OUTGOING SOLUTIONS VIA GEVREY-2 PROPERTIES

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ABSTRACT. Gajic–Warnick [GaWa19a] have recently proposed a definition of scattering resonances based on Gevrey-2 regularity at infinity and introduced a new class of potentials for which resonances can be defined. We show that standard methods based on complex scaling apply to a larger class of potentials and provide a definition of resonances in wider angles.

1. INTRODUCTION

One of the main issues in theoretical and numerical scattering theory is distinguishing outgoing parts of solutions modeling scattered waves. The simplest example is given by

$$P_V := D_x^2 + V(x), \quad x \in [1, \infty), \quad D_x := -i\partial_x, \tag{1.1}$$

where V is a *compactly* supported potential and we impose *Dirichlet* boundary condition at $x = 1$. Clearly,

$$(P_V - \lambda^2)u = f \in C_c^\infty([1, \infty)), \ u(1) = 0 \implies u(x) = ae^{i\lambda x} + be^{-i\lambda x}, \ x \gg 1. \tag{1.2}$$

The first term is called *outgoing* and the second term *incoming* (or vice versa, depending on your convention) – see [DyZw19, §2.1]. The method of *complex scaling* distinguishes incoming and outgoing solutions by deforming x into a curve in $\mathbb{C}$: the $e^{i\lambda x}$ term becomes exponentially decaying and the $e^{-i\lambda x}$ term, exponentially growing and hence outgoing solutions are characterized as L^2 solutions – see [DyZw19, §2.7] for a simple introduction. This method, introduced in the 70's by Aguilar–Combes, Balslev–Combes and Simon and then widely used in computational chemistry, reappeared in the 90's as the method of *perfectly matched layers* in numerical analysis – see [DyZw19, §4.7] for pointers to the literature. It applies to non-compactly supported potentials as long as analyticity and decay conditions on $V(x)$ in $|\operatorname{Im} x| \leq C|\operatorname{Re} x|$ are imposed for $|x| \gg 1$. In the simplest setting of (1.2) *scattering resonances* are $\lambda \in \mathbb{C}$ for which there exists a solution with $f = 0$ and $b = 0$.

Gajic and Warnick [GaWa19a] have recently proposed an intriguing alternative for distinguishing incoming and outgoing solutions using Gevrey-2 regularity (see (1.4)) at infinity. In the setting of compactly supported potentials their approach can be described as follows: put $Q_V(\lambda) := x^2 e^{-i\lambda x}(P_V - \lambda^2)e^{i\lambda x} = x^2(D_x^2 + 2\lambda D_x + V)$ and

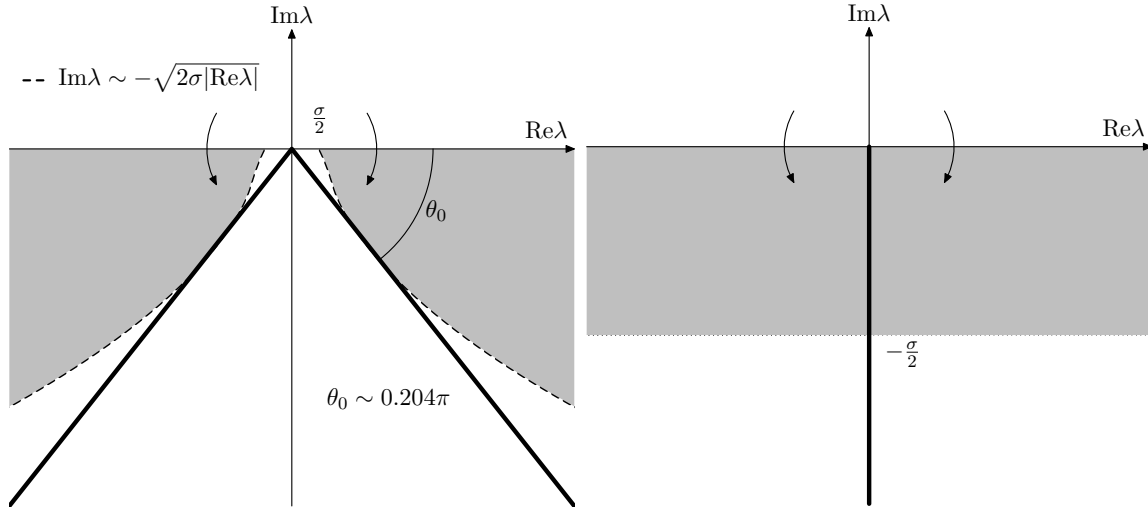FIGURE 1. We fix a $\sigma > 0$ and on the left show the region, Ω_σ, in which resonances are defined in [GaWa19a] (note the difference of conventions: here we followed the standard convention of scattering theory [DyZw19]) and on the right the illustration of the region, $\mathcal{M}_\sigma$, in this paper. The potential satisfies $V(x) = x^{-\gamma}W(1/x)$, $W \in G^{2,\sigma}([0,1])$, with $\gamma = 2$ in the case of [GaWa19a] and $\gamma = 1$ in this paper. For a fixed σ the region on the left is larger for $|\operatorname{Re}\lambda| \gg 1$ but as $\sigma \to \infty$ the conic regions are dramatically different. The asymptotic boundary of Ω_σ is heuristically explained using (1.3) and the fact that $e^{-2i\lambda/x} \in \widehat{G}^{2,|\lambda|-|\operatorname{Re}\lambda|}$ (see the appendix) so that for a fixed σ, we obtain the curve $|\lambda| - |\operatorname{Re}\lambda| = \sigma$, that is $2\sigma|\operatorname{Re}\lambda| = |\operatorname{Im}\lambda|^2 - \sigma^2$.

