitution $s \mapsto \delta s$. Thus,

$$\int_0^\infty e^{\frac{1}{2}is^2}\frac{s^2}{s^2+\delta^2}\,ds = \sqrt{\frac{\pi}{2}}e^{i\pi/4} + \mathcal{O}(\delta)\,.$$

In (5.21), make the substitution $s = t^{1/2}k$ and appeal to the above formula to obtain

$$\frac{f_1(0)+\mathcal{O}(q^2)}{\sqrt{t}}\left(\sqrt{\frac{\pi}{2}}e^{i\pi/4} + \mathcal{O}(q\sqrt{t})\right) + \mathcal{O}\left(\frac{q^2}{t^{1/2}}\right) + \mathcal{O}\left(\frac{q}{t^{3/2}}\right) \\ = \frac{1}{2}t^{-1/2}e^{i\pi/4}\int_0^\infty w_0(x)\,dx + \mathcal{O}(q^2) + \mathcal{O}\left(\frac{q}{t^{3/2}}\right).$$

□

5.5. **Proof of Theorem 1.** We will now combine Theorem 2 with the results of this section to proof Theorem 1. We start with the following lemma

Lemma 5.12. *Suppose that w_0 satisfies the assumptions of Theorem 1 and that*

$$\lambda_0 \stackrel{\text{def}}{=} 1 + \int_{\mathbb{R}} w_0(x)v_1(x)dx$$

is the nonlinear eigenvalue specified in Theorem 1. Then for the projection P_{λ_0} defined by (3.1) *(with q supressed in the subscript),*

$$P_{\lambda_0}(v_{\lambda_0} - v_1 - w_0) = \mathcal{O}(q^2)\,, \tag{5.22}$$

and consequently the solution, λ, to $P_\lambda(v_\lambda - v_1 - w_0) = 0$ satisfies

$$\lambda - \lambda_0 = \mathcal{O}(q^2)\,. \tag{5.23}$$

Proof. The definition (3.1) means that we need to show that $\omega(v_{\lambda_0} - v_1 - w_0, iv_{\lambda_0}) = \mathcal{O}(q^2)$ since the other term vanishes by the reality of w_0. Now, using the definition of λ_0 and the fact that $\lambda_0 = 1 + \mathcal{O}(q)$, we see that

$$\begin{aligned}\omega(v_{\lambda_0} - v_1 - w_0, iv_{\lambda_0}) &= 2(\lambda_0 - q) - \int v_{\lambda_0}(x)v_1(x)dx - \int w_0(x)v_{\lambda_0}(x)dx \\ &= 2(\lambda_0 - q) - \int v_{\lambda_0}(x)v_1(x)dx - \int w_0(x)v_1(x)dx + \mathcal{O}(q^2) \\ &= \int w_0(x)v_1(x) - (\lambda_0 - 1)\int \partial_\lambda(v_\lambda)|_{\lambda=1}(x)v_1(x)dx + \mathcal{O}(q^2)\,.\end{aligned}$$

The estimate (5.22) follows from

$$\int \partial_\lambda(v_\lambda)|_{\lambda=1}(x)v_1(x)dx = \frac{1}{2}\partial_\lambda\|v_\lambda\|^2_{L^2}|_{\lambda=1} = 1\,.$$

The comparison (5.23) between the exact solution and the approximate one is obtained from the implicit function theorem as in the proof of Proposition 3.1. □

The lemma shows that the assumptions of Theorem 2 are satisfied for $h = Cq$, $\theta = 0$ (w_0 is real) and

$$\lambda = \lambda_0 + \mathcal{O}(q^2)\,.$$

We can then apply Corollary 3.5 to obtain

$$\|u(t) - e^{it\lambda_0^2/2}\Big(v_{\lambda_0} + e^{-it\mathcal{L}_{\lambda_0,q}}w_0\Big)\|_{H^1_x} \leq C(1+t)^2q^2\,, \tag{5.24}$$

for all $0 \leq t \ll q^{-1/2}$. We now write

$$\begin{aligned}e^{-it\mathcal{L}_{\lambda_0,q}}w_0 &= \begin{bmatrix}1 & 0\end{bmatrix} e^{-\frac{1}{2}itH_{\lambda_0,q}}\begin{bmatrix}w_0\\ w_0\end{bmatrix} \\ &= \begin{bmatrix}1 & 0\end{bmatrix} e^{-\frac{1}{2}itH_{\lambda_0,q}}P_d\begin{bmatrix}w_0\\ w_0\end{bmatrix} + \begin{bmatrix}1 & 0\end{bmatrix} e^{-\frac{1}{2}itH_{\lambda_0,q}}P_c\begin{bmatrix}w_0\\ w_0\end{bmatrix}\,.\end{aligned}$$

The first conclusion of Theorem 1 given in (1.4) is immediate from (5.24) and Proposition 5.4 once we show that

$$\begin{bmatrix}1 & 0\end{bmatrix} e^{-\frac{1}{2}itH_{\lambda_0,q}}P_d\begin{bmatrix}w_0\\ w_0\end{bmatrix} = \mathcal{O}_{H^1}(q^{3/2})\,, \quad 0 \leq t \ll q^{-1/2}\,. \tag{5.25}$$

