CLASSICAL–QUANTUM CORRESPONDENCE IN LINDBLAD EVOLUTION

JEFFREY GALKOWSKI AND MACIEJ ZWORSKI

Abstract. We show that for the Lindblad evolution defined using (at most) quadratically growing classical Hamiltonians and (at most) linearly growing classical jump functions (quantized into jump operators assumed to satisfy certain ellipticity conditions and modeling interaction with a larger system), the evolution of a quantum observable remains close to the classical Fokker–Planck evolution in the Hilbert–Schmidt norm for times vastly exceeding the Ehrenfest time (the limit of such agreement with no jump operators). The time scale is the same as in the recent papers [HRR23a, HRR23b] by Hernández–Ranard–Riedel but the statement and methods are different.

1. Introduction

In quantum mechanics a system is often described using a *density matrix*, that is a positive operator of trace one on a Hilbert space. In this paper the Hilbert space will be given by $L^2(\mathbb{R}^n)$ so that the density operator is then

$$Au(x) = \sum_j p_j \langle u, u_j \rangle u_j(x), \quad p_j \geq 0, \quad \sum_j p_j = 1, \quad \langle u_j, u_i \rangle = \delta_{ij}.$$

If the system evolves according to the Schrödinger equation $(ih\partial_t + P)v(t) = 0$, where P is a self-adjoint unbounded operator on $L^2(\mathbb{R}^n)$ then (note the sign convention) the density matrix evolves by the Schrödinger propagation of u_j's. That gives the following equation:

$$\partial_t A(t) = \mathcal{L}_0 A(t), \quad \mathcal{L}_0 A := \frac{i}{h}[P, A], \quad A(t) = e^{t\mathcal{L}_0} A(0) = e^{itP/h} A(0) e^{-itP/h}. \tag{1.1}$$

This evolution clearly preserves density matrices. Gorini–Kossakowski–Sudarshan [GKS76] and Lindblad [Li76] generalized this by showing (in the setting of matrices and of bounded operators, respectively) that semigroups preserving the trace and complete positivity are generated by operators of the form

$$\mathcal{L}A := \frac{i}{h}[P, A] + \frac{\gamma}{h} \sum_{j=1}^{J} \left(L_j A L_j^* - \tfrac{1}{2}(L_j^* L_j A + A L_j^* L_j) \right), \quad \gamma \geq 0. \tag{1.2}$$

The corresponding evolution equation is called the Lindblad master equation or the GKLS equation and, following the long tradition which favours short northern European names, we refer to $\mathcal{L}$ as the Lindbladian – see [ChPa17] for a history of this discovery and pointers to the literature. The operators L_j are called jump operators and they describe a dissipative (see (1.4) below) interaction of a system evolving according to (1.1) with a larger "open" system. (Hence the term "jump" as L_j describe the effect of moving to that larger system.)

1.1. **Assumptions on P and L_j and Fokker–Planck evolution.** In this paper we will consider (1.2) with P and L_j's given by *pseudodifferential operators* (see (2.1) for the notation $a^{\mathrm{w}}(x, hD)$), that is semiclassical quantizations of classical observables, satisfying the following assumptions:

$$\begin{aligned} P = p^{\mathrm{w}}(x, hD), \quad |\partial^\alpha p(x,\xi)| \leq C_\alpha, \quad |\alpha| \geq 2, \quad p = \bar{p}, \\ L_j = \ell_j^{\mathrm{w}}(x, hD), \quad |\partial^\alpha \ell_j(x,\xi)| \leq C_\alpha, \quad |\alpha| \geq 1, \quad 1 \leq j \leq J. \end{aligned} \tag{1.3}$$

If in (1.2), $A = a^{\mathrm{w}}(x, hD)$, then the leading part of the semiclassical expansion of $\mathcal{L}A$ (see the derivation in §5) is given by the action, Qa, of the following Fokker–Planck operator

$$Q := H_p + \frac{\gamma}{2i} \sum_{j=1}^{J} (2\{\ell_j, \bar{\ell}_j\} - \ell_j H_{\bar{\ell}_j} + \bar{\ell}_j H_{\ell_j}) + \frac{h\gamma}{4} \sum_{j=1}^{J} (H_{\bar{\ell}_j} H_{\ell_j} + H_{\ell_j} H_{\bar{\ell}_j}). \tag{1.4}$$

Here, $H_f := \sum_{j=1}^n \partial_{\xi_j} f \partial_{x_j} - \partial_{x_j} f \partial_{\xi_j}$ is the Hamiltonian vector field of $f = f(x,\xi)$, and $\{f, g\} := H_f g$ is the Poisson bracket. We note that H_p is anti-selfadjoint with respect to the standard measure on $\mathbb{R}^n \times \mathbb{R}^n$. Since

$$\tfrac{1}{2i} \left(2\{\ell_j, \bar{\ell}_j\} - \ell_j H_{\bar{\ell}_j} + \bar{\ell}_j H_{\ell_j}\right) =: \tfrac{1}{i}\{\ell_j, \bar{\ell}_j\} + B_j, \quad B_j^* = -B_j, \tag{1.5}$$

the self-adjoint contribution to the second term is given by the real valued function

$$\mu := \frac{1}{2i} \sum_{j=1}^{J} \{\ell_j, \bar{\ell}_j\}. \tag{1.6}$$

It is interpreted as friction. Finally, the last term in (1.4) is self-adjoint and non-negative. Assumptions (1.3) show that μ is bounded ($\mu \in S(1)$ in the notation of §2).

Example. Suppose $J = 2n$ and $\ell_j = x_j$, $\ell_{j+n} = \xi_j$ for $j \leq n$. Then

$$Q = H_p + \tfrac{1}{2}\gamma h(\Delta_x + \Delta_\xi). \tag{1.7}$$

When $\gamma = 0$ (that is, when we consider (1.1)) classical quantum correspondence in the evolution is described using *Egorov's theorem* – see [Zw12, Theorem 11.12, §11.5] and references given there. Here we present it slightly differently, using the Hilbert–Schmidt norm of the operator – see Theorem 3 for a general version. For the evolution

(1.1) with $A(0) = (2\pi h)^{n/2} a_0^{\rm w}(x, hD)$ where $a_0 \in C_c^\infty(\mathbb{R}^{2n})$ is h-independent (so that $\|A(0)\|_{\mathscr{L}_2} = \|a_0\|_{L^2}$) we have

$$\|A(t) - \mathrm{Op}((\exp tH_p)^* a_0)\|_{\mathscr{L}_2} \leq C e^{3\Gamma t} h^2, \tag{1.8}$$

where $\| \bullet \|_{\mathscr{L}_2}$ denotes the Hilbert–Schmidt norm and

$$\Gamma := \sup_{|\alpha|=2} \sup_{\mathbb{R}^{2n}} |\partial^\alpha p(x, \xi)|. \tag{1.9}$$

For a more precise version of Γ, under additional assumptions on p, in terms of Lyapunov exponents of the flow of H_p see [OlBo23, Appendix C] and references given there. For a relation between (1.9) and the flow see Lemma 3.1.

The estimate (1.18) is not optimal, but as $\|a^{\rm w}(x, hD)\|_{\mathscr{L}_2} = (2\pi h)^{-n/2} \|a\|_{L^2(\mathbb{R}^{2n})}$, (1.18) indicates the basic principle that the agreement with classical evolution breaks down at times proportional to $\log(1/h)$, the Ehrenfest time.

Motivated by recent papers [HRR23a, HRR23b] by Hernández–Ranard–Riedel we consider the question of an agreement with classical evolution for much longer times: the quantum evolution is given by $e^{t\mathcal{L}}$ where $\mathcal{L}$ is the Lindblad operator (1.2) and the classical evolution by e^{tQ}, where Q is the Fokker–Planck operator (1.4). The results are shown in Theorem 1 for the special case of h-independent symbols, and in Theorem 4 for the more general situation of initial condition in exotic symbol classes. We show that agreement holds in Hilbert–Schmidt norms. The main advantage lies in an easy characterization of Hilbert–Schmidt pseudodifferential operators and in the simplicity of L^2 estimates for the Fokker–Planck evolution defined using (1.4).

Remark. As was shown by Davies [Da77], the operator of the form (1.2) generates a positivity preserving contraction on the Banach space of self-adjoint trace class operators provided that

$$Y := iP - \tfrac{1}{2} \sum_{j=1}^{J} L_j^* L_j$$

is the infinitesimal generator of a strongly continuous one parameter contraction semigroup on $L^2(\mathbb{R}^n)$. In our case, this follows from the Hille–Yosida theorem and Proposition A.2 (see the proof of Proposition 4.6 for a similar argument with $\mathscr{L}_2$ playing the role of L^2).

As in [HRR23a, HRR23b] we make a strong non-degeneracy assumption:

$$\mathbf{H}\mathbf{H}^* \geq c I_{\mathbb{C}^{2n}}, \quad \mathbf{H} := [H_{\ell_1}, \cdots, H_{\ell_J}, H_{\bar{\ell}_1}, \cdots, H_{\bar{\ell}_J}] \in M_{2n \times 2J}(\mathbb{C}). \tag{1.10}$$

This cumbersome looking condition corresponds to ellipticity of the second order operator appearing in the classical Fokker–Planck equation (1.4) corresponding to (1.2) – see example (1.7) and Remark 5 after Theorem 1. We also need a more technical condition

$$|\partial^\alpha \operatorname{Im} \ell_j||\ell_j| + |\operatorname{Im} \ell_j||\partial^\alpha \ell_j| \leq C_\alpha, \quad |\alpha| \geq 2. \tag{1.11}$$

1.2. **Lindblad propagation for h-independent observables.** With this notation in place we have a special case of Theorem 4 in §6:

Theorem 1. *Suppose that $\mathcal{L}$ is given by (1.2), assumptions (1.3), (1.10), and (1.11) hold and $h^{\frac{1}{3}} \leq \gamma \leq h^{-1}$. If $a_0 \in C_c^\infty(\mathbb{R}^{2n})$ is h-independent and $A(t)$ satisfies*

$$\partial_t A(t) = \mathcal{L}A(t), \quad A(0) = (2\pi h)^{n/2} a_0^{\rm w}(x, hD), \quad \|A(0)\|_{\mathscr{L}_2} = \|a_0\|_{L^2(\mathbb{R}^{2n})},$$

then for some constant C,

$$\|A(t) - \operatorname{Op}(a(t))\|_{\mathscr{L}_2} \leq Ce^{(M_0 + C_0 h)\gamma t} t h^{\frac{1}{2}} \gamma^{-\frac{3}{2}} (1+\gamma)(1 + t\gamma^{\frac{3}{2}} h^{\frac{1}{2}}), \tag{1.12}$$

where

$$(\partial_t - Q)a(t) = 0, \quad a(0) = (2\pi h)^{n/2} a_0, \qquad M_0 := \sup \mu.$$

When $\mu \equiv 0$ (see (1.6)) then (1.12) improves to

$$\|A(t) - a(t)^{\rm w}(x, hD)\|_{\mathscr{L}_2} \leq Ce^{Ch^2\gamma t} t h^{\frac{1}{2}} \gamma^{-\frac{3}{2}} (1+\gamma). \tag{1.13}$$

Remarks. 1. When in (1.3) p is quadratic and ℓ_j's are linear than the agreement of the two evolutions is exact. This corresponds to the same phenomenon in the case of Egorov's theorem – see [Zw12, Theorem 11.9].

2. When $p(x, \xi)$ is confining (for instance $p(x, \xi) \geq |x|^2 + |\xi|^2$ and subharmonic outside of a compact set) then Proposition 7.4 shows that in Example (1.7) (and most likely in greater generality), $\|a^{\rm w}(t)\|_{\mathscr{L}_2} \geq \|a^{\rm w}(0)\|_{\mathscr{L}_2}/C$ for $t \leq h^{-\nu}$, $\nu > 0$. That means that the estimates (1.12) and (1.13) are meaningful for long times.

3. To see the reason for the powers of h, γ, and t in (1.12) consider the simplest case given in (1.7). The classical (Fokker–Planck) evolution is then

$$(\partial_t - H_p - \varepsilon^2 \Delta_{x,\xi}) a(t) = 0, \quad \varepsilon := \sqrt{\gamma h/2}, \quad a(t) = a(t, x, \xi).$$

The solutions satisfy the following estimate (immediate if $H_p = 0$), see Proposition 5.1 (see (5.6)):

$$\sum_{|\alpha| \leq k} \|(\varepsilon \partial_{x,\xi})^\alpha a(t)\|_{L^2_{x,\xi}} \leq C \sum_{|\alpha| \leq k} \|(\varepsilon \partial_{x,\xi})^\alpha a(0)\|_{L^2_{x,\xi}}. \tag{1.14}$$

The key fact is that there is *no* dependence on t – that is not the case for the evolution by H_p alone, see (3.7). The composition formula for pseudodifferential operators in Lemma 2.2 shows that $\mathcal{L}\left[a(t)^{\rm w}(x, hD)\right] \equiv (Qa(t))^{\rm w}(x, hD)$ modulo terms quantizing functions bounded by the size of $(1+\gamma)h^2 \partial^3 a(t)$. These can be estimated using (1.14) where in the case of (1.16) and for $|\alpha| = 3$,

$$\begin{aligned}(1+\gamma)h^2 \|\partial^\alpha_{x,\xi} a(t)\|_{L^2_{x,\xi}} &\leq C(1+\gamma)h^2 \varepsilon^{-3} \sum_{|\alpha| \leq 3} \|(\varepsilon \partial_{x,\xi})^\beta a(0)\|_{L^2_{x,\xi}} \\ &\leq (2\pi h)^{\frac{n}{2}} C(1+\gamma)\gamma^{-\frac{3}{2}} h^{\frac{1}{2}}.\end{aligned} \tag{1.15}$$

To get (1.21) we write

$$A(t) - a(t)^{\rm w}(x, hD) = \int_0^t e^{(t-s)\mathcal{L}}(\mathcal{L}a(t)^{\rm w}(x, hD) - (Qa(t))^{\rm w}(x, hD))ds,$$

which together with (1.15) and the fact that $\|A(0)\|_{\mathscr{L}_2} = \|a_0\|_{L^2}$, gives (1.21). The extra growth in (1.12) results from friction which is absent in this example. We used here the fact that in the example $e^{t\mathcal{L}}$ is a contraction – in general there could be exponential growth produced by the friction term; this is reflected by the exponential prefactor in (1.12).

4. The class of operators P satisfying (1.3) includes Schrödinger operator whose classical dynamics exhibits chaotic behaviour. In that case one expects optimality of $t \sim \log(1/h)$ limit for classical–quantum correspondence for (1.1). For instance we could take

$$p(x, \xi) = \xi_1^2 + \xi_2^2 + x_1^2 + x_2^2 + \lambda(x)(x_1^2 x_2 - \tfrac{1}{3}x_2^3),$$

where $\lambda \in C_{\rm c}^\infty(\mathbb{R}^2; [0, 1])$ and $\lambda = 1$ near 0.

5. Compared to the models used in the physics literature – see Unruh–Zurek [UnZu89] for the pioneering discussion of the classical/quantum correspondence for open systems – the ellipticity hypothesis (1.10) made in [HRR23a] and here is too strong. Rather than (1.7), one should consider $\ell_j = x_j$, $0 \le j \le J = n$ so that the Fokker–Planck operators is given by $Q = H_p + \frac{1}{2}\gamma h \Delta_\xi$. This would require more subtle subelliptic estimates (see Smith [Sm20] for a recent treatment with an asymptotic parameter) than (1.14). Gong–Brumer [GoBr99] showed numerically that for such operators with chaotic classical dynamics for p, the classical/quantum correspondence persists for long times.