change variables to $y = 1/x$. Then (1.2) becomes

$$Q_V(\lambda)u = (D_y y^2 D_y - 2\lambda D_y + y^{-2}V(1/y))u = f \in C_c^\infty((0,1]), \quad u(1) = 0,$$
$$u(y) = a + be^{-2i\lambda/y}, \quad 0 < y \ll 1. \tag{1.3}$$

For $\operatorname{Im}\lambda \geq 0$, the outgoing solutions are simply solutions which are smooth up to $y = 0$. For $\operatorname{Im}\lambda < 0$, $y \mapsto e^{-2i\lambda/y} \in C^\infty([0,1])$, however, it is *not* in the Gevrey-2 class $G^{2,|\lambda|-|\operatorname{Re}\lambda|+\epsilon}([0,1])$, for any $\epsilon > 0$ – see (1.4) and the appendix. That shows that $u \in G^{2,\sigma}([0,1])$, $\sigma \gg 1$, guarantees that u is outgoing. In particular, for $f \equiv 0$, it gives a criterion for λ being a scattering resonance. Through a delicate Gevrey class analysis this idea was used in [GaWa19a] to extend definition of resonances to potentials satisfying $y^{-2}V(1/y) \in G^{2,\sigma}([0,1])$, $\sigma \gg 1$. In that case, they show that $Q_V(\lambda)^{-1}$ is well defined as a meromorphic operator with finite rank poles on certain Gevrey type spaces. The region in which resonances are defined by their method is shown on the left of Figure 1.

In this note we observe that the potentials of the form considered in [GaWa19a] (in fact, potentials with slower decay at infinity) can be decomposed into a sum of a potential analytic in a large sector and an exponentially decaying potential. We then use complex scaling method and consider the exponentially decaying potential as a perturbation. Using essentially well-know results this easily gives meromorphic continuation to strips in $\mathbb{C}\setminus i(-\infty, 0]$). The methods used here do not, however, recover the parabolic high energy region of [GaWa19a] – see Figure 1 and Remark 4 below.

To formulate the result we define Gevrey classes, in particular the Beurling–Gevrey class, $\gamma^a([0,1])$:

$$G^{a,\sigma}([0,1]) := \{u \in C^\infty([0,1]) \,:\, \exists C, \ \sup_{x \in [0,1]} |D_x^n u(x)| \le C\sigma^{-n}(n!)^a\},$$

$$G^a := \bigcup_{\sigma > 0} G^{a,\sigma}, \quad \gamma^a := \bigcap_{\sigma > 0} G^{a,\sigma}, \quad \widehat{G}^{a,\sigma} := G^{a,\sigma} \setminus \bigcup_{\sigma' > \sigma} G^{a,\sigma'}, \quad a \ge 1, \ \sigma > 0, \tag{1.4}$$

see [HöI, §1.3] for some fundamental results and references. With this in place we state