For $q > 0$ we have six contributions to the discrete spectrum, while for $q < 0$ there are four. By Lemma 5.2 the non-zero eigenvalues in the neighbourhood of zero do not contribute as they are odd while w_0 is even. The contribution of the zero eigenvalues is

$\mathcal{O}(q^2 t)$ by the same arguments as in §3. For $q < 0$ the coefficients of the eigenfuctions (which are uniformly bounded in H^1) are estimated using Lemma 5.1 by

$$C q^{1/2} \int_{\mathbb{R}} |w_0(x)| e^{-q|x|} \, dx \leq C' q^{3/2} \, .$$

Hence (5.25) holds and in view of (5.24) we have established (1.4).

To obtain (1.5) we use the above estimate and Proposition 5.9. The combined error term for $0 \ll t \ll q^{-1/2}$ is

$$\mathcal{O}\left(\frac{|q|}{t^{3/2}} + |q|^{3/2} + q^2 t^2 \right) ,$$

and that is bounded by $C|q|/t^{3/2}$ for $t \leq C'|q|^{-2/7}$.

Appendix A. A derivation of Kaup's basis using MATLAB

The Kaup spectral decomposition of the linearized operator was based on the connection with the Zakharov-Shabat system and the complete integrability of the cubic NLS, see [19] and [30]. We rediscovered the structure of his basis of solutions through a numerical experiment and it might be of interest to indicate how that was done. The original motivation was to show that the threshold resonances for the linearization of the cubic nonlinear Schrödinger equation (NLS) on the line are *simple* which can be done by an explicit construction of a solution to a system of ODEs.

The explicit resonant state of the linearized operator H_0 at 1 is given by

$$u_1 = \begin{bmatrix} 1 - \operatorname{sech}^2 x \\ - \operatorname{sech}^2 x \end{bmatrix} . \tag{A.1}$$

To show that it is simple, we need to show that any other bounded solution is a multiple of u_1.

As in standard scattering theory, the four independent solutions of $Hu = u/2$ can be characterized by their behaviour as $x \to \infty$ – see the proof of [21, Lemma 5.19]. In particular the resonant states can only be given as linear combinations of the two solutions, u_1 and u_2, satisfying

$$u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \mathcal{O}(e^{-2x}) \, , \quad e^{\sqrt{2}x} u_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \mathcal{O}(e^{-2x}) \, , \quad x \longrightarrow +\infty \, . \tag{A.2}$$

We see that u_1 is given by (A.1) and $u_1 \in L^\infty$. Once we show that $u_2 \notin L^\infty$ we will see that the multiplicity of the resonance is one.

As we have already seen in §4.1 the solution u_2 can be written explicitly. An elementary calculation confirms that

$$u_2 = \frac{\exp(-\sqrt{2}x)}{(1+\sqrt{2})^2} \begin{bmatrix} - \operatorname{sech}^2 x \\ (\tanh x + \sqrt{2})^2 \end{bmatrix} . \tag{A.3}$$

This shows that

$$u_2 = \exp(-\sqrt{2}x)\frac{(-1+\sqrt{2})^2}{(1+\sqrt{2})^2}\left(\begin{bmatrix}0\\1\end{bmatrix} + \mathcal{O}(e^{-2|x|})\right), \quad x \longrightarrow -\infty,$$

and, in particular, that $u_2 \notin L^\infty$.

The exact expression (A.3) was arrived at through an attempt to produce a computer assisted proof of the fact that $u_2 \notin L^\infty$.

The solution $u_2(x)$ is obtained by solving the following Volterra integral equation (see [21, (5.4)], where one should let $\lambda \to 0$ and renormalize following $\partial_x^2 \mapsto \partial_x^2/2$):

$$u_2(x) = e^{-\sqrt{2}x}\begin{bmatrix}0\\1\end{bmatrix} - \sqrt{2}\int_x^\infty \begin{bmatrix}2\sqrt{2}(y-x) & \sqrt{2}(y-x)\\ \sinh(\sqrt{2}(y-x)) & 2\sinh(\sqrt{2}(y-x))\end{bmatrix}\operatorname{sech}^2 y\, u_2(y)dy\,.$$

Solving this equation by iteration could in principle show that the solution is unbounded.