1.3. **Lindblad propagation for mixtures of Gaussian states.** We now state a special case of our theorem where we consider mixtures of Gaussian states in the sense similar to that in [HRR23a]. For that we define the standard coherent states:

$$\psi_{(x_0,\xi_0)} = (2\pi h)^{-\frac{n}{4}} e^{-(x-x_0)^2/2h} e^{i(x-x_0)\xi_0/h}, \quad \|\psi_{(x_0,\xi_0)}\|_{L^2(\mathbb{R}^n)} = 1.$$

The corresponding density operator is

$$\begin{aligned} &A_{(x_0,\xi_0)}u := \psi_{(x_0,\xi_0)}\langle u, \psi_{(x_0,\xi_0)}\rangle, \qquad A_{(x_0,\xi_0)} = a_0^{\rm w}(x, hD), \\ &a_{(x_0,\xi_0)}(x, \xi) = 2^n \exp\left(-\frac{1}{h}\left((x - x_0)^2 + (\xi - \xi_0)^2\right)\right). \end{aligned} \tag{1.16}$$

We note that in our result the Gaussian $(2\pi h)^{-\frac{n}{4}} e^{-(x-x_0)^2/2h}$ could be replaced by $\alpha h^{-n/4}\psi((x - x_0)/\sqrt{h})$ where $\psi \in \mathscr{S}(\mathbb{R}^{2n})$ and $\alpha = 1/\|\psi\|_{L^2}$.

For a probability measure λ_h on $\mathbb{R}^n \times \mathbb{R}^n$ we define the mixture of Gaussian states:

$$\begin{aligned} A_{\lambda_h} u &:= \int \psi_{(x_0,\xi_0)} \langle u, \psi_{(x_0,\xi_0)} \rangle d\lambda_h(x_0,\xi_0), \qquad A_{\lambda_h} = a_{\lambda_h}^{\mathrm{w}}(x,hD), \\ a_{\lambda_h}(x,\xi) &= 2^n \int \exp\left(-\frac{1}{h} \left((x-x_0)^2 + (\xi-\xi_0)^2 \right) \right) d\lambda_h(x_0,\xi_0). \end{aligned} \tag{1.17}$$

We note that $\|A_{\lambda_h}\|_{\mathscr{L}_1} = 1$. For the Hilbert Schmidt norm we calculate

$$(2\pi h)^{-n} 2^{2n} \int e^{-\frac{1}{h}((x-x_0)^2+(x-y_0)^2+(\xi-\xi_0)^2+(\xi-\eta_0)^2)} dx d\xi = e^{-\frac{1}{h}((x_0-y_0)^2+(\xi_0-\eta_0)^2)},$$

so that

$$\|A_{\lambda_h}\|_{\mathscr{L}_2}^2 = \iint \exp\left(-\frac{1}{h} \left((x_0-y_0)^2 + (\xi_0-\eta_0)^2 \right) \right) d\lambda_h(x_0,\xi_0) d\lambda_h(y_0,\eta_0).$$

If $\lambda_h = \mu(x,\xi) dx d\xi$, where μ is smooth and h independent we are close to the case considered in Theorem 1 and $\|A_{\lambda_h}\|_{\mathscr{L}_2} \sim h^{n/2}$.

As in (1.18) when $\gamma = 0$, we obtain a version of Egorov's Theorem: for the solution of (1.1) with $A(0) = a_{\lambda_h}^{\mathrm{w}}(x,hD)$,

$$\|A(t) - \mathrm{Op}((\exp tH_p)^* a_{\lambda_h})\|_{\mathscr{L}_2} \leq C e^{3\Gamma t} h^{\frac{1}{2}} \|A(0)\|_{\mathscr{L}_2}. \tag{1.18}$$

On the other hand, for the Lindblad evolution, the quantum– classical agreement lasts substantially longer as can be seen from the following special case of Theorem 4

Theorem 2. *Suppose that $\mathcal{L}$ is given by (1.2), assumptions (1.3), (1.10), and (1.11) hold and $\gamma < h^{-1}$. If, in the notation of (1.17), $A(t)$ satisfies*

$$\partial_t A(t) = \mathcal{L} A(t), \quad A(0) = A_{\lambda_h}, \tag{1.19}$$

then for some constant C,

$$\|A(t) - a(t)^{\mathrm{w}}(x,hD)\|_{\mathscr{L}_2} \leq C e^{(M_0+Ch)\gamma t} t (\gamma + \gamma^{-\frac{3}{2}}) h^{\frac{1}{2}} (1 + t\gamma^{\frac{3}{2}} h^{\frac{1}{2}}) \|A_{\lambda_h}\|_{\mathscr{L}_2}, \tag{1.20}$$

where

$$(\partial_t - Q) a(t) = 0, \quad a(0) = a_{\lambda_h}, \qquad M_0 := \sup \mu.$$

When $\mu \equiv 0$ (see (1.6)) then (1.20) improves to

$$\|A(t) - a(t)^{\mathrm{w}}(x,hD)\|_{\mathscr{L}_2} \leq C e^{Ch^2\gamma t} t (\gamma + \gamma^{-\frac{3}{2}}) h^{\frac{1}{2}} \|A_{\lambda_h}\|_{\mathscr{L}_2}, \tag{1.21}$$

Remark. The time scales appearing in Theorems 1, 2 and 4 agree with the time scales in [HRR23a], as long as $\gamma \leq 1$: Theorem 3.1 there gives the bound $C \max(1, \gamma^{-\frac{3}{2}}) h^{\frac{1}{2}} t$ for a two tier comparison of evolution of specially constructed Gaussian states. Under the assumptions in Theorem 1 it reads as

$$\begin{aligned} \|A(t) - \tilde{a}(t)^{w}(x,hD)\|_{\mathscr{L}_1} &\leq C \max(1, \gamma^{-\frac{3}{2}}) h^{\frac{1}{2}} t, \\ h^{-n} \|\tilde{a}(t) - a(t)\|_{L^1(\mathbb{R}^{2n})} &\leq C \max(1, \gamma^{-\frac{3}{2}}) h^{\frac{1}{2}} t, \end{aligned} \tag{1.22}$$

where $\|\bullet\|_{\mathscr{L}_1}$ is the trace class norm. Remarkably, since the semigroup $e^{t\mathcal{L}}$ is contracting on trace class operators, there is no exponential growth even when friction is positive. The estimate does *not* provide a bound on $A(t) - a(t)^w(x, hD)$ in any norm, but has the following natural consequence [HRR23a, (1.7)]:

$$\begin{aligned}\operatorname{tr}\left(A(t)b^{\mathrm{w}}(x,hD)\right) - (2\pi h)^{-n} \int_{\mathbb{R}^{2n}} a(t,x,\xi)b(x,\xi)dxd\xi \\ = \mathcal{O}\big(t \max(1,\gamma^{-\frac{3}{2}})h^{\frac{1}{2}}\big)(\|b^{\mathrm{w}}(x,hD)\|_{L^2\to L^2} + \|b\|_{L^\infty}).\end{aligned}$$

This is (typically) stronger than the corresponding consequence of (1.20):

$$\begin{aligned}\operatorname{tr}\left(A(t)b^{\mathrm{w}}(x,hD)\right) - (2\pi h)^{-n} \int_{\mathbb{R}^{2n}} a(t,x,\xi)b(x,\xi)dxd\xi \\ = \mathcal{O}\big(e^{(M_0+Ch)\gamma t} t(\gamma+\gamma^{-\frac{3}{2}})h^{\frac{1}{2}}(1+t\gamma^{\frac{3}{2}}h^{\frac{1}{2}})\big)(2\pi h)^{-n/2}\|b\|_{L^2}.\end{aligned}$$

We stress, however, that Theorem 4 below applies to very general initial states $A(0)$ of which Gaussian states or their mixtures are an example. In addition, at the cost of further terms in the expansion, it gives approximation of the Lindblad evolution modulo $\mathcal{O}(e^{(M_0+Ch)\gamma t}(th^{1/2}\gamma^{-\frac{3}{2}})^N(1+\gamma)^N(1+t\gamma^{\frac{3}{2}}h^{\frac{1}{2}})^{\frac{N+2}{2}}$ for any N. For instance, when

$$\gamma = h^\delta, \quad \delta < \tfrac{1}{3}, \quad t \leq h^{-\nu}, \quad \nu < \tfrac{1}{2}\min(2\delta, 1-3\delta),$$

this gives an expansion modulo $\mathcal{O}(h^\infty)$.

This paper is self-contained except for some basic facts about semiclassical quantization from [Zw12, Chapter 4]. It is organized as follows. In §2 we review the definition of pseudodifferential operators and symbol classes. We introduce a new L^2-based symbol class which is natural for the study of Hilbert–Schmidt operators, and show the properties of the corresponding pseudodifferential calculus. In §3 we present a variant of Egorov's theorem with Hilbert–Schmidt norm and in §4 we prove mapping properties of $e^{t\mathcal{L}}$. §5 is then devoted to estimates on the Fokker–Planck evolution. A general result about agreement of classical and quantum dynamics in Hilbert–Schmidt norm is proved in §6. In §7, we consider situations where we can effectively control the Hilbert–Schmidt norm of the Lindblad evolution from below. Finally, in the appendix, we review some properties of pseudodifferential operators with quadratic symbol growth.

Acknowledgements. We would like to thank Simon Becker for pointing out reference [Da77] which clarified the trace class properties of $e^{t\mathcal{L}}$ and Zhenhao Li for helpful comments on early versions of this paper. We are also grateful to Felipe Hernández, Daniel Ranard and Jess Riedel for an illuminating discussion of the results and methods of [HRR23a].

JG acknowledges support from EPSRC grants EP/V001760/1 and EP/V051636/1 and MZ from the NSF grant DMS-1901462 and the Simons Foundation under a

"Moiré Materials Magic" grant. MZ would like to thank Paul Brumer for introducing him, many years ago in the context of discussing [GoBr99], to the problem of classical/quantum correspondence for Lindbladians. MZ is grateful to University College London for the generous hospitality during the writing of this paper.

2. Symbol spaces and quantization

The operators introduced in §1 are defined using *pseudodifferential operators* which are obtained by a Weyl quantization process: at first for $a \in \mathscr{S}(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ (here $\mathscr{S}$ denotes Schwartz functions, that is functions u for which $x^\alpha \partial^\beta u$ are bounded for all multiindices α and β; $\mathscr{S}'$ denotes its dual, the space of tempered distributions – see [Zw12, Chapter 3]) we define

$$\operatorname{Op}(a)u = a^{\mathrm{w}}(x, hD, h)u := \frac{1}{(2\pi h)^n} \int a\left(\frac{x+y}{2}, \xi\right) e^{\frac{i}{h}\langle x-y, \xi\rangle} u(y) dy d\xi. \tag{2.1}$$

The Hilbert–Schmidt norm has a clean expression in terms of the symbol a (PDE parlance for classical observables):

$$\| \operatorname{Op}(a) \|^2_{\mathscr{L}_2} = \operatorname{tr} \operatorname{Op}(a) \operatorname{Op}(a)^* = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^{2n}} |a(x, \xi)|^2 dx d\xi.$$

This is in contrast with the trace class norm which does not have an easy characterization in terms of a and its estimates require L^1 norm of derivatives of a – see [DiSj99, Chapter 9].

In this paper we consider different classes of symbols for which (2.1) remains valid and has interesting composition properties (as an operator $\operatorname{Op}(a) : \mathscr{S} \to \mathscr{S}'$ the operator (2.1) is well defined for $a \in \mathscr{S}'(\mathbb{R}^{2n})$ – see [Zw12, Theorem 4.2]). We first recall the standard symbol class: for $m : \mathbb{R}^{2n} \to [0, \infty)$ satisfying $m(z)/m(w) \leq C(1 + |z - w|)^N$,

$$a \in S_\delta(m) \implies |\partial_z^\alpha a(z, h)| \leq C_\alpha h^{-\delta|\alpha|} m(z), \quad z = (x, \xi) \in \mathbb{R}^{2n}. \tag{2.2}$$

When $\delta = 0$ we write $S(m)$ and when $m = 1$, S_δ.

The next class corresponds to the conditions in (1.3): for smooth function on $\mathbb{R}^{2n}$,

$$u(z, h) \in S_{(k)} \iff |\partial_z^\alpha u(z, h)| \leq C_\alpha, \quad |\alpha| \geq k, \tag{2.3}$$

with constants C_α independent of h. The seminorms are given by the best constants C_α.

In dealings with Hilbert–Schmidt operators it is natural to consider symbols whose bounds are defined using L^2 norms. For smooth functions on $C^\infty(\mathbb{R}^{2n})$ depending on parameters h we define, for $0 \leq \rho < 1$,

$$a \in S^{L^2}_\rho \iff h^{-\frac{n}{2}} \| \partial_z^\alpha a \|_{L^2(\mathbb{R}^{2n})} \leq C_\alpha h^{-\rho|\alpha|}. \tag{2.4}$$

with the obvious seminorms. We note that the Sobolev embedding theorem and an interpolation argument show that $|\partial_z^\alpha a| \leq C'_\alpha h^{-\rho(|\alpha|+n+\delta)+\frac{n}{2}}$, for any $\delta > 0$. Hence for $\rho = 0$ the L^2 based spaces are contained in $h^{\frac{n}{2}} S(1)$ defined above, and in general

$$S_\rho^{L^2} \subset h^{-\rho(n+)+\frac{n}{2}} S_\rho(1). \tag{2.5}$$

It will also be useful to consider mixed spaces obtained by taking tensor products:

$$c(z,w) \in S \otimes S_\rho^{L^2} \iff h^{-\frac{n}{2}} \| \sup_z \partial_z^\alpha \partial_w^\beta c(z, \bullet)\|_{L^2} \leq C_{\alpha\beta} h^{-\rho|\beta|}, \quad z, w \in \mathbb{R}^{2n}. \tag{2.6}$$

We stress that we always demand that $0 \leq \rho < 1$.

Remark Another choice of the norm could be given by $\sup_z \|\partial_z^\alpha \partial_w^\beta c(z, \bullet)\|_{L^2}$ and both agree on products. The choice in definition (2.6) is motivated by the fact that $\|f(w,w)\|_{L^2_w} \leq \| \sup_z |f(z,w)|\|_{L^2_w}$ which does not work for the other choice.

For the properties of operators which are quantizations of $a \in S_\delta(m)$ see [Zw12, Chapter 4]. The same methods apply to operators obtained from $a \in S_{(k)}$ and are reviewed in the appendix. In particular we obtain spectral properties of operators quantizing $S_{(2)}$. Since the properties of $S_\rho^{L^2}$ and $0 \leq \rho < 1$ are more unusual we present them in this section. We start with

Lemma 2.1. *Suppose that $Q : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}$ is a non-degenerate bilinear quadratic form. Then, using definition (2.6),*

$$e^{ihQ(D_z, D_w)} : S \otimes S_\rho^{L^2} \to S \otimes S_\rho^{L^2}, \tag{2.7}$$

is continuous and for every N

$$e^{ihQ(D_z,D_w)} a - e^{\frac{i\pi}{4} \operatorname{sgn} Q} \sum_{k=0}^{N-1} \left(\frac{h}{i}\right)^k \frac{1}{k!} Q(D_z, D_w)^k a(z,w) \in h^{N(1-\rho)} S \otimes S_\rho^{L^2}, \tag{2.8}$$

where $\operatorname{sgn} Q$ is the signature of Q considered as a quadratic form on $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$.