Theorem. *Suppose that $V(x) = x^{-1}W(1/x)$ with $W \in G^{2,\sigma}([0,1])$. Then the scattering resolvent,*

$$R_V(\lambda) := (P_V - \lambda^2)^{-1} : L^2_{\text{comp}}([1,\infty)) \to L^2_{\text{loc}}([1,\infty)), \quad \operatorname{Im}\lambda \gg 1,$$

continues to a meromorphic family of operators with poles of finite rank on

$$\mathcal{M}_\sigma := \{\lambda \in \mathbb{C} \,:\, \operatorname{Im}\lambda > -\tfrac{\sigma}{2}\} \setminus i(-\infty, 0].$$

In particular, when $W \in \gamma^2([0,1])$, the resolvent continues to $\mathbb{C} \setminus i(-\infty, 0]$.

Remarks. 1. The motivation in [GaWa19a] came from the study of quasinormal modes of black holes [GaWa19b]. In the case considered there, unlike in de Sitter space situations (positive cosmological constant – see Bony–Häfner [BoHa08], Dyatlov [Dy12] and Vasy [Va13] for detailed studies and references), the issue of seeing the effects of quasinormal modes on wave evolution is complicated by the behaviour at 0. However, as shown by Dyatlov [Dy15], some results can be obtained by considering frequency cut-offs. For fine decay results related to the behaviour of the resolvent at zero see Hintz [Hi20] and references given there.

2. The necessity of a cut at 0 or a continuation to (a part of) the Riemann surface of $\lambda \mapsto \log\lambda$ is a genuine phenomenon as can be seen by considering $V(x) = \alpha/x^2$. For instance, for $\alpha = n^2 - \frac{1}{4}$, P_V corresponds to the radial component of the Dirichlet Laplacian on $\mathbb{R}^2\setminus B(0,1)$ acting on the nth Fourier component – see Christiansen [Ch17] for recent advances in scattering by obstacles in even dimensions and references. The proof here would allow moving to an angle π on the Riemann surface (rather than $\pi/2$ as stated) at the expense of shrinking the strip (which gives the whole Riemann surface $-\pi < \arg\lambda < 2\pi$). We opted for a simpler presentation which avoids the introduction of Riemann surfaces and of the dependence of the size of the strip on the angle.

3. Defining resonances for potentials which are not analytic in conic neighbourhoods of infinity, or rather seeing their impact on "observable" phenomena, has been an object of recent studies. Martinez–Ramond–Sjöstrand [MRS09] proved that in some non-analytic situations resonances are invariantly defined up to any power of their imaginary part. Bony–Michel–Ramond [BMR19] showed that many phenomena associated to resonances can be seen for potentials decaying at infinity: the "interaction region" (where, say, the trapped set lies in the case of AdS black holes) is responsible for "observable" effects.

4. We expect that the improved angle estimate of this note and the better large $\operatorname{Re} \lambda$ region can be combined. One approach would require the methods of Helffer–Sjöstrand [HeSj86] recently revisited by Guedes Bonthonneau–Jézéquel [GuJe20] and the authors [GaZw19]. However, without stronger motivation, we have restricted ourselves to more straightforward methods.

The paper is organized as follows. In §2 we revisit well known asymptotic results of Borel, Ritt and Watson to decompose potentials in the theorem. In §3.1 we show that having a meromorphic extension guarantees exponential growth estimates for the resolvent. That meromorphic continuation is obtained using the method of complex scaling for the analytic part of the potential. Finally, in §3.2, we use this exponential estimates to show that adding an exponentially decay potential given meromorphy in a strip in $\mathbb{C} \setminus i(-\infty, 0]$. Except for invoking the now standard method of complex scaling, the paper is essentially self contained.

2. Gevrey-2 property at infinity

In this section, we decompose a $G^{2,\sigma}$ potential into one which is analytic in a large sector and one which decays exponentially towards infinity.

Proposition 1. *For* $V \in C^\infty([1, \infty))$ *satisfying* $V(x) = W(1/x)$, $W \in G^{2,\sigma}([0,1])$ *and any* $\rho < \sigma$ *and* $\epsilon > 0$ *we have*

$$V(x) = V_1(x) + V_2(x), \quad V_1 \in C^\infty([0, \infty)), \quad \exists C \quad |V_1(x)| \leq Ce^{-\rho x},$$

$$V_2(z) \text{ is holomorphic in } \mathcal{C}_\epsilon \text{ and} \tag{2.1}$$

$$V_2(z) \sim \sum_{\ell=0}^\infty \frac{W^{(\ell)}(0)}{\ell!} z^{-\ell}, \ z \to \infty, \ z \in \mathcal{C}_\epsilon,$$

where $\mathcal{C}_\epsilon := \{z : |\arg z| < \pi/2 - \epsilon\}$.