Elimination of the exponential growth in the integral equation is helpful for the theoretical estimates of [1] and [21] and seems essential for a succesful numerical scheme. Thus we consider

$$v(x) \stackrel{\text{def}}{=} \exp(\sqrt{2}x)u_2(x)\,,$$

which solves

(A.4)

$$v(x) = \begin{bmatrix}0\\1\end{bmatrix} + \mathcal{K}v(x)\,, \quad \mathcal{K}v(x) \stackrel{\text{def}}{=} \int_x^\infty K(x,y)v(y)dy\,,$$

$$K(x,y) \stackrel{\text{def}}{=} -\sqrt{2}\begin{bmatrix}2\sqrt{2}(y-x) & \sqrt{2}(y-x)\\ \sinh(\sqrt{2}(y-x)) & 2\sinh(\sqrt{2}(y-x))\end{bmatrix}\operatorname{sech}^2 y\, \exp(\sqrt{2}(x-y))\,.$$

It is not hard to see the convergence of

$$v(x) = \sum_{n=0}^\infty \mathcal{K}^n\left(\begin{bmatrix}0\\1\end{bmatrix}\right)(x)\,, \tag{A.5}$$

in, say $C^k(\mathbb{R})$, for any k. Hence showing that $v(x) \not\to 0$, $x \to -\infty$ is in principle possible by a numerical computation.

We easily implement the operator $\mathcal{K}$ in MATLAB. The input is an array which is a discretised $\mathbb{R}^2$-valued function on $[-10, 30]$ with N grid points. Because of the Volterra structure of the equation the left limit, -10, is not important. The right cutoff, 30, is chosen large enough to make the effect of the potential negligible. The integrals are computed using the built-in trapezium rule and the errors can be estimates. We used $N = 10^4$ which would have to be even larger for rigorous estimates, while experimentally it was clearly an "overkill".

```
function KT = KT(u)
mu =sqrt(2);
```

```
[M,N]=size(u);
x = linspace(-10,30,N);
for j=1:N-1
y = linspace(x(j),30,N-j+1); v = sech(y).*sech(y);
u1=u(:,[j:N]);
uu(1,:)=v.*(y-x(j)).*(2*u1(1,:)+u1(2,:));
uu(2,:)=v.*(sinh(mu*(y-x(j)))/mu).*(u1(1,:)+2*u1(2,:));
uu(1,:)=-4*exp(mu*(x(j)-y)).*uu(1,:);
uu(2,:)=-4*exp(mu*(x(j)-y)).*uu(2,:);
KT(1,j)=trapz(y,uu(1,:));
KT(2,j)=trapz(y,uu(2,:));
clear uu
end
KT(:,N)=[0;0];
```

When the numerical solution obtained using (A.5) with $n = 10$ was plotted (see Fig.7) we noticed that the plot of the first component looked remarkably like a plot of $-\alpha \operatorname{sech}^2 x$, $\alpha > 0$ and the fit based on the minimum of first component (experimental $-\alpha$) was almost exact. From the operator H_0 it is clear that having one component of the solution we obtain the other and that quickly led to the exact solution (A.3). This then suggests the form of general solution for other values of k and μ as given in §4.1.

It would be difficult in general, and by our method in particular, to show the existence of a resonance.

Appendix B. Perturbation of eigenvalues at zero energy

Here we present the perturbation theory for $H_q - z$ at $z = 0$. Even though we could in principle obtain the same results from careful analysis of the matrix $\mathcal{A}(k, q)$ described in detail in Appendix C, the method used here is more general and does not depend on explicit formulæ. It is of course close to the similar study in the semi-classical case, see [11] and references given there. However, since the delta function is clearly different from a slowly varying potential with a nondegenerate minimum we give a selfcontained argument.

B.1. **Grushin problem.** We recall that the linearized operator acting on

$$\begin{bmatrix} \operatorname{Re} w \\ \operatorname{Im} w \end{bmatrix} \in \mathbb{R}^2 \subset \mathbb{C}^2 \,,$$

is given by

$$F_q \stackrel{\text{def}}{=} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} L_{q+} & 0 \\ 0 & L_{q-} \end{bmatrix} , \qquad \begin{aligned} L_{q+} &= 1 - \partial_x^2 - 6v^2 - 2q\delta_0 \\ L_{q-} &= 1 - \partial_x^2 - 2v^2 - 2q\delta_0 \,, \end{aligned} \tag{B.1}$$