Proof. We denote by B the symmetric matrix corresponding to our quadratic form: $Q(\zeta, \omega) = \frac{1}{2}\langle B(\zeta,\omega), (\zeta,\omega)\rangle$. For $a \in S(1) \otimes S_\rho^{L^2} \subset h^{-\rho(n+)+\frac{n}{2}} S(1) \otimes S_\rho(1)$, hence the expression

$$c(z,w) := e^{ihQ(D_z,D_w)} a(z,w)$$

makes sense as an element in $\mathscr{S}'$ (to see this, we apply e.g. [Zw12, Theorem 4.17] with for each fixed value of h) and by [Zw12, Theorem 4.8], for $a \in \mathscr{S}$,

$$c(z,w) = \frac{|\det B|^{-\frac{1}{2}}}{(2\pi h)^{2n}} \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} e^{\frac{i}{h}\varphi(z_1,z_2)} a(z+z_1, w+w_1) dz_1 dw_1,$$

where

$$\varphi(z_1, w_1) = -\tfrac{1}{2}\langle B^{-1}(z_1,w_1), (z_1,w_1)\rangle.$$

Since $a \in h^{-\rho(n+)+\frac{n}{2}}S(1) \otimes S_\rho(1)$, this integral can be understood in the sense of oscillatory integrals and defines an element of $\mathscr{S}'$ – see [Zw12, §3.6]. Recall also that oscillatory integrals allow for integrations by parts.

Set $v_1 = h^{-\rho} w_1$, and $\chi \in C_c^\infty(\mathbb{R}^{2n} \times \mathbb{R}^{2n})$ with $\chi \equiv 1$ near 0 and $\operatorname{supp} \chi \subset B(0,1)$. Then using the fact that $w_1 \mapsto \varphi(z_1, w_1)$ is linear, we obtain

$$\begin{aligned}
c(z,w) &= \frac{|\det B|^{-\frac{1}{2}}}{(2\pi h^{1-\rho})^{2n}} \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} e^{\frac{i}{h^{1-\rho}}\varphi(z_1,v_1)} a(z+z_1, w+h^\rho v_1) dz_1 dv_1, \\
&= \frac{|\det B|^{-\frac{1}{2}}}{(2\pi h^{1-\rho})^{2n}} \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} e^{\frac{i}{h^{1-\rho}}\varphi(z_1,v_1)} \chi(z_1,v_1) a(z+z_1, w+h^\rho v_1) dz_1 dv_1 \\
&\quad + \frac{|\det B|^{-\frac{1}{2}}}{(2\pi h^{1-\rho})^{2n}} \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} e^{\frac{i}{h^{1-\rho}}\varphi(z_1,v_1)} (1-\chi(z_1,v_1)) a(z+z_1, w+h^\rho v_1) dz_1 dv_1 \\
&=: c_1(z,w) + c_2(z,w)
\end{aligned}$$

We start by considering c_1. In this case, the integrand is compactly supported and we may apply the method of stationary phase [Zw12, Theorem 3.16 and Theorem 3.17]. That gives

$$\begin{aligned}
&\Big|\partial_z^{\alpha_1}\partial_w^{\alpha_2}\Big(c_1(z,w) - e^{\frac{i\pi}{4}\operatorname{sgn} B} \sum_{k=0}^{N-1} \Big(\frac{h^{1-\rho}}{i}\Big)^k \frac{1}{k!}\Big(Q(D_{z_1}, D_{v_1})^k a(z+z_1, w+h^\rho v_1)|_{z_1=v_1=0}\Big)\Big)\Big| \\
&\le C_N h^{(1-\rho)N} \sum_{|\beta_1|+|\beta_2|\le 2N+4n+1} h^{-\rho|\alpha_2|} \sup_{|(z_1,v_1)|<1} |\partial_{z_1}^{\beta_1+\alpha_1}\partial_{v_1}^{\beta_2+\alpha_2} a(z+z_1, w+h^\rho v_1)| \\
&=: C_N h^{(1-\rho)N} \sum_{|\beta_1|+|\beta_2|\le 2N+4n+1} R_{\alpha\beta}(z,w),
\end{aligned}$$

with the estimates on the remainder provided by Sobolev's embedding:

$$|R_{\alpha\beta}(z,w)|^2 \le h^{-2\rho|\alpha_2|} \sum_{\gamma \le 2n+1} \|\partial_{z_1}^{\beta_1+\alpha_1+\gamma_1}\partial_{v_1}^{\beta_2+\alpha_2+\gamma_2} a(z+\cdot, w+h^\rho\cdot)\|^2_{L^2(B_{\mathbb{R}^{2n}}(0,1))}.$$

Hence, with $B := B_{\mathbb{R}^{4n}}(0,1)$,

$$\begin{aligned}
\int_{\mathbb{R}^{2n}} \sup_z |R_{\alpha\beta}(z,w)|^2 dw &\le h^{-2\rho|\alpha_2|} \sum_{\gamma\le 2n+1} \int_{\mathbb{R}^{2n}} \int_B \sup_z |\partial_{(z_1,v_1)}^{\alpha+\beta+\gamma} a(z+z_1, w+h^\rho v_1)|^2 dz_1 dv_1 dw \\
&\le \int_B h^{2\rho(|\gamma_2|+|\beta_2|)} \|\sup_z |\partial_{(z,w)}^{\alpha+\beta+\gamma} a(z,\cdot)|\|^2_{L^2} dz_1 dv_1 \\
&\le C \sum_{\gamma\le 2n+1} h^{2\rho(|\gamma_2|+|\beta_2|)} \|\sup_z |\partial_{(z,w)}^{\alpha+\beta+\gamma} a(z,\cdot)|\|^2_{L^2}.
\end{aligned}$$

In particular, this implies that

$$c_1(z,w) - e^{\frac{i\pi}{4}\operatorname{sgn} B} \sum_{k=0}^{N-1} \Big(\frac{h^{1-\rho}}{i}\Big)^k \frac{1}{k!}\Big(Q(D_{z_1}, D_{v_1})^k a(z+z_1, w+h^\rho v_1)\Big)|_{v_1=z_1=0}$$

is in $h^{(1-\rho)N} S \otimes S_\rho^{L^2}$.

We now consider the remaining term in c, c_2, and note that on $\mathrm{supp}(1-\chi)$, $|\partial_{(z_1,v_1)}\varphi(z_1,v_1)| \geq c\langle (z_1,v_1)\rangle$. Hence, integration by parts (justified by the definition of the oscillatory integral) yields, for $N > 2n+1$,

$$\begin{aligned}
&h^{(\rho-1)(2N-4n)} \| \sup_z \partial^\alpha_{(z,w)} c_2(z,\cdot)\|^2_{L^2}\\
&\leq C_N \int \sup_z \Big(\iint \sum_{|\beta_1|+|\beta_2|\leq N} \langle (z_1,v_1)\rangle^{-N} \Big| \partial^\alpha_{(z,w)} \Big(\partial_z^{\beta_1} (h^\rho \partial_w)^{\beta_2} a(z+z_1, w+h^\rho v_1)\Big)\Big| dz_1 dv_1\Big)^2 dw\\
&\leq C_N \iiint \sum_{|\beta_1|+|\beta_2|\leq N} \langle (z_1,v_1)\rangle^{-2N+2n+1} \sup_z \Big| \partial^\alpha_{(z,w)} \partial_z^{\beta_1} (h^\rho\partial_w)^{\beta_2} a(z+z_1,w+h^\rho v_1)\Big|^2 dz_1 dv_1 dw\\
&\leq C_N \iint \sum_{|\beta_1|+|\beta_2|\leq N} \langle (z_1,v_1)\rangle^{-2N+2n+1} \Big\| \sup_z \partial^\alpha_{(z,w)} \partial_z^{\beta_1} (h^\rho \partial_w)^{\beta_2} a(z,\cdot)\Big\|^2_{L^2} dz_1 dv_1\\
&\leq C_N \sum_{|\beta_1|+|\beta_2|\leq N} \Big\| \sup_z \partial^\alpha_{(z,w)} \partial_z^{\beta_1} (h^\rho\partial_w)^{\beta_2} a(z,\cdot)\Big\|^2_{L^2}.
\end{aligned}$$

Hence, we have $c_2 \in h^{(N-2n)(1-\rho)} S \otimes S_\rho^{L^2}$ for arbitrary N and $c \in S \otimes S_\rho^{L^2}$. The argument also shows that that the map from a to c is continuous, and (2.8) holds. ☐

We can write the composition law for operators in $S_\rho^{L^2}$ with $S_{(k)}$.

Lemma 2.2. *Let $0 \leq \rho < 1$, $k \geq 0$, $a \in S_{(k)}, b \in S_\rho^{L^2}$. Then,*

$$\mathrm{Op}(a)\,\mathrm{Op}(b) = \mathrm{Op}(c),$$

where c has the following expansion: for $N \geq k$,

$$c(x,\xi) - \sum_{j=0}^{N-1} \frac{1}{j!} \left(\frac{h}{2i} \sigma(D_x, D_\xi, D_y, D_\eta)^j a(x,\xi) b(y,\eta)\right) |_{\substack{y=x\\ \eta=\xi}} \in h^{N(1-\rho)} S_\rho^{L^2}. \tag{2.9}$$

Proof. Writing $z = (x,\xi)$, $w = (y,\eta)$, we have

$$\mathrm{Op}(a)\,\mathrm{Op}(b) = \mathrm{Op}(c), \qquad c(z) := e^{ihA(D_{z,w})} a(z) b(w)|_{z=w},$$

where $A(D_{z,w}) := -\frac{1}{2}\sigma(D_x, D_\xi, D_y, D_\eta)$. By Taylor's formula

$$c(z,h) = \sum_{\ell=0}^{N-1} \frac{1}{\ell!} (ihA(D))^\ell (a(z)b(w))|_{z=w} + R_N(z,h)$$

where

$$R_N(z,h) : \frac{1}{(N-1)!} \int_0^1 (1-t)^{N-1} e^{ithA(D)} (ihA(D))^N ((a(z)b(w))|_{z,w} dt.$$

For $N \geq k$,

$$A(D_{z,w})^N a(z) b(w) \in h^{-N\rho} S \otimes S_\rho^{L^2}.$$

Hence, Lemma 2.1 applies and $e^{ihtA(D)} : S \otimes S_\rho^{L^2} \to S \otimes S_\rho^{L^2}$ has uniform bounds in $t \in [0,1]$. Now, for $e \in S \otimes S_\rho^{L^2}$, we have

$$\|\partial_w^\alpha e(w,w)\|_{L^2} \leq C \sum_{|\beta|\leq|\alpha|} \|\partial^\beta_{(z,w)} e(z,w)|_{w=z}\|_{L^2_w} \leq C \sum_{|\beta|\leq|\alpha|} \| \sup_z |\partial^\beta_{(z,w)} e(z,\cdot)|\|_{L^2}.$$

We conclude that $R_N \in h^{(1-\rho)N} S_\rho^{L^2}$ which is (2.9). $\square$

3. Egorov's theorem revisited

We give a variant of Egorov's theorem which is analogous to Theorems 1 and 4 and uses propagation of quantum observables in symbol classes $S_\rho^{L^2}$ introduced in §2. In fact, the proof of Theorem 4 follows the same strategy with improved estimates coming from diffusion estimates: Lemma 3.1 below (see also (3.7)) is replaced by Proposition 5.1.

We start with a lemma relating the constant Γ in (1.9) to the properties of the flow (see [Zw12, Lemma 11.11] for a slightly different version)

Lemma 3.1. *Let $\varphi_t := \exp tH_p$ where p satisfies (1.3). Then*

$$|\partial^\alpha \varphi_t(x,\xi)|_{\ell^\infty(\mathbb{R}^{2n})} \leq C_\alpha e^{\Gamma|\alpha|t}, \quad \alpha \in \mathbb{N}^{2n}, \quad |\alpha| > 0. \tag{3.1}$$

In the proof of Lemma 3.1 we use the following version of Grönwall's inequality:

Lemma 3.2. *Let $\Gamma \in \mathbb{R}$ and suppose that $u : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies*

$$u(t) \leq v(t) + \Gamma \int_0^t u(s)ds. \tag{3.2}$$

Then,

$$u(t) \leq v(t) + \Gamma \int_0^t e^{\Gamma(t-s)} v(s)ds, \quad t \geq 0.$$

Proof. Define $w(t) := \int_0^t u(s)ds$. Then, w is continuously differentiable and satisfies

$$w'(t) \leq v(t) + \Gamma w(t), \quad w(0) = 0.$$

Hence, conjugating by $e^{-\Gamma t}$ and integrating gives

$$w(t) \leq \int_0^t e^{\Gamma(t-s)} v(s)ds,$$

which, after substitution in (3.2), finishes the proof. $\square$

Proof of Lemma 3.1. The proof of (3.1) is an induction on $|\alpha|$. The first step is the case of $|\alpha| = 1$. Since $(d/dt)\varphi_t = H_p(\varphi_t)$,

$$\frac{d}{dt}(\partial^\alpha \varphi_t) = \partial H_p(\varphi_t)\partial^\alpha \varphi_t, \quad \partial^\alpha \varphi(0) = \alpha. \tag{3.3}$$

Since the entries of the matrix ∂H_p are bounded by Γ, integration gives

$$\sup_{\mathbb{R}^{2n}} |\partial^\alpha \varphi_t|_{\ell^\infty} \leq 1 + \Gamma \int_0^t \sup_{\mathbb{R}^{2n}} |\partial^\alpha \varphi_s|_{\ell^\infty} ds.$$

Lemma 3.2 then gives (3.1) for $|\alpha| = 1$.