Proof. We observe that the decomposition (2.1) follows from the following decomposition of W: for all $\rho < \sigma$,

$$W(x) = W_1(x) + W_2(x), \quad W_1 \in G^{2,\rho}([0,1]), \quad \rho < \sigma \ \ \forall \ell \in \mathbb{N} \ \ W_1^{(\ell)}(0) = 0,$$

$$W_2(z) \text{ is holomorphic in } \mathcal{C}_\epsilon \text{ and } \ W_2(z) \sim \sum_{\ell=0}^{\infty} \frac{W^{(\ell)}(0)}{\ell!} z^\ell, \ z \to 0, \ z \in \mathcal{C}_\epsilon. \tag{2.2}$$

In fact, if $V_2(z) := W_2(1/z)$ then the expansion of V_2 follows. On the other hand, Taylor expansion at 0 and the Gevrey property imply that

$$|W_1(x)| \leq \min_{\ell \in \mathbb{N}} x^\ell \sup_{[0,x]} |W_1^{(\ell)}|/\ell! \leq C \min_{\ell \in \mathbb{N}} (x/\rho)^\ell \ell! \leq C' e^{-\rho/x},$$

which gives the required estimate for $V_1(x) = W_1(1/x)$.

Hence we need to establish (2.2) and for that we use the classical approach of Watson [Wa11] (see [Ba94, Chapter 2] for a modern presentation). To implement it, we put

$$f_n := \frac{W^{(n)}(0)}{n!}, \quad |f_n| \leq K\sigma^{-n} n!, \quad g(\zeta) := \sum_{n=0}^{\infty} \frac{f_n \zeta^n}{n!}. \tag{2.3}$$

The function g is then holomorphic in $|\zeta| < \sigma$ and for $0 < \rho_1 < \sigma$ we define

$$W_2(z) := z^{-1} \int_0^{\rho_1} g(\zeta) e^{-\zeta/z} d\zeta, \tag{2.4}$$

which is holomorphic for $z \in \Lambda$, the Riemann surface of $\log z$. Then for $\cos(\arg z) \geq \epsilon$,

$$\left| W_2(z) - \sum_{n=0}^{N-1} f_n z^n \right| \leq C_{\rho_1} K N \epsilon^{-1} |z|^N N! (\epsilon \rho_1)^{-N}.$$

Indeed,

$$\left| W_2(z) - \sum_{n=0}^{N-1} f_n z^n \right| \leq \left| z^{-1} \int_0^{\rho_1} g(\zeta) e^{-\zeta/z} d\zeta - \sum_{n=0}^{N-1} \int_0^\infty z^{-1} \frac{f_n}{n!} \zeta^n e^{-\zeta/z} d\zeta \right|$$

$$\leq \frac{\sigma}{\sigma - \rho_1} \int_0^{\rho_1} |z|^{-1} K \zeta^N \sigma^{-N} e^{-\zeta\epsilon/|z|} d\zeta + \int_{\rho_1}^\infty |z|^{-1} K \sum_{n=0}^{N-1} \sigma^{-n} \zeta^n e^{-\zeta\epsilon/|z|} d\zeta$$

$$\leq \frac{\sigma}{\sigma - \rho_1} K \epsilon^{-1} |z|^N N! (\epsilon\sigma)^{-N} + \int_{\rho_1}^\infty K N |z|^{-1} (1 + (\sigma^{-1}\zeta)^N) e^{-\zeta\epsilon/|z|} d\zeta$$

$$\leq \frac{\sigma}{\sigma - \rho_1} K \epsilon^{-1} |z|^N N! (\epsilon\sigma)^{-N} + K \epsilon^{-1} N e^{-\epsilon\rho_1/|z|} + K N \epsilon^{-1} |z|^N N! (\epsilon\sigma)^{-N}$$

$$\leq C_{\rho_1} \epsilon^{-1} |z|^N N N! (\epsilon\rho_1)^{-N}$$

(See [Ba94, Exercise 3, p.16].) This shows the existence of the expansion in (2.1).