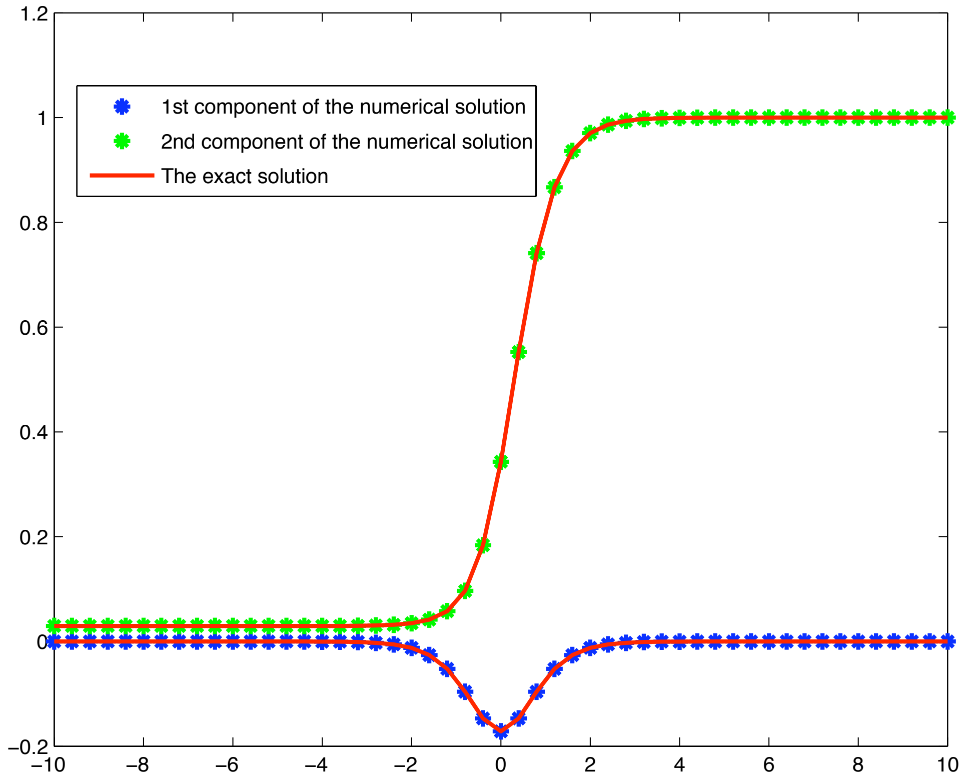

FIGURE 7. The plots of components of $\exp(\sqrt{2}x)u_2(x)$, following the numerical computation and the exact solutions. We see that $\lim_{x\to-\infty}\exp(\sqrt{2}x)u_2^2(x) = (3-2\sqrt{2})/(3+2\sqrt{2}) \neq 0$, where u_2^2 is the second component of the solution. We used $N = 10^4$ grid point and the plot shows the sampling of 100 points.

where v is the nonlinear ground state. We take elements of $H^2(\mathbb{R};\mathbb{C})$ and write them as column vectors of real and imaginary parts giving an identification

$$H^2(\mathbb{R};\mathbb{C}) \simeq H^2(\mathbb{R};\mathbb{R}) \oplus H^2(\mathbb{R};\mathbb{R})$$

The elements $e_j\eta$ take the 2-vector form

$$e_1\eta = \begin{bmatrix} -\eta \\ 0 \end{bmatrix}, \quad e_2\eta = \begin{bmatrix} 0 \\ x\eta \end{bmatrix}, \quad e_3\eta = \begin{bmatrix} 0 \\ \eta \end{bmatrix}, \quad e_4\eta = \begin{bmatrix} \eta + x\eta' \\ 0 \end{bmatrix}$$

The symplectic form, in vector notation, becomes

$$\omega\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = \int (-u_1 v_2 + v_1 u_2) \tag{B.2}$$

In the matrix notation, the relations (2.21) become

$$\begin{bmatrix} 0 & L_- \\ -L_+ & 0 \end{bmatrix} \begin{bmatrix} -\eta' \\ 0 \end{bmatrix} = 0, \qquad \begin{bmatrix} 0 & L_- \\ -L_+ & 0 \end{bmatrix} \begin{bmatrix} 0 \\ x\eta \end{bmatrix} = \begin{bmatrix} -\eta' \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & L_- \\ -L_+ & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \eta \end{bmatrix} = 0, \qquad \begin{bmatrix} 0 & L_- \\ -L_+ & 0 \end{bmatrix} \begin{bmatrix} x\eta' + \eta \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \eta \end{bmatrix}$$

To perform spectral analysis, we complexify the space and work on

$$H \stackrel{\text{def}}{=} H^2(\mathbb{R};\mathbb{C}) \oplus H^2(\mathbb{R};\mathbb{C})$$

The symplectic form ω (B.2) extends to H, by analytic continuation (with exactly the same expression as in (B.2); we do not insert any complex conjugations).

Following the standard procedure (see [25]) we build an *invertible* matrix in block form

$$\mathcal{G}_q = \begin{bmatrix} F_q - z & R_- \\ R_+ & 0 \end{bmatrix}$$

with suitably chosen

$$R_- : \mathbb{C}^2 \to H\,, \quad R_+ : H \to \mathbb{C}^2\,.$$

We will select R_-, R_+ to be constant (independent of q and z) operators such that $\mathcal{G}$ is invertible with inverse represented in block form as

$$\mathcal{G}^{-1} = \begin{bmatrix} E & E_+ \\ E_- & E_{-+} \end{bmatrix}$$

The components depend on q and z and have the following mapping properties:

$$\begin{aligned} &E : H \to H\,, & &E_- : H \to \mathbb{C}^2\,, \\ &E_+ : \mathbb{C}^2 \to H\,, & &E_{-+} : \mathbb{C}^2 \to \mathbb{C}^2\,. \end{aligned}$$

To find R_- and R_+ and to compute $E^0(z)$, $E^0_+(z)$, $E^0_-(z)$ and $E^0_{-+}(z)$, we first consider $(F_0 - z)|_{\mathfrak{g}\cdot\eta}$, that is $F_0 - z$ acting on the generalized kernel. Ordering the basis using $e_j \cdot \eta$, $j = 1,2,3,4$, we see that

$$(F_0 - z)|_{\mathfrak{g}\cdot\eta} = \begin{bmatrix} -z & 1 & & \\ 0 & -z & & \\ & & -z & 1 \\ & & 0 & -z \end{bmatrix}.$$