Now assume $|\alpha| = \ell$ and suppose the estimate (3.1) is valid for all multiindices β with $1 \leq |\beta| < \ell$. We differentiate (3.3), to find

$$\frac{d}{dt}\left(\partial^\alpha \varphi_t\right) = \partial H_p(\varphi_t)\partial^\alpha \varphi_t + g(t), \tag{3.4}$$

where $g(t)$ is a sum of terms having the form

$$g_{\alpha\beta} \circ \varphi_t \, \partial^{\beta_1}\varphi_t \cdots \partial^{\beta_k}\varphi_t, \quad g_{\alpha\beta} \in S(1),$$

for $\beta_1 + \cdots \beta_k = \alpha$ and $0 < |\beta_j| < |\alpha| = \ell$ $(j = 1, \ldots, k)$. The induction hypothesis implies $\sup_{\mathbb{R}^{2n}} |g(t)|_{\ell^\infty} \leq Ce^{\Gamma|\alpha||t|}$. Integrating as above, we obtain

$$\sup_{\mathbb{R}^{2n}} |\partial^\alpha \varphi_t|_{\ell^\infty} \leq Ce^{\Gamma|\alpha|t} + \Gamma \int_0^t \sup_{\mathbb{R}^{2n}} |\partial^\alpha \varphi_s|_{\ell^\infty} ds.$$

and we can use Lemma 3.2 to obtain (3.1). ☐

Theorem 3. *Suppose that $\mathcal{L}_0$ is given by (1.1) with P satisfying (1.3) and $0 \leq \rho < \frac{2}{3}$. If $A(t)$ satisfies (in the notation of §2)*

$$\partial_t A(t) = \mathcal{L}_0 A(t), \quad A(0) = \operatorname{Op}(a_0), \quad a_0 \in S_\rho^{L^2},$$

Then, for every N there exist $C_N > 0$ and $a(t) \in S_{\rho(t)}^{L^2}$ such that for Γ given by (1.9) and

$$\rho(t) := \rho + \frac{\Gamma t}{|\log h|} \leq \frac{2}{3}, \tag{3.5}$$

$a(t) - (\exp tH_p)^* a_0 \in h^{2-3\rho} e^{3\Gamma t} S_{\rho(t)}^{L^2}$ and

$$\|A(t) - \operatorname{Op}(a(t))\|_{\mathscr{L}_2} \leq C_N e^{3N\Gamma t} h^{N(2-3\rho)}. \tag{3.6}$$

Proof. We define

$$U_0(t)b := (\exp tH_p)^* b, \quad \partial_t U_0(t) = H_p U_0(t), \quad U_0(0) = I,$$

and note that using the definition (3.5) and Lemma 3.1 we have

$$U(t-s) : S_{\rho(s)}^{L^2} \to S_{\rho(t)}^{L^2}, \tag{3.7}$$

To construct $a(t)$ we start with $a_0(t) := U_0(t) a_0$ so that $a_0(t) \in S_{\rho(t)}^{L^2}$. Set $A_0(t) := \operatorname{Op}(a_0(t))$. Then, using Lemma 2.2 we obtain

$$\dot{A}_0(t) = \operatorname{Op}(\dot{a}_0(t)) = \operatorname{Op}(H_p a_0(t)) = \mathcal{L}_0 A_0(t) + \operatorname{Op}(e_0(t)), \quad e_0(t) \in h^{(2-3\rho)} e^{3\Gamma t} S_{\rho(t)}^{L^2}.$$

Suppose now that we found

$$a_j(t) \in h^{(2-3\rho)j} e^{3j\Gamma t} S^{L^2}_{\rho(t)} \quad j = 0, \dots, N-1$$

such that, with $A_{N-1} := \sum_{j=0}^{N-1} \operatorname{Op}(a_j(t))$, we have

$$\dot{A}_{N-1} = \mathcal{L}_0 A_{N-1}(t) + \operatorname{Op}(e_N(t)), \quad e_N(t) \in h^{(2-3\rho)N} e^{3N\Gamma t} S^{L^2}_{\rho(t)}.$$

Using e_N we define

$$a_N(t) := -\int_0^t U_0(t-s) e_N(s) ds, \quad \partial_t a_N = H_p a_N - e_N, \quad a_N(0) = 0.$$

Then, using (3.7),

$$a_N(t) \in h^{(2-3\rho)N} e^{3N\Gamma t} S^{L^2}_{\rho(t)},$$

and hence, with $A_N(t) = A_{N-1}(t) + \operatorname{Op}(a_N(t))$, we have

$$\begin{aligned}\dot{A}_N(t) &= \mathcal{L}_0 A_{N-1}(t) + \operatorname{Op}(e_N(t)) + \operatorname{Op}(\dot{a}_N(t)) \\ &= \mathcal{L}_0 A_{N-1}(t) + \operatorname{Op}(H_p a_N(t)) \\ &= \mathcal{L}_0 A_N(t) + \operatorname{Op}(e_{N+1}(t)), \quad e_{N+1}(t) \in h^{(2-3\rho)(N+1)} e^{3(N+1)\Gamma t} S^{L^2}_{\rho(t)}.\end{aligned}$$

Note that in the last line we used Lemma 2.2 to obtain the estimates on e_{N+1}. This gives $a = \sum_{j \le N} a_j$.

To compare $A_N(t) := \operatorname{Op}(a(t))$ to $A(t)$, we use the fact that $e^{t\mathcal{L}_0}$ preserves the Hilbert–Schmidt norm (see (1.1)):

$$\big\| A(t) - A_N(t) \big\|_{\mathscr{L}_2} \le \int_0^t \Big\| e^{(t-s)\mathcal{L}_0} \operatorname{Op}(e_{N+1}(s)) \Big\|_{\mathscr{L}_2} ds \le h^{(2-3\rho)(N+1)} e^{3(N+1)\Gamma t}.$$

This completes the proof □

4. The semigroup generated by the Lindbladian.

We prove here that the Lindblad evolution is well defined in the space of Hilbert–Schmidt operators. This is done under the assumption (1.3) alone.

To describe the action of $\mathcal{L}$ on operators $\mathscr{S} \to \mathscr{S}'$, we identify such operators with their Schwartz kernels in $\mathbb{R}^n \times \mathbb{R}^n$ and consider

$$\mathcal{L}_1 : \mathscr{S}'(\mathbb{R}^n \times \mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n \times \mathbb{R}^n), \quad \mathcal{L}_0 : \mathscr{S}(\mathbb{R}^n \times \mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n \times \mathbb{R}^n). \tag{4.1}$$

More precisely, for $K \in \mathscr{S}'(\mathbb{R}^n \times \mathbb{R}^n)$, and $\chi \in \mathscr{S}(\mathbb{R}^n \times \mathbb{R}^n)$ we denote by $K(\chi)$ the distributional pairing, formally equal to $\int K(x,y)\chi(x,y)dxdy$. Then for $A, B : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ we define, $(A \otimes B)K \in \mathscr{S}'(\mathbb{R}^n \times \mathbb{R}^n)$ by

$$(A \otimes B)K(\varphi \otimes \psi) := K(A^t \varphi \otimes B\psi), \quad \varphi, \psi \in \mathscr{S}(\mathbb{R}^n), \quad (\varphi \otimes \psi)(x,y) := \varphi(x)\psi(y),$$

where A^t is the transpose of A: for $f, g \in \mathscr{S}(\mathbb{R}^n)$, $(Af)(g) = f(A^t g)$ (this also defines the action of A on $\mathscr{S}'$). We note that if we identify the Schwartz kernels with operators then $(A \times B)K = AKB$.

In this notation

$$\begin{aligned}\mathcal{L}_1 := \frac{i}{h}\big(\operatorname{Op}(p)\otimes I - I \otimes \operatorname{Op}(p)\big) + \frac{\gamma}{h}\sum_j \big((\operatorname{Op}(\ell_j)\otimes I)(I\otimes \operatorname{Op}(\bar\ell_j))\big) \\ - \tfrac{1}{2}\sum_j \big(\operatorname{Op}(\bar\ell_j)\operatorname{Op}(\ell_j)\otimes I + I \otimes \operatorname{Op}(\bar\ell_j)\operatorname{Op}(\ell_j)\big),\end{aligned} \tag{4.2}$$

and $\mathcal{L}_0 := \mathcal{L}_1|_{\mathscr{S}(\mathbb{R}^{2n}}$.

The following lemma describes $\mathcal{L}_1$ in a way that allows an application of Proposition A.2, which in turn provides the definition of $\mathcal{L}$ as an unbounded operator on $\mathscr{L}_2$.

Lemma 4.1. *The operator $\mathcal{L}_1 : \mathscr{S}'(\mathbb{R}^{2n}) \to \mathscr{S}'(\mathbb{R}^{2n})$ defined by (4.2) is given by $\mathcal{L}_1 = \operatorname{Op}(L)$, where where $L = L(x,\xi,y,\eta) \in C^\infty(\mathbb{R}^{4n})$ satisfies*

$$|\partial^\alpha L| \le C_\alpha(1 + |x| + |\xi| + |y| + |\eta|), \qquad |\alpha| \ge 1. \tag{4.3}$$

Moreover, identifying the Hilbert–Schmidt class $\mathscr{L}_2(L^2(\mathbb{R}^n))$ with $L^2(\mathbb{R}^n \times \mathbb{R}^n)$ using Schwartz kernels, the Lindbladian $\mathcal{L}$ with the domain

$$\mathcal{D}(\mathcal{L}) := \{A \in \mathscr{L}_2(L^2(\mathbb{R}^n)) : \mathcal{L}_1 A \in \mathscr{L}_2(L^2(\mathbb{R}^n))\}, \tag{4.4}$$

satisfies

$$\mathcal{L} = \overline{\mathcal{L}_0}, \quad \mathcal{L}^* = \overline{\mathcal{L}_0^*},$$

where $\mathcal{L}_0^* : \mathscr{S}(\mathbb{R}^n \times \mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n \times \mathbb{R}^n)$ is the formal adjoint of $\mathcal{L}_0$.

Proof. Using coordinates $((x,\xi),(y,\eta)) \in \mathbb{R}^{2n} \times \mathbb{R}^{2n}$ and denoting $\operatorname{Op}_{\mathbb{R}^{2n}}$ the Weyl quantization on $\mathbb{R}^{2n}$, the definitions above show that

$$\operatorname{Op}(a)\otimes I = \operatorname{Op}_{\mathbb{R}^{2n}}(a(x,\xi)), \qquad I \otimes \operatorname{Op}(a) = \operatorname{Op}_{\mathbb{R}^{2n}}(\tilde a(y,\eta)), \quad \tilde a := e^{i\langle hD_y, D_\eta\rangle} a.$$

(See [Zw12, Theorem 4.13]: if $\operatorname{Op}(a) = \operatorname{Op}_1(a_1)$, then $\operatorname{Op}(\tilde a) = \operatorname{Op}(a)^t = \operatorname{Op}_0(a_1)$.) Consequently $a \in C^\infty(\mathbb{R}^{2n})$ satisfies, $|\partial^\alpha_{(x,\xi)} a(x,\xi)| \le C_\alpha(1 + |x| + |\xi|)$ for $|\alpha| \ge 1$, and, by [Zw12, Theorem 4.17], so does $\tilde a$.

Since $\ell_j \in S_{(1)}$, by Proposition A.1 we have $\operatorname{Op}(\bar\ell_j)\operatorname{Op}(\ell_j) = \operatorname{Op}(c_j)$, for c_j satisfying $|\partial^\alpha_{(x,\xi)} c_j(x,\xi)| \le C_\alpha(1 + |x| + |\xi|)$ for $|\alpha| \ge 1$. Together with the facts that $\ell_j \in S_{(1)}$ and $p \in S_{(2)}$, this implies $\mathcal{L}_1 = \operatorname{Op}_{\mathbb{R}^{2n}}(L)$, where L satisfies (4.3). Thus we can apply Proposition A.2 and the lemma follows. ∎

The next lemma describe the adjoint of $\mathcal{L}$:

Lemma 4.2. *The adjoint of the Lindblad operator $\mathcal{L}$, $\mathcal{L}^*$, is given by*

$$\mathcal{L}^* B = -\frac{i}{h}[P, B] + \frac{\gamma}{h} \sum_j L_j^* B L_j - \frac{1}{2}(L_j^* L_j B + B L_j^* L_j), \tag{4.5}$$

with domain

$$\mathcal{D}(\mathcal{L}^*) = \{A \in \mathscr{L}_2 \, : \, \mathcal{L}^* A \in \mathscr{L}_2\},$$

where for any $A \in \mathscr{L}_2$, $\mathcal{L}^* A$ is defined as an operator $\mathscr{S} \to \mathscr{S}'$.

Proof. By Proposition A.2 it is sufficient to compute the formal adjoint in the action on operators $\mathscr{S}' \to \mathscr{S}$. Observe that, using cyclicity of the trace, for $A, B : \mathscr{S}' \to \mathscr{S}$,

$$\begin{aligned} \left\langle \tfrac{i}{h}[P, A], B \right\rangle_{\mathscr{L}_2} &= \operatorname{tr}\left(\tfrac{i}{h}[P, A] B^* \right) = \tfrac{i}{h} \operatorname{tr}\left((PA - AP) B^* \right) = \tfrac{i}{h} \operatorname{tr}\left(A [P, B]^* \right) \\ &= \operatorname{tr}\left(A \big(-\tfrac{i}{h}[P, B] \big)^* \right) = \langle A, -\tfrac{i}{h}[P, B] \rangle_{\mathscr{L}_2}, \\ \langle L_j A L_j^*, B \rangle_{\mathscr{L}_2} &= \operatorname{tr}\left(L_j A L_j^* B^* \right) = \operatorname{tr}\left(A L_j^* B^* L_j \right) = \langle A, L_j^* B L_j \rangle_{\mathscr{L}_2}, \\ \langle L_j^* L_j A, B \rangle_{\mathscr{L}_2} &= \operatorname{tr}\left(L_j^* L_j A B^* \right) = \operatorname{tr}\left(A B^* L_j^* L_j \right) = \langle A, L_j^* L_j B \rangle_{\mathscr{L}_2}, \end{aligned}$$

and similarly for $\langle A L_j^* L_j, B \rangle_{\mathscr{L}_2}$. ☐

We next record some properties of $\mathcal{L}$ and its adjoint.

Lemma 4.3. *For $A : \mathscr{S}' \to \mathscr{S}$,*

$$2 \operatorname{Re} \langle \mathcal{L} A, A \rangle_{\mathscr{L}_2} = -\frac{\gamma}{h} \sum_j \| [L_j, A] \|_{\mathscr{L}_2}^2 + \frac{\gamma}{h} \langle \sum_j [L_j, L_j^*] A^*, A^* \rangle_{\mathscr{L}_2}, \tag{4.6}$$

and

$$2 \operatorname{Re} \langle \mathcal{L}^* A, A \rangle_{\mathscr{L}_2} = -\frac{\gamma}{h} \sum_j \| [L_j^*, A] \|_{\mathscr{L}_2}^2 + \frac{\gamma}{h} \langle \sum_j [L_j, L_j^*] A, A \rangle_{\mathscr{L}_2}. \tag{4.7}$$

Proof. First, observe that (1.2) and (4.5) show

$$(\mathcal{L} A)^* = \mathcal{L} A^*, \qquad (\mathcal{L}^* A)^* = \mathcal{L}^* A^*.$$

Thus, we compute

$$\begin{aligned}
2\operatorname{Re}\langle \mathcal{L}A, A\rangle_{\mathscr{L}_2} &= \operatorname{tr}((\mathcal{L}A)A^* + A(\mathcal{L}A^*)) \\
&= \operatorname{tr}\Big(\frac{i}{h}[P,A]A^* + A\frac{i}{h}[P,A^*] + \frac{\gamma}{h}\sum_j \Big(L_jAL_j^*A^* + AL_jA^*L_j^*\Big) \\
&\qquad - \frac{\gamma}{2h}\sum_j (L_j^*L_jAA^* + AL_j^*L_jA^* + AL_j^*L_jA^* + AA^*L_j^*L_j)\Big) \\
&= \operatorname{tr}\Big(\frac{i}{h}[P,A]A^* + A\frac{i}{h}[P,A^*]\Big) \\
&\qquad + \frac{\gamma}{h}\sum_j \operatorname{tr}\Big(L_jAL_j^*A^* + AL_jA^*L_j^* - L_j^*L_jAA^* - AL_j^*L_jA^*\Big).
\end{aligned}$$

Now,

$$\operatorname{tr}\Big([P,A]A^* + A[P,A^*]\Big) = \operatorname{tr}\Big(PAA^* - AA^*P\Big) = 0,$$

and

$$\begin{aligned}
&\operatorname{tr}\Big(L_jAL_j^*A^* + AL_jA^*L_j^* - L_j^*L_jAA^* - AL_j^*L_jA^*\Big) \\
&= \operatorname{tr}\Big(- [L_j, A]([L_j, A])^* + L_jAA^*L_j^* + AL_jL_j^*A^* - L_j^*L_jAA^* - AL_j^*L_jA^*) \\
&= \operatorname{tr}\Big(- [L_j, A]([L_j, A])^* + [L_j, L_j^*]A^*A).
\end{aligned}$$

Hence, (4.6) follows.

The computation for (4.7) is similar. Since the commutator part of $\mathcal{L}^*$ has the same form as that of $\mathcal{L}$, we only need to compute

$$\begin{aligned}
&\operatorname{tr}\Big(L_j^*AL_jA^* + AL_j^*A^*L_j - L_j^*L_jAA^* - AL_j^*L_jA^*\Big) \\
&= \operatorname{tr}\Big(- [L_j^*, A]([L_j^*, A])^* + L_j^*AA^*L_j - L_j^*L_jAA^*\Big) \\
&= \operatorname{tr}\Big(- [L_j^*, A]([L_j^*, A])^* + [L_j, L_j^*]AA^*\Big),
\end{aligned}$$

and (4.7) follows. □

The next lemma will be used to control the second terms on the right hand sides of (4.6) and (4.7).

Lemma 4.4. *Let $C_0 \in \mathbb{R}$ and suppose that $E : \mathscr{S} \to L^2$ is a self-adjoint operator on $L^2(\mathbb{R}^n)$ satisfying $E \leq C_0$. Then, for $B : \mathscr{S}' \to \mathscr{S}$,*

$$\Big\langle EB, B\Big\rangle_{\mathscr{L}_2} \leq C_0\|B\|^2_{\mathscr{L}_2}. \tag{4.8}$$

Proof. To see this, observe that exists an L^2-orthonormal basis u_j and $\lambda_j \geq 0$,

$$BB^* = \sum_j \lambda_j u_j \otimes u_j, \quad (f \otimes g)(\varphi) := f\langle \varphi, g\rangle.$$

We also note that if $\lambda_j > 0$ then $u_j \in \mathscr{S} \subset \mathcal{D}(E)$. Then,

$$\begin{aligned}\Big\langle EB, B\Big\rangle_{\mathscr{L}_2} &= \operatorname{tr}(EBB^*) = \sum_j \langle EBB^* u_j, u_j\rangle_{L^2} = \sum_j \lambda_j \langle Eu_j, u_j\rangle_{L^2} \\ &\leq C_0 \sum_j \lambda_j = C_0 \|B\|^2_{\mathscr{L}_2},\end{aligned}$$

which is (4.8). □

Next, we provide an estimate

Lemma 4.5. *Suppose that, as a bounded self-adjoint operator on $L^2(\mathbb{R}^n)$ (see (1.6))*

$$\sum_j [L_j, L_j^*] \leq \frac{2Mh}{\gamma}. \tag{4.9}$$

Then, for $A : \mathscr{S} \to \mathscr{S}'$ and $\lambda > 0$,

$$\lambda \|A\|_{\mathscr{L}_2} \leq \|(\mathcal{L} - M - \lambda)A\|_{\mathscr{L}_2}, \quad \lambda \|A\|_{\mathscr{L}_2} \leq \|(\mathcal{L}^* - M - \lambda)A\|_{\mathscr{L}_2}, \tag{4.10}$$

Proof. Observe that by Lemma 4.3, and Lemma 4.4

$$\begin{aligned}2 \operatorname{Re}\langle (\mathcal{L} - M - \lambda)A, A\rangle_{\mathscr{L}_2} &\leq -2\lambda \|A\|_{\mathscr{L}_2} - 2M\|A\|_{\mathscr{L}_2} + \frac{\gamma}{h}\Big\langle \sum_j [L_j, L_j^*]A^*, A^*\Big\rangle_{\mathscr{L}_2} \\ &\leq -2\lambda \|A\|_{\mathscr{L}_2}.\end{aligned}$$

Hence,

$$2\lambda \|A\|_{\mathscr{L}_2} \leq |2 \operatorname{Re}\langle (\mathcal{L} - M - \lambda)A, A\rangle_{\mathscr{L}_2}| \leq 2\|(\mathcal{L} - M - \lambda)A\|_{\mathscr{L}_2} \|A\|_{\mathscr{L}_2},$$

from which the first estimate in (4.10) follows. The argument for the second estimate is identical. □

Proposition 4.6. *Suppose that (4.9) holds. Then the operator $\mathcal{L}$ with domain $\mathcal{D}(\mathcal{L}) := \{A \in \mathscr{L}_2 : \mathcal{L}A \in \mathscr{L}_2\}$ generates a strongly continuous semigroup*

$$e^{t\mathcal{L}} : \mathscr{L}_2 \to \mathscr{L}_2 \quad \textit{and} \quad \|e^{t\mathcal{L}}\|_{\mathscr{L}_2 \to \mathscr{L}_2} \leq e^{Mt}, \qquad t \geq 0.$$

Proof. By Proposition A.2, or rather its proof (see (A.4)), and Lemma 4.1, for $A \in \mathcal{D}(\mathcal{L})$ there exists a sequence of operators $A_n : \mathscr{S}' \to \mathscr{S}$ such that $A_n \xrightarrow{\mathscr{L}_2} A$ and $\mathcal{L}A_n \xrightarrow{\mathscr{L}_2} \mathcal{L}A$. Hence, for $A \in \mathcal{D}(\mathcal{L})$ and $\lambda > 0$, Lemma 4.5 gives

$$\begin{aligned}\lambda \|A\|_{\mathscr{L}_2} &= \lambda \lim_{n\to\infty} \|A_n\|_{\mathscr{L}_2} \\ &\leq \lim_{n\to\infty} \|(\mathcal{L} - M - \lambda)A_n\|_{\mathscr{L}_2} = \|(\mathcal{L} - M - \lambda)A\|_{\mathscr{L}_2}.\end{aligned}$$

Similarly, for $A \in \mathcal{D}(\mathcal{L}^*)$, we have $A_n : \mathscr{S}' \to \mathscr{S}$ such that $A_n \xrightarrow{\mathscr{L}_2} A$ and $\mathcal{L}^* A_n \xrightarrow{\mathscr{L}_2} \mathcal{L}^* A$. This implies that for $A \in \mathcal{D}(\mathcal{L}^*)$, and $\lambda > 0$

$$\lambda \|A\|_{\mathscr{L}_2} \leq \|(\mathcal{L}^* - M - \lambda)A\|_{\mathscr{L}_2}.$$

In particular, $(\mathcal{L} - M - \lambda)^{-1} : \mathscr{L}_2 \to \mathcal{D}(\mathcal{L})$ exists and satisfies,

$$\|(\mathcal{L} - M - \lambda)^{-1}\|_{\mathscr{L}_2 \to \mathscr{L}_2} \leq \lambda^{-1}, \quad \lambda > 0.$$

The Hille-Yosida theorem then implies that $\mathcal{L} - M$ generates a strongly continuous semigroup $e^{t(\mathcal{L}-M)}$ satisfying

$$\|e^{t(\mathcal{L}-M)}\|_{\mathscr{L}_2 \to \mathscr{L}_2} \leq 1,$$

from which the proposition follows. □

We conclude this section by showing how condition (4.9) is related to a lower bound on the friction (1.6)

Lemma 4.7. *Let*

$$M_0 := \sup_{\mathbb{R}^{2n}} \mu, \quad \mu := \frac{1}{2i} \sum_{j=1}^{J} \{\ell_j, \bar{\ell}_j\}. \tag{4.11}$$

Then there is $C_0 > 0$ such that (4.9) holds with

$$M = \gamma M_0 + C_0 h \gamma, \tag{4.12}$$

for $0 < h < 1$. Furthermore, if $\mu \equiv 0$, then (4.9) holds with

$$M = C_0 h^2 \gamma \tag{4.13}$$

for $0 < h < 1$.

Proof. Since (1.3) shows that $\mu \in S(1)$, the first estimate is a straightforward application of sharp Gårding inequality for the class $S(1)$ – see [DiSj99, Theorem 7.1] or [Zw12, §4.7.2]. When $\mu \equiv 0$, we use that $[L_j, L_j^*] = \operatorname{Op}(\frac{h}{2i}\{\ell_j, \bar{\ell}_j\} + h^3 e)$ for some $e \in S(1)$ and hence the second estimate follows. □

5. The Classical Dynamics

It will be convenient to rewrite the Lindbladian as

$$\mathcal{L}A = \frac{i}{h}[P, A] + \frac{\gamma}{2h} \sum_j \left([L_j A, L_j^*] + [L_j, A L_j^*]\right).$$

Our first goal is to motivate the classical Fokker–Planck equation (1.4) from the evolution equation for $\mathcal{L}$.

Observe that for $0 \leq \rho < 1$, and $a \in S_\rho^{L^2}$,

$$L_j A = \operatorname{Op}(\ell_j a + \frac{h}{2i}\{\ell_j, a\} + h^{2-2\rho} e_1), \qquad A L_j^* = \operatorname{Op}(a \bar{\ell}_j + \frac{h}{2i}\{a, \bar{\ell}_j\} + h^{2-2\rho} e_2),$$

with $e_j \in S_\rho^{L^2}$. Hence for $a \in S_\rho^{L^2}$

$$\begin{aligned}\mathcal{L}A = \operatorname{Op}(H_p a) + \frac{\gamma}{2i}\sum_j \operatorname{Op}((2\{\ell_j, \bar\ell_j\}a - \ell_j H_{\bar\ell_j} a + \bar\ell H_\ell a)) \\ + \frac{h\gamma}{4}\sum_j \operatorname{Op}(H_{\bar\ell_j} H_{\ell_j} a + H_{\ell_j} H_{\bar\ell_j} a) + h^{2-3\rho}(1+\gamma)\operatorname{Op}(e),\end{aligned} \tag{5.1}$$

with $e \in S_\rho^{L^2}$. Heuristic arguments in the physics literature – see [HRR23a] and the discussion and references given there – suggest that the natural classical evolution should be given by the equation up to the diffusion term $\sum_j H_{\bar\ell_j} H_{\ell_j} + H_{\ell_j} H_{\bar\ell_j}$ which is a non-positive differential operator acting on the classical observable a (see (1.7) for a striking example). Hence as the generator of the classical flow (a form of Fokker–Planck operator) we take $Q \in \operatorname{Diff}^2(\mathbb{R}^{2n})$ given by

$$Q := H_p + \frac{\gamma}{2i}\sum_j (2\{\ell_j, \bar\ell_j\} - \ell_j H_{\bar\ell_j} + \bar\ell_j H_{\ell_j}) + \frac{h\gamma}{4}\sum_j (H_{\bar\ell_j} H_{\ell_j} + H_{\ell_j} H_{\bar\ell_j}).$$

The key estimate for evolution by Q is given as follows. We need here the additional technical assumption (1.11). To state the next estimate we recall the definition of semiclassical Sobolev norms:

$$\|u\|^2_{H^s_\varepsilon} := \int (1 + |\varepsilon\zeta|^2)^s |\widehat{u}(\zeta)|^2 d\zeta, \quad \widehat{u}(\zeta) := \int u(z) e^{-iz\zeta} dz. \tag{5.2}$$

Proposition 5.1. *Suppose that (1.11) holds, and $\gamma \le h^{-1}$. Let $U(t) : L^2(\mathbb{R}^{2n}) \to L^2(\mathbb{R}^{2n})$ be defined by*

$$(\partial_t - Q)U(t) = 0, \qquad U(0) = \operatorname{Id}. \tag{5.3}$$

Then, for all $s \ge 0$, there is $C > 0$ such that for all $t \ge 0$,

$$\|U(t)\|_{H^s_\varepsilon \to H^s_\varepsilon} \le C e^{M_0 \gamma t} \sqrt{1 + t\gamma\varepsilon}, \qquad \|U(t)\|_{L^2 \to L^2} \le C e^{M_0 \gamma t}. \tag{5.4}$$

where M_0 is given in (4.11), the norms are defined in (5.2), and

$$\varepsilon := \sqrt{\gamma h}. \tag{5.5}$$

If, $\sum_j \{\bar\ell_j, \ell_j\} \equiv 0$, that is there is no friction (1.6), then

$$\|U(t)\|_{H^s_\varepsilon \to H^s_\varepsilon} \le C. \tag{5.6}$$

Remark. The estimates (5.4) and (5.6) do not address the smoothing effect of the evolution by (5.3). Obtaining quantitative estimates seems to require stronger assumptions than (1.3) and we restrict ourselves to that case.

Proof. Recall from (1.4) and (1.5) that Q is given by $H_p + \gamma \sum_j B_j + \mu$ plus a second order divergence form operator and the first two terms are anti-selfadjoint. Hence, for $u \in H^2$,

$$\mathrm{Re}\langle Qu, u\rangle = \frac{\gamma}{2i}\sum_j \langle \{\ell_j, \bar\ell_j\}u, u\rangle - \frac{h\gamma}{4}\sum_j \left(\|H_{\ell_j}u\|_{L^2}^2 + \|H_{\bar\ell_j}u\|_{L^2}^2\right).$$

We start with an estimate on the solution, v, to

$$e^{-tM_0\gamma}(\partial_t - Q)(e^{tM_0\gamma}v(t)) = (\partial_t - Q + M_0\gamma)v(t) = f, \qquad v(0) = v_0. \tag{5.7}$$

We have

$$\begin{aligned}\langle f, v\rangle &= \mathrm{Re}\langle(\partial_t - Q + M_0\gamma)v, v\rangle \\ &= \frac{1}{2}\partial_t\|v\|_{L^2}^2 + \gamma\langle(M_0 - \sum_j \tfrac{1}{2i}\{\ell_j, \bar\ell_j\})v, v\rangle + \frac{h\gamma}{4}\sum_j(\|H_{\ell_j}v\|_{L^2}^2 + \|H_{\bar\ell_j}v\|_{L^2}^2).\end{aligned}$$

Hence,

$$\partial_t\|v\|_{L^2}^2 + \frac{h\gamma}{2}\sum_j\left(\|H_{\ell_j}u\|_{L^2}^2 + \|H_{\bar\ell_j}u\|_{L^2}^2\right) \le 2|\langle f, v\rangle|.$$

For $T > 0$ the ellipticity hypothesis (1.10) then gives

$$\begin{aligned}\|v(T)\|_{L^2}^2 + \gamma hc\int_0^T \|\nabla v\|_{L^2}^2 &\le \|v(T)\|_{L^2}^2 + \frac{h\gamma}{2}\int_0^T\sum_j\left(\|H_{\ell_j}v\|_{L^2}^2 + \|H_{\bar\ell_j}v\|_{L^2}^2\right) \\ &\le 2\int_0^T \|f(t)\|_{L^2}\|v(t)\|_{L^2}dt + \|v_0\|_{L^2}^2.\end{aligned} \tag{5.8}$$

Now let u solve

$$(\partial_t - Q + M_0\gamma)u = 0, \qquad u(0) = u_0.$$

Then, applying (5.8), we obtain

$$\|u(T)\|_{L^2}^2 + c\int_0^T \|\varepsilon\nabla u(t)\|_{L^2}^2 ds \le \|u_0\|_{L^2}^2, \quad \varepsilon = \sqrt{\gamma h}. \tag{5.9}$$

To proceed by induction let us assume that for $k \ge 0$

$$\begin{aligned}\sum_{0\le|\alpha|\le k}\|(\varepsilon\partial)^\alpha u(T)\|_{L^2}^2 &+ \int_0^T \sum_{1\le|\alpha|\le k+1}\|(\varepsilon\partial)^\alpha u(t)\|_{L^2}^2 \\ &\le C\sum_{|\beta|\le k}\|(\varepsilon\partial)^\beta u_0\|_{L^2}^2 + CT\gamma\varepsilon\|u_0\|_{L^2}^2.\end{aligned} \tag{5.10}$$

We set

$$Q_1 := \sum_j i\{\bar\ell_j, \ell_j\}, \quad Q_2 := \frac{1}{2i}\sum_j (-\ell_j H_{\bar\ell_j} + \bar\ell_j H_{\ell_j}),$$
$$Q_3 := \frac{1}{4}\sum_j (H_{\bar\ell_j} H_{\ell_j} + H_{\ell_j} H_{\bar\ell_j}).$$

so that

$$\begin{aligned} (\partial_t - Q + M_0)\partial^\alpha u &= [H_p, \partial^\alpha]u + \gamma[Q_1, \partial^\alpha]u + \gamma[Q_2, \partial^\alpha]u + \gamma h[Q_3, \partial^\alpha]u, \\ \partial^\alpha(0) &= \partial^\alpha u_0. \end{aligned} \tag{5.11}$$

We have the following estimates on the commutators appearing on the right hand side:

$$\begin{aligned} &\|[H_p, \partial^\alpha]u\|_{L^2} \le C \sum_{1\le|\beta|\le|\alpha|} \|\partial^\beta u\|_{L^2}, \qquad \|[Q_1, \partial^\alpha]u\|_{L^2} \le C \sum_{0\le|\beta|\le|\alpha|-1} \|\partial^\beta u\|_{L^2}, \\ &\|[Q_2, \partial^\alpha]u\|_{L^2} \le C \sum_{1\le|\beta|\le|\alpha|} \|\partial^\beta u\|_{L^2}, \qquad \|[Q_3, \partial^\alpha]u\|_{L^2} \le C \sum_{1\le|\beta|\le|\alpha|+1} \|\partial^\beta u\|_{L^2}. \end{aligned} \tag{5.12}$$

It is important here that in the estimates *not* involving Q_1, we have $|\beta| \ge 1$ on the right hand sides. To obtain the estimate on commutators with Q_2, we use assumption (1.11).