The computation (see [25, §2.2])

$$\begin{bmatrix} -z & 1 & & & 0 & 0 \\ 0 & -z & & & 1 & 0 \\ & & -z & 1 & 0 & 0 \\ & & 0 & -z & 0 & 1 \\ 1 & 0 & 0 & 0 & & \\ 0 & 0 & 1 & 0 & & \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 & & & 1 & 0 \\ 1 & 0 & & & z & 0 \\ & & 0 & 0 & 0 & 1 \\ & & 1 & 0 & 0 & z \\ z & 1 & 0 & 0 & z^2 & \\ 0 & 0 & z & 1 & & z^2 \end{bmatrix}$$

gives us $R_\pm$ for which $\mathcal{G}_0$ is invertible:

$$R_- \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} = \zeta_1 e_2 \eta + \zeta_2 e_4 \eta \,, \quad R_+ u = \begin{bmatrix} P_1 u \\ P_3 u \end{bmatrix} \,, \quad Pu = \sum_{j=1}^{4} P_j u \, e_j \eta \,.$$

This tells us that

$$E_+^0 \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} = \begin{bmatrix} \zeta_1 \\ z\zeta_1 \\ \zeta_2 \\ z\zeta_2 \end{bmatrix} = \zeta_1 e_1 \eta + z\zeta_1 e_2 \eta + \zeta_2 e_3 \eta + z \zeta_2 e_4 \eta$$

or, more explicitly,

$$E_+^0 \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} = \begin{bmatrix} -\eta' & z(\eta + x\eta') \\ zx\eta & \eta \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} \tag{B.3}$$

We also find that

$$E_-^0 \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} z\alpha_1 + \alpha_2 \\ z\alpha_3 + \alpha_4 \end{bmatrix}$$

or in other words $E_-^0 : H \to \mathbb{C}^2$ is expressed as

$$E_-^0 \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -z \int ux\eta - \int v\eta' \\ z \int v(x\eta)' - \int u\eta \end{bmatrix} \tag{B.4}$$

We use the following formula to compute $E_{-+}^q(z)$:

$$E_{-+}^q = E_{-+}^0 - E_-^0 (F_q - F_0) E_+^0 + \mathcal{O}(q^2)$$

By the Schur complement formula, $F_q - z$ is invertible if and only if $E_{-+}^q(z)$ is invertible, so we want to find z (in terms of q) such that $\det(E_{-+}^q(z)) = 0$. We know that $E_{-+}^0 : \mathbb{C}^2 \to \mathbb{C}^2$ is

$$E_{-+}^0 = \begin{bmatrix} z^2 & 0 \\ 0 & z^2 \end{bmatrix}$$

and thus have all the ingredients to analyze the perturbation.

B.2. **Substitutions.** We have

$$\operatorname{sech}^2(|x|+q) = \operatorname{sech}^2 x - 2q \operatorname{sech}^2 x \tanh |x| + \cdots$$

and thus (to first order in q)

$$\begin{aligned} L_+^q - L_+^0 &= 6q \operatorname{sech}^2 x \tanh |x| - q\delta_0(x) \\ L_-^q - L_-^0 &= 2q \operatorname{sech}^2 x \tanh |x| - q\delta_0(x) \end{aligned}$$

and therefore (to first order in q),

$$F_q - F_0 = \begin{bmatrix} 0 & 2q \operatorname{sech}^2 x \tanh |x| - q\delta_0(x) \\ -6q \operatorname{sech}^2 x \tanh |x| + q\delta_0(x) & 0 \end{bmatrix} .$$

We will use the notation $\eta = \operatorname{sech} x$ and $\sigma = \tanh x$. Using (B.3) we see that $(F_q - F_0)E_+^0 : \mathbb{C}^2 \to H$ takes the form

$$(F_q - F_0)E_+^0 = \begin{bmatrix} 2qzx\eta^3\sigma \operatorname{sgn} x & 2q\eta^3\sigma \operatorname{sgn} x - q\delta_0 \\ -6q\eta^3\sigma^2 \operatorname{sgn} x & -6qz\eta^3\sigma(1-x\sigma) \operatorname{sgn} x + qz\delta_0 \end{bmatrix}$$

From this, and (B.4), we compute $E_-^0(F_q - F_0)E_+^0 : \mathbb{C}^2 \to \mathbb{C}^2$ takes the form

$$E_-^0(F_q - F_0)E_+^0 = \begin{bmatrix} -qz^2\alpha - q\beta & 0 \\ 0 & q\gamma + qz^2\delta \end{bmatrix}$$

where

$$\beta = 6 \int \eta^4\sigma^3 \operatorname{sgn} x = 1 \,, \quad \gamma = 1 - 2\int \eta^4 \sigma \operatorname{sgn} x = 0 \,,$$

and thus

$$E_{-+}^q(z) = \begin{bmatrix} (1+q\alpha)z^2 + q & 0 \\ 0 & (1-q\delta)z^2 \end{bmatrix} + \mathcal{O}(q^2) \,. \tag{B.5}$$