Applying (5.8) to (5.11) and using (5.12) we obtain

$$\begin{aligned} &\sum_{|\alpha|=k+1} \|\partial^\alpha u(T)\|_{L^2}^2 + c\gamma h \int_0^T \sum_{|\alpha|=k+1} \|\nabla \partial^\alpha u\|_{L^2}^2 dt \\ &\quad \le C \int_0^T \sum_{|\alpha|=k+1} \Big(\sum_{1\le|\beta|\le k+1} \|\partial^\beta u\|_{L^2} + \gamma\|u\|_{L^2} + \gamma h \sum_{1\le|\beta'|\le k+2} \|\partial^{\beta'} u\|_{L^2}\Big) \|\partial^\alpha u\|_{L^2} dt \\ &\qquad\qquad + \sum_{|\alpha|=k+1} \|\partial^\alpha u_0\|_{L^2}^2 \\ &\quad \le C \int_0^T \sum_{|\alpha|=k+1} \Big(\sum_{1\le|\beta|\le k+1} \|\partial^\beta u\|_{L^2} + \gamma h \sum_{|\beta'|=k+2} \|\partial^{\beta'} u\|_{L^2}\Big) \|\partial^\alpha u\|_{L^2} dt \\ &\qquad\qquad + \sum_{|\alpha|=k+1} \|\partial^\alpha u_0\|_{L^2}^2 + CT\gamma\|u_0\|_{L^2}^2. \end{aligned}$$

Young's inequality ($2ab \le \delta^{-1}a^2 + \delta b^2$) allows us to move the highest order terms from the right hand side to the left hand side and that gives

$$\begin{aligned} &\sum_{|\alpha|=k+1} \|\partial^\alpha u(T)\|_{L^2}^2 + c\gamma h \int_0^T \sum_{|\alpha|=k+1} \|\nabla \partial^\alpha u\|_{L^2}^2 dt \\ &\quad \le C \int_0^T \sum_{1\le|\beta|\le k+1} \|\partial^\beta u\|_{L^2}^2 dt + \sum_{|\alpha|=k+1} \|\partial^\alpha u_0\|_{L^2}^2 + CT\gamma\|u_0\|_{L^2}^2. \end{aligned}$$

We now use the inductive hypothesis (5.10) (with $\varepsilon = \sqrt{\gamma h} \leq 1$) to obtain

$$
\begin{aligned}
&\sum_{|\alpha|=k+1} \|(\varepsilon\partial)^\alpha u(T)\|_{L^2}^2 + c\int_0^T \sum_{|\beta|=k+2} \|(\varepsilon\partial)^\beta u\|_{L^2}^2 dt \\
&\quad \leq C \sum_{1\leq|\beta|\leq k+1} \varepsilon^{k+1-|\beta|} \int_0^T \|(\varepsilon\partial)^\beta u\|_{L^2}^2 dt + \sum_{|\alpha|=k+1} \|(\varepsilon\partial)^\alpha u_0\|_{L^2}^2 \\
&\qquad + CT\gamma\varepsilon^{k+1}\|u_0\|_{L^2}^2. \\
&\quad \leq C\Big(\sum_{1\leq|\beta|\leq k+1} \varepsilon^{k+1-|\beta|} \big(\sum_{|\alpha|\leq|\beta|} \|(\varepsilon\partial)^\alpha u_0\|_{L^2}^2 + CT\gamma(\gamma h)\|u_0\|_{L^2}^2 \big) \\
&\qquad + \sum_{|\alpha|=k+1} \|(\varepsilon\partial)^\alpha u_0\|_{L^2}^2 + CT\gamma\varepsilon^{k+1}\|u_0\|_{L^2}^2 \\
&\quad \leq C \sum_{|\alpha|\leq k+1} \|(\varepsilon\partial)^\alpha u_0\|_{L^2}^2 + CT\gamma\varepsilon\|u_0\|_{L^2}^2.
\end{aligned}
$$

Combined with the inductive hypothesis this shows that (5.10) holds with k replaced by $k+1$.

Returning to (5.7) we see that (5.10) gives (5.4). When $\sum_j \{\bar{\ell}_j, \ell_j\} \equiv 0$ then we can take M_0 and $Q_2 \equiv 0$ in the proof and that gives (5.6) (note that in this case Q_1 vanishes and hence the last term on the right hand side of (5.10) does not appear). ☐

6. Agreement of quantum and classical dynamics

In this section we obtain an accurate approximation to the solution of the Lindblad master equation which is a far reaching strengthening of Theorem 1 in §1.

Theorem 4. *Suppose that $\mathcal{L}$ is given by (1.2), assumptions (1.3), (1.10), and (1.11) hold, $h^{2\rho-1} \leq \gamma \leq h^{-1}$ for some $0 \leq \rho \leq \frac{2}{3}$. There is $C_0 > 0$ such that if $A(t)$ satisfies (in the notation of §2)*

$$\partial_t A(t) = \mathcal{L}A(t), \quad A(0) = \mathrm{Op}(a_0), \quad a_0 \in S_\rho^{L^2},$$

then, for every N there exist $C_N > 0$ and $a(t) \in S_\rho^{L^2}$ such that

$$
\begin{aligned}
\|A(t) - \mathrm{Op}(a(t))\|_{\mathscr{L}_2} &\leq C_N e^{(M_0 + C_0 h)\gamma t}(1+\gamma)^{N+1}(1+\gamma^{\frac{3}{2}}h^{\frac{1}{2}}t)^{\frac{N+2}{2}} t^{N+1} h^{(2-3\rho)(N+1)}, \\
a(t) - U(t)a_0 &\in e^{M_0\gamma t} t h^{(2-3\rho)}(1+\gamma)(1+t\gamma^{\frac{3}{2}}h^{\frac{1}{2}}) S_\rho^{L^2},
\end{aligned}
\tag{6.1}
$$

where $U(t)$ was defined by (5.3).

If $\sum_j \{\ell_j, \bar{\ell}_j\} \equiv 0$, that is there is no friction (1.6), then

$$
\begin{aligned}
\|A(t) - \mathrm{Op}(a(t))\|_{\mathscr{L}_2} &\leq C_N e^{C_0 h^2 \gamma t}(1+\gamma)^{N+1} t^{N+1} h^{(2-3\rho)(N+1)}, \\
a(t) - U(t)a_0 &\in h^{(2-3\rho)} t(1+\gamma) S_\rho^{L^2},
\end{aligned}
\tag{6.2}
$$

Proof of Theorem 2 assuming Theorem 4. Let $a_0 \in S^{L_2}_{1/2}$. Then observe that by Proposition 4.6 and Lemma 4.7, together with the fact that for $a_0 \in S^{L_2}_{1/2}$, $\|\operatorname{Op}(a_0)\|_{\mathscr{L}_2} \leq C$,

$$\|A(t)\|_{\mathscr{L}_2} \leq Ce^{(M_0+C_0h)\gamma t}.$$

Next, using Proposition 5.1

$$\|\operatorname{Op}(U(t)a_0)\|_{\mathscr{L}_2} \leq Ce^{M_0\gamma t}.$$

Therefore, since our estimates are trivially valid when $t(\gamma+\gamma^{-\frac{3}{2}})h^{\frac{1}{2}}$, we may assume without loss of generality that $t(\gamma+\gamma^{-\frac{3}{2}})h^{\frac{1}{2}} \leq 1$.

We now consider two cases: $\gamma = h^{2\rho-1}$ for some $\rho \geq \frac{1}{2}$ and $\rho = \frac{1}{2}$ with $\gamma \geq 1$. Observe that when $\gamma = h^{2\rho-1}$ for some $\rho \geq \frac{1}{2}$, then, using that $\gamma \leq 1$, the estimate (6.1) reads

$$\begin{aligned} \|A(t) - \operatorname{Op}(a(t))\|_{\mathscr{L}_2} &\leq C_N e^{(M_0+C_0h)\gamma t}(th^{\frac{1}{2}}\gamma^{-\frac{3}{2}})^{N+1}(1+t\gamma^{\frac{3}{2}}h^{\frac{1}{2}})^{\frac{N+2}{2}} \\ a(t) - U(t)a_{\lambda_h} &\in e^{(M_0+C_0h)\gamma t}(th^{\frac{1}{2}}\gamma^{-\frac{3}{2}})(1+t\gamma^{\frac{3}{2}}h^{\frac{1}{2}})S^{L^2}_{\rho}. \end{aligned} \tag{6.3}$$

Hence, since $t\gamma^{-\frac{3}{2}}h^{\frac{1}{2}} \leq 1$, the estimate (1.20) follows in this case. On the other hand, when $\gamma \geq 1$ and we set $\rho = \frac{1}{2}$, the estimate (6.1) reads

$$\begin{aligned} \|A(t) - \operatorname{Op}(a(t))\|_{\mathscr{L}_2} &\leq C_N e^{(M_0+C_0h)\gamma t} th^{\frac{1}{2}}\gamma^{N+1}t^{N+1}h^{\frac{1}{2}(N+1)}(1+t\gamma^{\frac{3}{2}}h^{\frac{1}{2}})^{\frac{N+2}{2}}, \\ a(t) - U(t)a_{\lambda_h} &\in e^{(M_0+C_0h)\gamma t} th^{\frac{1}{2}}\gamma(1+t\gamma^{\frac{3}{2}}h^{\frac{1}{2}})S^{L^2}_{\frac{1}{2}}, \end{aligned} \tag{6.4}$$

Taking $N = 0$ and using $t\gamma h^{\frac{1}{2}} \leq 1$, we obtain

$$\|A(t) - \operatorname{Op}(U(t)a_{\lambda_h})\|_{\mathscr{L}_2} \leq Ce^{(M_0+C_0h)\gamma t} th^{\frac{1}{2}}(1+t\gamma^{\frac{3}{2}}h^{\frac{1}{2}}).$$

□

Proof of Theorem 4. Define $a_0(t) := U(t)a_0$, with U given in (5.3). Then, recalling that $\varepsilon = \sqrt{\gamma h/2}$, $h^\rho \leq \varepsilon \leq 1$, (5.4) gives

$$a_0(t) \in e^{M_0\gamma t}(1+t\gamma^{\frac{3}{2}}h^{\frac{1}{2}})^{\frac{1}{2}}S^{L^2}_{\rho}, \quad \text{uniformly in } t \geq 0.$$

Set $A_0(t) := \operatorname{Op}(a_0(t))$. Then, using Lemma 2.2 as in the derivation of (5.1), we obtain

$$\dot{A}_0(t) = \operatorname{Op}(\dot{a}_0(t)) = \operatorname{Op}(Qa_0(t)) = \mathcal{L}A_0(t) + \operatorname{Op}(e_1(t)),$$

where

$$e_1(t) \in h^{2-3\rho}(1+\gamma)e^{M_0\gamma t}(1+t\gamma^{\frac{3}{2}}h^{\frac{1}{2}})^{\frac{1}{2}}S^{L^2}_{\rho}.$$

Suppose, by induction that we have found

$$a_j(t) \in e^{M_0\gamma t}t^j h^{(2-3\rho)j}(1+\gamma)^j(1+t\gamma^{\frac{3}{2}}h^{\frac{1}{2}})^{\frac{j+1}{2}}S^{L^2}_{\rho}, \qquad j = 0,\ldots,N-1$$

such that, with $A_{N-1} := \sum_{j=0}^{N-1}\operatorname{Op}(a_j(t))$, we have

$$\dot{A}_{N-1} = \mathcal{L}A_{N-1}(t) + \operatorname{Op}(e_N(t)),$$

with

$$e_N(t) \in e^{M_0\gamma t} t^{N-1}(1+t\gamma^{\frac{3}{2}}h^{\frac{1}{2}})^{\frac{N}{2}}(1+\gamma)^N S_\rho^{L^2}.$$

Using e_N we define

$$a_N(t) = -\int_0^t U(t-s)e_N(s)ds.$$

Since,

$$\int_0^t s^{N-1}(1+\gamma^{\frac{3}{2}}h^{\frac{1}{2}}s)^{\frac{N}{2}}(1+\gamma^{\frac{3}{2}}h^{\frac{1}{2}}(t-s))^{\frac{1}{2}}ds \leq (1+\gamma^{\frac{3}{2}}h^{\frac{1}{2}}t)^{\frac{N+1}{2}}t^N/(N+1),$$

$$a_N(t) \in e^{M_0\gamma t}t^N h^{(2-3\rho)N}(1+\gamma^{\frac{3}{2}}h^{\frac{1}{2}}t)^{\frac{N+1}{2}} S_\rho^{L^2},$$

and hence, with $A_N(t) = A_{N-1}(t) + \mathrm{Op}(a_N(t))$, we have

$$\begin{aligned}\dot{A}_N(t) &= \mathcal{L}A_{N-1}(t) + \mathrm{Op}(e_N(t)) + \mathrm{Op}(\dot{a}_N(t))\\ &= \mathcal{L}A_{N-1}(t) + \mathrm{Op}(Qa_N(t))\\ &= \mathcal{L}A_N(t) + \mathrm{Op}(e_{N+1}(t)),\end{aligned}$$

with

$$e_{N+1} \in e^{M_0\gamma t}t^N(1+\gamma)^{N+1}h^{(2-3\rho)(N+1)}(1+\gamma^{\frac{3}{2}}h^{\frac{1}{2}}t)^{\frac{N+1}{2}} S_\rho^{L^2}.$$

Note that in the last line we used Lemma 2.2 to obtain the estimates on e_{N+1}. This gives $a = \sum_{j\leq N} a_j$.

We next use Proposition 4.6 and Lemma 4.7 to compare $A(t)$ and $A_N(t)$:

$$\begin{aligned}\left\|A(t) - A_N(t)\right\|_{\mathscr{L}_2} &\leq \int_0^t \left\|e^{(t-s)\mathcal{L}}\,\mathrm{Op}(e_{N+1}(s))\right\|_{\mathscr{L}_2} ds\\ &\leq C_N e^{(M_0+C_0h)\gamma t}(1+\gamma)^{N+1}t^{N+1}h^{(2-3\rho)(N+1)}(1+\gamma^{\frac{3}{2}}h^{\frac{1}{2}}t)^{\frac{N+2}{2}}.\end{aligned}$$

The stronger version under the assumption that $\sum_j\{\ell_j,\bar{\ell}_j\} = 0$ follows from the stronger estimates in (4.13) and (5.6). □

7. Bounds on the Hilbert–Schmidt norm of Lindblad Evolution

In this section we use Theorem 4 to give lower bounds on the Hilbert–Schmidt norm of the Lindblad evolution in the case of Example (1.7). We will consider two special cases: quadratic hamiltonians and confining Hamiltonians.

7.1. **Quadratic Hamiltonians.** We first show that when p is quadratic and the initial condition is Gaussian, it is possible to solve (1.4) exactly. For the purposes of this section, we let B be a real, symmetric, matrix and suppose that

$$p(x,\xi) := \frac{1}{2}\langle B\rho,\rho\rangle, \qquad \rho := \begin{pmatrix} x \\ \xi \end{pmatrix}. \tag{7.1}$$

We also use the notation $\Omega := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ for the standard symplectic form.

Lemma 7.1. *Let p as in (7.1), A_0 be a real, symmteric, positivie definite matrix. $\rho_0 \in \mathbb{R}^{2n}$, and u solve*

$$(\partial_t - H_p - \tfrac{1}{2}\gamma h \Delta_{x,\xi})u = 0, \qquad u(0) = \exp\Big(-\frac{1}{2h}\langle A_0(\rho-\rho_0, \rho-\rho_0)\rangle\Big). \tag{7.2}$$

Then,

$$u(t) = e^{f(t)} \exp\Big(-\frac{1}{2h}\langle A(t)(\rho-\rho_0(t), \rho-\rho_0(t))\rangle\Big),$$

where $A(0) = A_0$, $\rho_0(0) = \rho_0$, $f(0) = 0$, and

$$\begin{aligned} \dot\rho_0(t) &= -\Omega B\rho_0 \\ \dot A(t) = (A+A^t)\Omega B - \tfrac{\gamma}{4}(A+A^t)^2 \qquad &\dot f(t) = -\tfrac{\gamma}{4}\operatorname{tr}(A(t)+A^t(t)). \end{aligned} \tag{7.3}$$

Proof. We compute

$$\begin{aligned} \partial_t u &= u\Big(\dot f - \frac{1}{2h}\langle \dot A(\rho-\rho_0), \rho-\rho_0\rangle + \frac{1}{2h}\langle A\dot\rho_0, \rho-\rho_0\rangle + \frac{1}{2h}\langle A(\rho-\rho_0),\dot\rho_0\rangle\Big), \\ &= u\Big(\dot f - \frac{1}{2h}\langle \dot A(\rho-\rho_0), \rho-\rho_0\rangle + \frac{1}{2h}\langle (A+A^t)\dot\rho_0, \rho-\rho_0\rangle\Big), \\ H_p u &= u\Big(-\frac{1}{2h}\langle A\Omega B\rho, (\rho-\rho_0)\rangle - \frac{1}{2h}\langle A(\rho-\rho_0), \Omega B\rho\rangle\Big) \\ &= u\Big(-\frac{1}{2h}\langle (A+A^t)\Omega B\rho_0, \rho-\rho_0\rangle - \frac{1}{2h}\langle (A+A^t)\Omega B(\rho-\rho_0), \rho-\rho_0\rangle\Big) \\ \gamma h\Delta u &= u\gamma h\Big(\frac{1}{4h^2}\langle (A+A^t)(\rho-\rho_0), (A+A^t)(\rho-\rho_0)\rangle - \frac{1}{2h}2\operatorname{tr}(A+A^t)\Big) \\ &= u\gamma h\Big(\frac{1}{4h^2}\langle (A+A^t)^2(\rho-\rho_0), \rho-\rho_0\rangle - \frac{1}{2h}\operatorname{tr}(A+A^t)\Big). \end{aligned}$$

Then, using that u satisfies (7.2), and equating terms by homogeneity in $\rho-\rho_0$, we obtain (7.3). ☐

Remark. As an easy corollary of Lemma 7.1, we see that if $A_0 = 2I$, $B = 0$, then

$$u(t) = \frac{1}{(1+2\gamma t)^n} e^{-\frac{1}{h(1+2\gamma t)}\langle \rho-\rho_0, \rho-\rho_0\rangle}, \qquad \|u(t)\|_{L^2} = \Big(\frac{\pi h}{2(1+2\gamma t)}\Big)^{\frac{n}{2}}.$$

When $p = 0$, the Lindblad evolution is exactly given by the Fokker–Planck evolution, and thus the solution $A(t)$ to (1.19), satisfies

$$\|A(t)\|_{\mathscr{L}_2} = \Big(\frac{1}{1+2\gamma t}\Big)^{\frac{n}{2}}.$$

7.2. **Confining Hamiltonians.** We next consider the case where the Hamiltonian p is confining. We assume in this subsection that there are $c, m > 0$ and $M \in \mathbb{R}$ such that

$$\begin{aligned} \Delta p \geq 0, \qquad p \geq c|\nabla p|^2, \qquad \text{on } |p| \geq M, \\ |p| \geq c\langle(x,\xi)\rangle^m - 1/c, \qquad (x,\xi) \in \mathbb{R}^{2n}. \end{aligned} \tag{7.4}$$

We show in Proposition 7.4 that under this assumption, for sufficiently dispersed initial data, the Hilbert–Schmidt norm of the Lindblad evolution is bounded from below in the Hilbert–Schmidt norm.

We will use a maximum principle type argument that, in the presence of a confining hamiltonian, the Fokker–Planck evolution remains well confined in L^1 for long times. We start by constructing an effective barrier with which to apply the maximum principle.

Lemma 7.2. *Suppose that (7.4) holds. Then for any $f \in C^\infty(\mathbb{R})$, such that* $\operatorname{supp} f' \subset (M,\infty)$*, $f, f'' \geq 0$, $f'' \in C_c^\infty$, there is $C > 0$ such that, defining*

$$g(t) := \frac{g(0)}{C\gamma h t g(0) + 1}, \qquad v(t,x,\xi) := \exp(-g(t)f(p(x,\xi))), \tag{7.5}$$

we have

$$(\partial_t - H_p - \tfrac{1}{2}\gamma h \Delta) v \geq 0, \quad t \geq 0. \tag{7.6}$$

Proof. We calculate

$$\begin{aligned} v_t - \gamma h \Delta v &= [-g'(t)f(p) + \gamma h((-g^2[f'(p)]^2 + gf''(p))|\nabla p|^2 + gf'(p)\Delta p)]v \\ &\geq [-g'(t)f(p) + \gamma h(-g^2[f'(p)]^2|\nabla p|^2)]v \\ &\geq [-g'(t)f(p) + C\gamma h(-g^2[f'(p)]^2 p)]v. \end{aligned} \tag{7.7}$$

Since $f \geq 0$ and $|f''(p)| \leq C$,

$$|f'(p)| \leq C\sqrt{f(p)}.$$

Hence, using that $f'' \in C_c^\infty$, we have that there is p_0 such that $f'(p) = L$ for $p \geq p_0$ large enough and $f'(p) = 0$ for $p < M$,

$$f(p) \geq \max(c[f'(p)]^2, L(p-p_0) + f(p_0)) \geq c[f'(p)]^2 p.$$

Thus, for $C > 0$ large enough, and g given in (7.5) (so that $-g'(t) - C\gamma h g^2 = 0$) the last inequality in (7.7) gives (7.6). ☐

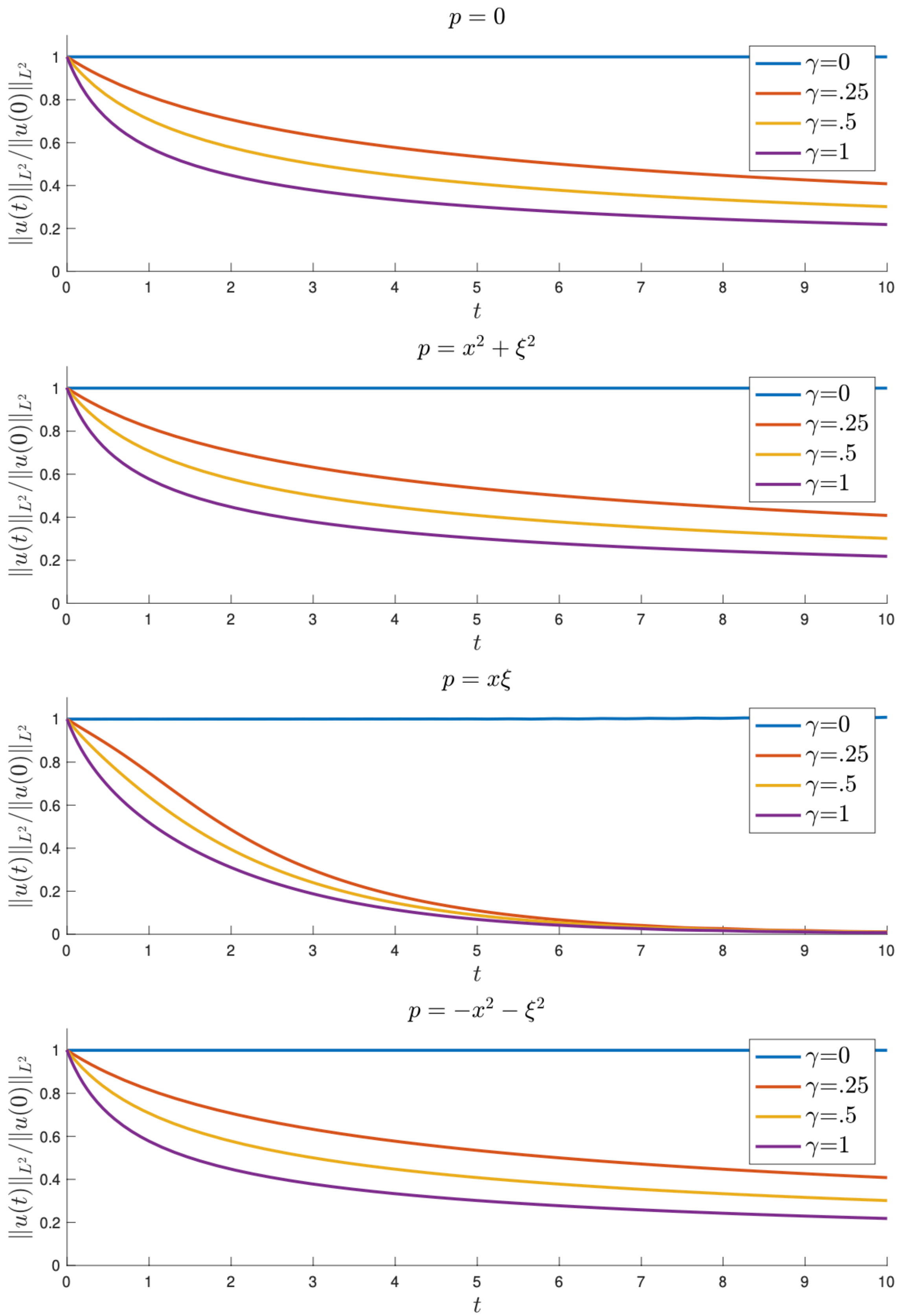

FIGURE 1. The $\|u(t)\|_{L^2}/\|u(0)\|_{L^2}$ for the solution to (7.2) with $A = 2I$ for various choices of B in (7.1).

In the next lemma, we show that, given some apriori assumptions on the Fokker–Planck solution, we are able to confine the majority of its L^1 mass to a bounded set. As a consequence, we obtain that the L^2 norm cannot decay for long times.

Lemma 7.3. *Suppose that (7.4) holds. Then,* $\forall\, R_0, c_1 > 0\, \exists\, R_1 > 0\, \forall\, \delta, N > 0\, \exists\, C_{N,\delta}$ *such that if*

$$0 \le a_0(x,\xi) \le e^{-c_1\langle (x,\xi)\rangle^2/h}, \qquad |(x,\xi)| \ge R_0, \qquad 0 \le a_0 \le 1 \tag{7.8}$$

then for any solution $a(t) \in L^\infty(\{t>0\}\times \mathbb{R}^{2n}_{(x,\xi)})$ *to*

$$(\partial_t - H_p - \tfrac{1}{2}\gamma h \Delta) a = 0, \qquad a(0) = a_0,$$

we have

$$\|a(t)\|_{L^1(\mathbb{R}^d\setminus B(0,R_1))} = C_{N,\delta} h^N \qquad 0 \le t \le h^{-1+\delta}\gamma^{-1}. \tag{7.9}$$

In particular,

$$\|a_0\|_{L^1} - C_{N,\delta} h^N \le C\|a(t)\|_{L^2}, \tag{7.10}$$

Proof. Let $M_1 \ge 0$ such that

$$p \ge M_1 \implies c\langle (x,\xi)\rangle^m/2 \le p \le C_1 \langle (x,\xi)\rangle^2 \text{ and } |(x,\xi)| \ge R_0.$$

Then, set $M_0 = \max(M, M_1)$, let$\psi \in C_c^\infty((M_0, M_0+1); [0,\infty)$ with $\int \psi = 1$ and define $f(x) := \int_0^x \int_0^s \psi(t) dt ds$ so that $f''(x) = \psi(x)$, $\operatorname{supp} f \subset (M_0, \infty)$ with $f, f'' \ge 0$. Let $R_1 \ge 0$ such that $p \ge M_0 + 1$ on $|(x,\xi)| \ge R_1$.

Since $f(p(x,\xi)) = 0$ on $|(x,\xi)| \le R_0$, $f(p) \le Cp \le C\langle (x,\xi)\rangle^2$ there exists $c_0 > 0$ such that

$$\exp(-c_0 f(p(x,\xi))/h) \ge a_0.$$

We now apply Lemma 7.2 with $g(0) = c_0/h$: for v in (7.5)

$$(\partial_t - H_p - \tfrac{1}{2}\gamma h\Delta)(v - a) \ge 0, \qquad t \ge 0.$$

The maximum principle [Co80, Theorem 1] then shows that $0 \le a \le v$ and consequently, using that $f(p) \ge cp \ge c\langle (x,\xi)\rangle^m$ on $\mathbb{R}^{2n} \setminus B(0,R_1)$,

$$\begin{aligned}\|a(t)\|_{L^1(\mathbb{R}^{2n}\setminus B(0,R_1))} &\le \|v(t)\|_{L^1(\mathbb{R}^{2n}\setminus B(0,R_1)} \\ &\le \int_{\mathbb{R}^{2n}\setminus B(0,R_1)} e^{-c_0\langle (x,\xi)\rangle^m/(C\gamma t + h)} dx d\xi \le C e^{-c_0/(C\gamma t h + h)},\end{aligned}$$

from which (7.9) follows.