By expanding $\det E_{-+}^q(x)$ as see that E_{-+}^q fails to be invertible when $z = \pm i q^{1/2} + \mathcal{O}(q^{3/2})$. The explicit generalized kernel of F_q given in the beginning of §3 shows that the double eigenvalue at 0 persists under perturbation. We can now give

Proof of Lemma 5.2: We only need to check the properties of $w_q^\pm$. The equation $\sigma_3 w_q^\pm = w_q^\mp$ follows from the fact that $\sigma_3 F_q \sigma_3 = -F_q$. Since the eigenfuctions are simple and F_q commutes with $u(x) \mapsto u(-x)$, and because of the σ_3 symmetry, they are either both odd or both even. Schur's formula (see for instance [25, §1]) shows that

$$\operatorname{Res}_{z=\lambda_q^\pm}(F_q - z)^{-1} = \operatorname{Res}_{z=\lambda_q^\pm} E_+^q(z) E_{-+}^q(z)^{-1} E_-^q(z) \,, \tag{B.6}$$

and we note that

$$\mathbb{C} \cdot w_q^\pm = \text{Image } \operatorname{Res}_{z=\lambda_q^\pm}(F_q - z)^{-1} \,. \tag{B.7}$$

In addition to (B.5) we also have, using (B.3) and (B.4),

$$
\begin{aligned}
E_+^q(z)\begin{bmatrix}\zeta_1\\ \zeta_2\end{bmatrix} &= \begin{bmatrix}-\eta' & z(\eta+x\eta')\\ zx\eta & \eta\end{bmatrix}\begin{bmatrix}\zeta_1\\ \zeta_2\end{bmatrix} + \mathcal{O}_H(q|\zeta|_{\mathbb{C}^2})\,,\\
E_-^q(z)\begin{bmatrix}u_1\\ u_2\end{bmatrix} &= \begin{bmatrix}-z\int u_1 x\eta - \int u_2\eta'\\ z\int u_2(x\eta)' - \int u_1\eta\end{bmatrix} + \mathcal{O}_{\mathbb{C}^2}\left(q\left\|u\right\|_H\right)\,.
\end{aligned}
$$

This, (B.6), and (B.7) show that

$$
w_q^\pm = \begin{bmatrix}\eta'\\ \pm i q^{1/2} x\eta\end{bmatrix} + \mathcal{O}_H\left(q\right)\,.
$$

Hence $w_q^\pm$ is approximately odd, and consequently odd. $\square$

Appendix C. The system of equations for A, B, C, D

Here we describe how to solve for the coefficents $A(k)$, $B(k)$, $C(k)$, and $D(k)$ in (5.1). Define

$$
\begin{bmatrix}f(x,k)\\ g(x,k)\end{bmatrix} \stackrel{\text{def}}{=} \psi(x,k)
$$

Set $\tilde{A} = e^{(ik-\mu)\theta}A$, $\tilde{B} = e^{2ik\theta}B$, $\tilde{C} = C$, and $\tilde{D} = e^{(ik-\mu)\theta}D$, $\theta = \tanh^{-1} q$. Denote $f(0\pm) = \lim_{x\to 0\pm} f(x)$, etc. Using that $s(0) = (1-q^2)^{1/2}$ and $t(\pm 0) = \pm q$, we obtain

$$
\begin{aligned}
e^{ik\theta} f(0-) &= (q+ik)^2 - \tilde{A}(1-q^2)\\
e^{ik\theta} f(0+) &= \tilde{B}(q-ik)^2 + \tilde{C}(q+ik)^2 - \tilde{D}(1-q^2)\\
e^{ik\theta} g(0-) &= -(1-q^2) + \tilde{A}(q+\mu)^2\\
e^{ik\theta} g(0+) &= -\tilde{B}(1-q^2) - \tilde{C}(1-q^2) + \tilde{D}(q+\mu)^2
\end{aligned}
$$

The two equations we obtain by requiring continuity at $x = 0$ are

$$
f(0) \stackrel{\text{def}}{=} f(0-) = f(0+), \quad \text{and} \quad g(0) \stackrel{\text{def}}{=} g(0-) = g(0+)\,. \tag{C.1}
$$

We further compute, from the formula for ψ, that

$$
\begin{aligned}
e^{ik\theta} f'(0-) &= (q+ik)^2 ik - 2(1-q^2)(q+ik) + \tilde{A}\Big(-(1-q^2)\mu - 2(1-q^2)q \Big) \\
e^{ik\theta} f'(0+) &= \tilde{B}\Big((q-ik)^2 ik + 2(1-q^2)(q-ik) \Big) \\
&\quad + \tilde{C}\Big(-(q+ik)^2 ik + 2(1-q^2)(q+ik) \Big) \\
&\quad + \tilde{D}\Big(\mu(1-q^2) + 2(1-q^2)q \Big) \\
e^{ik\theta} g'(0-) &= \Big(-2(1-q^2)q - (1-q^2)ik \Big) + \tilde{A}\Big(-2(q+\mu)(1-q^2) + \mu(q+\mu)^2 \Big) \\
e^{ik\theta} g'(0+) &= \tilde{B}\Big(2(1-q^2)q - (1-q^2)ik \Big) + \tilde{C}\Big(2(1-q^2)q + (1-q^2)ik \Big) \\
&\quad + \tilde{D}\Big(2(1-q^2)(q+\mu) - \mu(q+\mu)^2 \Big)
\end{aligned}
$$

The form of the derivative compatibility conditions is:

$$
f'(0-) - f'(0+) = 2qf(0)\,, \quad g'(0-) - g'(0+) = 2qg(0)\,. \tag{C.2}
$$

The four equations (C.1),(C.2) give rise to the 4×4 system

$$
\mathcal{A}(k,q) \begin{bmatrix} \tilde{A} \\ \tilde{B}-1 \\ \tilde{C} \\ \tilde{D} \end{bmatrix} = q \begin{bmatrix} 4ik \\ 0 \\ -2(\mu-q) \\ -2(1-q^2) \end{bmatrix} , \tag{C.3}
$$

with the coefficient matrix $\mathcal{A}(k,q)$ given by

$$
\begin{bmatrix}
1-q^2 & (q-ik)^2 & (q+ik)^2 & -(1-q^2) \\
(q+\mu)^2 & 1-q^2 & 1-q^2 & -(q+\mu)^2 \\
(1-q^2) & (q-ik)(\mu-q) & (q+ik)(\mu-q) & (1-q^2) \\
-(q+\mu)(k^2+q^2) & (1-q^2)(q-ik) & (1-q^2)(q+ik) & -(q+\mu)(k^2+q^2)
\end{bmatrix} .
$$

C.1. **Exact solutions.** The solution is obtained from `Mathematica` or, in principle, by Gaussian elimination. Recalling that $\theta = \tanh^{-1} q$, we have

$$
\begin{aligned}
A &= -\frac{2e^{(-ik+\mu)\theta} q(ik+q)\left(-1+q^2\right)}{1+q^2\left(-2+k^2+2q^2\right)+2q\left(k^2+q^2\right)\mu+\left(k^2+q^2\right)\mu^2} \\
B &= \frac{e^{-2ik\theta}(k-iq)(i+k(q+\mu)-iq(2q+\mu))(k(q+\mu)-i(1+q\mu))}{k\left(1+q^2\left(-2+k^2+2q^2\right)+2q\left(k^2+q^2\right)\mu+\left(k^2+q^2\right)\mu^2\right)} \\
C &= -\frac{iq\left(-1+\left(2+k^2\right)q^2+2q\left(k^2+q^2\right)\mu+\left(k^2+q^2\right)\mu^2\right)}{k\left(1+q^2\left(-2+k^2+2q^2\right)+2q\left(k^2+q^2\right)\mu+\left(k^2+q^2\right)\mu^2\right)} \\
D &= \frac{2e^{(-ik+\mu)\theta} q(ik+q)\left(-1+q^2\right)}{1+q^2\left(-2+k^2+2q^2\right)+2q\left(k^2+q^2\right)\mu+\left(k^2+q^2\right)\mu^2}
\end{aligned}
$$

The numerator in the expression for B is $(k-iq)v(k)w(k)$, where

$$v(k) = (i+k\mu) + q(k-i\mu) - 2iq^2, \quad w(k) = (-i+k\mu) + q(k-i\mu)\,.$$

We clearly see that $k = iq$ is a root of B, and we further find that at $k = iq$,

$$A(iq) = 0, \quad B(iq) = 0, \quad C(iq) = 1, \quad D(iq) = 0\,.$$

Thus,

$$\psi(x) = \begin{bmatrix} (\tanh(|x|+\theta) - q)^2 \\ -\operatorname{sech}^2(|x|+\theta) \end{bmatrix} e^{q(|x|+\theta)}$$

solves the equation

$$H_q\psi = (1-q^2)\psi\,,$$

giving an eigenvalue when $q < 0$.

We will now specify a branch of $\mu = \sqrt{2+k^2}$, and study the roots of $v(k)$ and $w(k)$ to check for consistency with Appendix B. Since $2+k^2$ has roots at $\pm i\sqrt{2}$, we will cut along the imaginary axis, and take μ as the branch defined on the domain

$$\mathbb{C}\setminus(-i\infty, -i\sqrt{2}] \cup [i\sqrt{2}, +i\infty)\,,$$

that is real and positive for $k > 0$.