To obtain (7.10), observe that

$$\partial_t \int a dx d\xi = \int (H_p + \tfrac{1}{2}\gamma h \Delta) a dx d\xi = 0.$$

Hence,

$$\begin{aligned}\|a_0\|_{L^1} = \|a(t)\|_{L^1} &\leq \|a(t)\|_{L^1(B(0,R_1))} + C_{N,\delta}h^N \\ &\leq CR_1^n\|a(t)\|_{L^2(B(0,R_1))} + C_{N,\delta}h^N \\ &\leq CR_1^n\|a(t)\|_{L^2} + C_{N,\delta}h^N.\end{aligned}$$

□

Finally, we show that the Hilbert–Schmidt norm of the Lindblad evolution with a confining Hamiltonian can, in many cases, be effectively controlled from below. We

Proposition 7.4. *Suppose that $\mathcal{L}$ is given by (1.2), assumptions (1.3), and (7.4) hold, that ℓ_j's are as in (1.7). If $h^{2\rho-1} \leq \gamma \leq h^{-1}$ for some $0 \leq \rho \leq \frac{2}{3}$ then*

$\exists\, C_0 > 0\, \forall\, c_1 > 0, R_0 > 0, a_0 \in S_\rho^{L^2}$ *with* $a_0/\|a_0\|_{L^\infty}$ *satisfying* (7.8), $N, \delta > 0\, \exists\, C > 0$

such that if $A(t)$ satisfies (in the notation of §2)

$$\partial_t A(t) = \mathcal{L}A(t), \quad A(0) = \mathrm{Op}(a_0),$$

then, for $0 \leq t \leq h^{-1+\delta}\gamma^{-1}$,

$$(2\pi h)^{\frac{n}{2}} \operatorname{tr} A(0) - Ch^N\|a_0\|_{L^\infty} - Ce^{C_0 h^2\gamma t}(1+\gamma)th^{(2-3\rho)} \leq C\|A(t)\|_{\mathscr{L}_2}.$$

Proof. By Theorem 4,

$$\|A(t) - \mathrm{Op}(a(t))\|_{\mathscr{L}_2} \leq Ce^{C_0 h^2\gamma t}(1+\gamma)th^{(2-3\rho)} \tag{7.11}$$

where $a(t) = U(t)a_0$. In particular, $a(t)$ satisfies

$$(\partial_t - H_p - \tfrac{1}{2}\gamma h\Delta)a = 0, \qquad a(0) = 0,$$

and $a(t) \in S_\rho^{L^2}$. Since $a(t) \in S_\rho^{L_2}$, uniformly in $t > 0$, by the Sobolev embedding, $a(t) \in L^\infty(\{t > 0\} \times \mathbb{R}^{2n}_{(x,\xi)})$ and hence applying Lemma 7.3 then yields

$$\begin{aligned}(2\pi h)^n \operatorname{tr} A(0) - Ch^N\|a_0\|_{L^\infty} &= \|a_0\|_{L^1} - Ch^N\|a_0\|_{L^\infty} \\ &\leq C\|a(t)\|_{L^2} = C(2\pi h)^{\frac{n}{2}}\|\mathrm{Op}(a(t))\|_{\mathscr{L}_2}.\end{aligned} \tag{7.12}$$

The Proposition now follows from combining (7.11) and (7.12). □

Appendix: operators with quadratic symbol growth

We start with the composition formula of operators quantizing symbols in $S_{(k)}$ where that space was defined in (2.3).

Proposition A.1. *Suppose that* $a_j \in S_{(k_j)}$, $j = 1, 2$. *Then* $\mathrm{Op}(a_1)\,\mathrm{Op}(a_2) = \mathrm{Op}(b)$, *where for any* $N \geq \max(k_1, k_2)$,

$$b(x,\xi,h) - \sum_{\ell=0}^{N-1} \frac{1}{\ell!}\left(\frac{h}{2i}\sigma(D_x, D_\xi, D_y, D_\eta)\right)^\ell a_1(x,\xi)a_2(y,\eta)|_{x=y,\xi=\eta} \in h^N S_0, \qquad \text{(A.1)}$$

where σ *is the standard symplectic form on* $\mathbb{R}^{2n}$.

Remark. Note that b in the statement of the proposition is not necessarily in an $S_{(k)}$ class since they are not closed under multiplication.

Proof. We observe that $S_{(k)} \subset S(m_k)$, $m_k(x,\xi) = (1 + |x| + |\xi|)^k$. Hence [Zw12, Theorem 4.18] applies and, writing $z = (x, \xi)$, $w = (y, \eta)$,

$$b(z,h) = \exp(ihA(D))(a_1(x)a_2(w))|_{z=w}, \quad A(D_{z,w}) = -\tfrac{1}{2}\sigma(D_x, D_\xi, D_y, D_\eta).$$

By Taylor's formula,

$$b(z,h) = \sum_{\ell=0}^{N-1} \frac{1}{\ell!}(ihA(D))^\ell(a_1(z)a_2(w))|_{z=w} + R_N(z,h)$$

where

$$R_N(z,h) : \frac{1}{(N-1)!}(1-t)^{N-1}e^{ithA(D)}(ihA(D))^N((a_1(z)a_2(w))|_{z,w}.$$

For $N \geq \max(k_1, k_2)$, $A(D)^N a_1(z)a_2(w) \in S_0(\mathbb{R}^{4n}_{z,w})$ and since $e^{ihtA(D)} : S_0(\mathbb{R}^{4n}) \to S(\mathbb{R}^{4n})$ (with uniform bounds for $0 \leq t \leq 1$ – see [Zw12, Theorem 4.17]) we conclude that $R_N \in h^N S_0(\mathbb{R}^{2n})$ which is (A.1). □

We now present a general spectral result following the proof of a special case in [Hö95] (see the example in [Zw12, §C.2.2]):

Proposition A.2. *Suppose that* $p(x,\xi) \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ *satisfies*

$$|\partial^\alpha p(x,\xi)| \leq C_\alpha(1 + |x| + |\xi|), \quad |\alpha| \geq 1, \qquad \text{(A.2)}$$

and define

$$N_p u = p^{\mathrm{w}}(x,D)u, \quad \mathcal{D}(N_p) := \mathscr{S}(\mathbb{R}^d),$$
$$M_p u = p^{\mathrm{w}}(x,D)u, \quad \mathcal{D}(M_p) := \{u \in L^2(\mathbb{R}^d) : p^{\mathrm{w}}(x,D)u \in L^2(\mathbb{R}^d)\},$$

where in the case of $u \in L^2(\mathbb{R}^d) \subset \mathscr{S}'(\mathbb{R}^d)$ *we consider* $p^{\mathrm{w}}(x,D)u \in \mathscr{S}'(\mathbb{R}^d)$. *Then* M_p *is closed and*

$$M_p = \overline{N_p}, \quad M_p^* = M_{\bar{p}}. \qquad \text{(A.3)}$$

Proof. We recall that $p^{\rm w}(x,D) : \mathscr{S}' \to \mathscr{S}'$ is continuous and hence, if $u_j \to u$ and $p^{\rm w}(x,D)u_j \to v$ in L^2, then $u_j \to u$ in $\mathscr{S}'$. Consequently, $v = p^{\rm w}(x,D)u \in L^2$, $u \in \mathcal{D}(M_p)$ and $M_p u = v$. This shows that M_p is closed.

To show that M_p is the closure of N_p we have to show that for any $u \in \mathcal{D}(M_p)$ there exists a family $u_\varepsilon \in \mathscr{S}$ such that $u_\varepsilon \to u$ and

$$p^{\rm w}(x,D)u_\varepsilon \to p^{\rm w}(x,D)u \quad \text{in } L^2 \text{ as } \varepsilon \to 0. \tag{A.4}$$

To construct u_ε we take $\chi \in C^\infty_{\rm c}(\mathbb{R}^{2d})$ equal to one in $B_{\mathbb{R}^{2n}}(0,1)$, and put

$$u_\varepsilon := \chi_\varepsilon^{\rm w}(x,D)u \in \mathscr{S}, \quad \chi_\varepsilon(x,\xi) := \chi(\varepsilon x, \varepsilon \xi), \quad u_\varepsilon \to u \ \text{ in } L^2.$$

Then $p^{\rm w} u_\varepsilon = \chi_\varepsilon^{\rm w} p^{\rm w} u_\varepsilon + [p^{\rm w}, \chi_\varepsilon^{\rm w}]u_\varepsilon$ and as $\chi_\varepsilon^{\rm w} p^{\rm w} u \to p^{\rm w}$ in L^2, we need to show that

$$[p^{\rm w}(x,D), \chi_\varepsilon^{\rm w}(x,D)]u \to 0 \ \text{ in } L^2 \ , \quad \varepsilon \to 0. \tag{A.5}$$

To see this we note that [Zw12, Theorem 4.18] and the two term Taylor expansion of $e^{iA(D)}$ give

$$[p^{\rm w}(x,D), \chi_\varepsilon(x,D)] = a_\varepsilon^{\rm w}(x,D), \quad a_\varepsilon(x,\xi) = i\{\chi_\varepsilon, p\}(x,\xi) + b_\varepsilon(x,\xi), \tag{A.6}$$

where

$$b_\varepsilon(x,\xi) := \int_0^1 (1-t)\left(e^{itA(D)}(iA(D))^2\left(p(x,\xi)\chi_\varepsilon(y,\eta) - p(y,\eta)\chi_\varepsilon(x,\xi)\right)\right)|_{x=y,\xi=\eta} dt,$$

and $A(D) := \sigma(D_x, D_\xi; D_y, D_\eta)$. In view of (A.2),

$$\{\chi_\varepsilon, p\}(x,\xi) = \varepsilon \sum_{j=1}^n \left(\partial_{x_j} p(x,\xi)(\partial_{\xi_j}\chi)(\varepsilon x, \varepsilon \xi) - \partial_{\xi_j} p(x,\xi)(\partial_{x_j}\chi)(\varepsilon x, \varepsilon\xi)\right)$$

is bounded in $S(1)$, uniformly as $\varepsilon \to 0$.

To obtain estimates on b_ε we observe that, for some $c_{\alpha\beta} \in \mathbb{C}$,

$$\begin{aligned} &(iA(D))^2(p(x,\xi)\chi_\varepsilon(y,\eta) - p(y,\eta)\chi_\varepsilon(x,\xi)) \\ &= \sum_{|\alpha|=|\beta|=2} c_{\alpha\beta}\varepsilon^2(\partial^\alpha p(x,\xi)\partial^\beta\chi(\varepsilon y, \varepsilon\eta) - \partial^\alpha\chi(\varepsilon x, \varepsilon\xi)\partial^\beta p(y,\eta)) \in \varepsilon(S(m) + S(1/m)), \end{aligned}$$

where the order function is given by $m(x,\xi,y,\eta) := \langle x,\xi\rangle\langle y,\eta\rangle^{-1}$. The inclusion follows from the fact $\varepsilon \le C\langle\varepsilon(x,\xi)\rangle^{-1}$ for $(x,\xi) \in \operatorname{supp}\chi$ and from the assumption (A.2). By [Zw12, Theorem 4.17], the operators $e^{ithA(D)} : S(m^{\pm1}) \to S(m^{\pm1})$ are bounded uniformly in t. Since $m|_{x=y,\xi=\eta} = 1$, $b_\varepsilon \in \varepsilon S(1)$, and [Zw12, Theorem 4.23] gives, uniformly as $\varepsilon \to 0$,

$$\|b_\varepsilon^{\rm w}(x,D)\|_{L^2\to L^2} \le C\varepsilon. \tag{A.7}$$

We now choose $\psi \in C^\infty_{\rm c}(\mathbb{R}^n)$ supported in $B_{\mathbb{R}^{2n}}(0,1)$, equal to one near 0, and put $\psi_\varepsilon(x,\xi) = \psi(\varepsilon x, \varepsilon\xi)$. Then $\{\chi_\varepsilon, p\}(x,\xi)\psi_\varepsilon(x,\xi) \equiv 0$, and [Zw12, Theorems 4.18 and 4.23] imply

$$\|\{\chi_\varepsilon, p\}^{\rm w}(x,D)\psi_\varepsilon^{\rm w}(x,D)\|_{L^2\to L^2} \le C\varepsilon.$$

This and (A.7) give

$$\begin{aligned}[p^{\mathrm{w}}(x,D),\chi_\varepsilon^{\mathrm{w}}(x,D)] &= \{\chi_\varepsilon,p\}^{\mathrm{w}}(x,D)(1-\psi_\varepsilon^{\mathrm{w}}(x,D)) + \{\chi_\varepsilon,p\}^{\mathrm{w}}(x,D)\psi_\varepsilon^{\mathrm{w}}(x,D) + b_\varepsilon^{\mathrm{w}}(x,D)\\ &= \{\chi_\varepsilon,p\}^{\mathrm{w}}(x,D)(1-\psi_\epsilon^{\mathrm{w}}(x,D)) + \mathcal{O}(\varepsilon)_{L^2\to L^2}.\end{aligned}$$

Since $\psi_\varepsilon^{\mathrm{w}}(x,D)u \to u$ in L^2 and $\{\chi_\varepsilon,p\} \in S(1)$ (hence by [Zw12, Theorem 4.23] $\|\{\chi_\varepsilon,p\}^{\mathrm{w}}(x,D)\|_{L^2\to L^2}$ is uniformly bounded), this and (A.7) give (A.5).

It remains to show the last assertion in (A.3). For that we recall that $v \in \mathcal{D}(M_p^*)$ if and only if there exists $C = C(v)$ such that for all $u \in \mathcal{D}(M_p)$

$$\langle M_p u, v\rangle \leq C\|u\|_{L^2}. \tag{A.8}$$

For $u \in \mathscr{S} \subset \mathcal{D}(M_p)$ we have $\langle M_p u, v\rangle = \langle u, \bar{p}^{\mathrm{w}}(x,D)v\rangle$, where $\bar{p}^{\mathrm{w}}(x,D)v \in \mathscr{S}'$ and (A.8) implies that $\bar{p}^{\mathrm{w}}(x,D)v \in L^2$. Hence $M_p^* \subset M_{\bar{p}}$. Since M_p^* is closed, $N_{\bar{p}} \subset N_p^* = \bar{N}_p^* = M_p^*$. It follows that $M_{\bar{p}} = \bar{N}_{\bar{p}} \subset M_p^*$ and that $M_p^* = M_{\bar{p}}$. ☐

References

[ChPa17] D. Chruściński and S. Pascazio, *A brief history of the GKLS equation,* Open Systems & Information Dynamics **24**(2017), 1740001.

[Co80] C.Cosner, *Asymptotic behavior of solutions of second order parabolic partial differential equations with unbounded coefficients,* Journal of Differential Equations, **35**(1980), 407–428.

[Da77] E.B. Davies, *Quantum dynamical semigroups and the neutron diffusion equation,* Reports on Mathematical Physics **11**(1977), 169–188.

[DiSj99] M. Dimassi and J. Sjöstrand, *Spectral Asymptotics in the Semi-Classical Limit*, Cambridge U Press, 1999.

[GoBr99] J. Gong and P. Brumer, *Decoherence and correspondence in conservative chaotic dynamics,* Phys. Rev. E **60**(1999), 1643–1647.

[GKS76] V. Gorini, A. Kossakowski and E.C.G. Sudarshan, *Completely positive dynamical semigroups of N-level systems,* J. Math. Phys. **17**(1976), 821–825.

[HRR23a] F. Hernández, D. Ranard, C.J. Riedel, *The* $\hbar \to 0$ *limit of open quantum systems with general Lindbladians: vanishing noise ensures classicality beyond the Ehrenfest time.* arXiv:2307.05326

[HRR23b] F. Hernández, D. Ranard, C.J. Riedel, *Decoherence ensures classicality beyond the Ehrenfest time as* $\hbar \to 0$*,* arXiv:2306.13717

[Hö95] L. Hörmander, *Symplectic classification of quadratic forms and general Mehler formulas,* Math. Z. **219**(1995), 413–449.

[Li76] G. Lindblad, *On the generators of quantum dynamical semigroups,* Commun. Math. Phys. **48**(1976), 119–130.

[OlBo23] I. Oltman and Y. Borns-Weil, *Particle trajectories for quantum maps,* Ann. H. Poincaré (2023) published online, arXiv:2210.03224.

[Sm20] H.F. Smith, *Parametrix for a semiclassical subelliptic operator,* Anal.&PDE **7**(2020), 2375–2398.

[UnZu89] W.G. Unruh and W.H. Zurek, *Reduction of a wave packet in quantum orangeian motion,* Phys. Rev. D **40**(1989), 1071–1094.

[Zw12] M. Zworski, *Semiclassical Analysis*, Graduate Studies in Mathematics **138**, AMS, 2012.

Email address: j.galkowski@ucl.ac.uk

Department of Mathematics, University College London, WC1H 0AY, UK

Email address: zworski@math.berkeley.edu

Department of Mathematics, University of California, Berkeley, CA 94720