We now examine $v(k)$ for $0 < |q| \ll 1$. Setting $k = -i + \kappa q^{1/2}$, we find that $\mu = 1 - i\kappa q^{1/2} + \kappa^2 q + \mathcal{O}(q^{3/2})$. Substituting yields

$$v(k) = -2i(\kappa^2+1)q + \mathcal{O}(q^{3/2})\,,$$

and thus a root occurs at $\kappa = \pm i$, i.e. when $k = -i \pm iq^{1/2} + \mathcal{O}(q)$. Substituting $k = -i \pm iq^{1/2}$ into the numerator of the formula for B, we obtain $\mathcal{O}(q^{3/2})$, while substituting into the denominator, we obtain $\mathcal{O}(q)$, and thus we have found an approximate root of B. This implies that we have eigenvalues at $1+k^2 = \pm 2q^{1/2} + \mathcal{O}(q)$. The roots of $w(k)$ occur near $k = +i$, giving nonphysical poles of the resolvent $H_q - (k^2+1))^{-1}$.

From the above formulas, we have

$$\begin{aligned}
\frac{A}{B} &= -\frac{2ie^{(ik+u)\theta}kq\left(-1+q^2\right)}{(i+k(q+\mu)-iq(2q+\mu))(k(q+\mu)-i(1+q\mu))} \\
\frac{1}{B} &= \frac{e^{2ik\theta}k\left(1+q^2\left(-2+k^2+2q^2\right)+2q\left(k^2+q^2\right)\mu+\left(k^2+q^2\right)\mu^2\right)}{(k-iq)(i+k(q+\mu)-iq(2q+\mu))(k(q+\mu)-i(1+q\mu))} \\
\frac{C}{B} &= -\frac{ie^{2ik\theta}q\left(-1+\left(2+k^2\right)q^2+2q\left(k^2+q^2\right)\mu+\left(k^2+q^2\right)\mu^2\right)}{(k-iq)(i+k(q+\mu)-iq(2q+\mu))(k(q+\mu)-i(1+q\mu))} \\
\frac{D}{B} &= \frac{2ie^{(ik+u)\theta}kq\left(-1+q^2\right)}{(i+k(q+\mu)-iq(2q+\mu))(k(q+\mu)-i(1+q\mu))}
\end{aligned}$$

C.2. **Behaviour for large k.** The behaviour for large values of k could be deduced from general principles of scattering theory. Here we proceed directly using the matrix $\mathcal{A}(k,q)$ which we write as $\mathcal{A}(k,q) = \mathcal{A}_0(k) + q\mathcal{B}(k,q)$, where

$$\mathcal{A}_0(k) = \begin{bmatrix} 1 & -k^2 & -k^2 & -1 \\ \mu^2 & 1 & 1 & -\mu^2 \\ 1 & -ik\mu & ik\mu & 1 \\ -\mu k^2 & -ik & ik & -\mu k^2 \end{bmatrix},$$

and

$$\mathcal{A}_0^{-1} = \frac{1}{2(1+k^2)^2} \begin{bmatrix} 1 & k^2 & 1 & -\mu \\ -\mu^2 & 1 & ik\mu & i/k \\ -\mu^2 & 1 & -ik\mu & -i/k \\ -1 & -k^2 & 1 & -\mu \end{bmatrix},$$

$\mu = \sqrt{2+k^2}$. For $|k| > \epsilon > 0$, we have

$$\mathcal{A}_0^{-1} = \mathcal{O}_{\mathbb{C}^4 \to \mathbb{C}^4}(1/\langle k \rangle^2), \quad \mathcal{B} = \mathcal{O}_{\mathbb{C}^4 \to \mathbb{C}^4}(\langle k \rangle^2),$$

with the implicit constant in the first estimate dependent on ϵ. Hence

$$q\mathcal{A}_0^{-1}\mathcal{B} = \mathcal{O}_{\mathbb{C}^4 \to \mathbb{C}^4}(q).$$

For q small enough, depending of ϵ, we can used the Neumann series inversion of $I + q\mathcal{A}_0^{-1}\mathcal{B}$ to obtain, and consequently, for $|k| > \epsilon$,

$$\begin{bmatrix} A \\ B-1 \\ C \\ D \end{bmatrix} = q(I + q\mathcal{A}_0^{-1}\mathcal{B})^{-1}\mathcal{A}_0^{-1} \begin{bmatrix} 4ik \\ 0 \\ -2\mu \\ -6 \end{bmatrix} = q\mathcal{A}_0^{-1} \begin{bmatrix} 4ik \\ 0 \\ -2\mu \\ -6 \end{bmatrix} + \mathcal{O}(q^2/\langle k \rangle). \tag{C.4}$$

$$\begin{bmatrix} A \\ B-1 \\ C \\ D \end{bmatrix} = \frac{q}{(1+k^2)^2} \begin{bmatrix} 2ik + 2\mu \\ -3i/k - 3ik\mu^2 \\ 3i/k - ik\mu^2 \\ -2ik + 2\mu \end{bmatrix} + \mathcal{O}(q^2/\langle k \rangle) \tag{C.5}$$

This provides the estimates needed in Lemma 5.5.

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E-mail address: `holmer@math.brown.edu`

Department of Mathematics, Brown University, 151 Thayer Street, Providence, RI 02912, USA

E-mail address: `zworski@math.berkeley.edu`

Mathematics Department, University of California, Evans Hall, Berkeley, CA 94720, USA