Since this expansion holds with all derivatives, on the real axis (we can then take ϵ arbitrarily close to 1) we have $|W_2^{(\ell)}(x)| \leq C_\rho (\ell!)^2 \rho^{-\ell}$ for any $\rho < \rho_1$. Since the expansion also shows that $W_2^{(\ell)}(0) = W^{(\ell)}(0)$, (2.2), and hence (2.1), follow. $\square$

3. Meromorphic continuation

Our main theorem is a consequence of the following proposition together with Proposition 1.

Proposition 2. *Suppose that $V \in L^\infty([1,\infty); \mathbb{C})$ can be decomposed as follows*

$$V(x) = V_1(x) + V_2(x), \quad |V_1(x)| \leq Ce^{-\gamma x},$$

$$V_2(z) \text{ is holomorphic for } |\arg z| < \alpha < \pi \text{ and } V_2(z) \underset{z \to \infty}{\longrightarrow} 0 \text{ there.} \tag{3.1}$$

Then $R(\lambda)$ has a meromorphic continuation to $\{\lambda : -\alpha < \arg \lambda < \pi + \alpha, \ 2\operatorname{Im}\lambda > -\gamma\}$.

Throughout this section, for $V \in L^\infty(\mathbb{R}; \mathbb{C})$, we let $P_V := D_x^2 + V$ be defined by the quadratic form

$$Q(u,v) = \langle D_x u, D_x v \rangle_{L^2(1,\infty)} + \langle Vu, v \rangle_{L^2(1,\infty)}$$

with form domain $H_0^1(1,\infty)$. Then, for $\operatorname{Im}\lambda \gg 1$,

$$R_V(\lambda) := (P_V - \lambda^2)^{-1} : L^2 \to L^2$$

the *resolvent of P_V* is defined by spectral theory.

In order to prove Proposition 2 we first observe that the now standard method of complex scaling (see Sjöstrand [Sj96, §5] for a presentation from a PDE point of view and references) shows that

$$R_{V_2}(\lambda) : L^2_{\mathrm{comp}}([1,\infty)) \to L^2_{\mathrm{loc}}([1,\infty))$$

continues from $\operatorname{Im}\lambda \gg 1$ to $-\alpha < \arg \lambda < \pi + \alpha$.

3.1. **Exponential estimates on R_{V_2}.** The goal of this section is to prove the following proposition. A general case would follow from Helffer–Sjöstrand theory [HeSj86] but we opt for a quick one dimensional argument.

Proposition 3. *Fix $\gamma > 0$. Then, the meromorphic family $\lambda \mapsto R_{V_2}(\lambda) : L^2_{\mathrm{comp}} \to L^2_{\mathrm{loc}}$ extends to a meromorphic family*

$$\lambda \mapsto R_{V_2}(\lambda) : e^{-\gamma\langle x\rangle} L^2 \to e^{\gamma\langle x\rangle} H^2, \tag{3.2}$$

for

$$\{\lambda \in \mathbb{C} : \operatorname{Im}\lambda > -\gamma\} \cap \mathcal{W}_\alpha, \quad \mathcal{W}_\alpha := \{-\alpha < \arg \lambda < \pi + \alpha\}.$$

We first need the following apriori estimates.

Lemma 4. *Let $V \in L^\infty([1,\infty);\mathbb{C})$ such that $\lim_{x\to\infty} V(x) = 0$ and fix $\gamma > 0$ Then, for all λ with $\gamma > |\operatorname{Im}\lambda|$, $\lambda \neq 0$, there is $C(\lambda) > 0$ such that for all $u \in H^2_{\mathrm{loc}}$,*

$$\|e^{-\gamma\langle x\rangle}u\|_{H^2} \leq C(\lambda)\left(\|e^{\gamma\langle x\rangle}(P_V - \lambda^2)u\|_{L^2} + \|u\|_{L^2(1,4)}\right). \tag{3.3}$$

Proof. First, suppose that $u \in C_c^\infty$ and $(P_V - \lambda^2)u = f$. Then, with

$$w_\lambda(x) := \begin{pmatrix} u(x) \\ \lambda^{-1}D_x u(x) \end{pmatrix},$$

we have

$$D_x w_\lambda = A(x)w_\lambda + \begin{pmatrix} 0 \\ \lambda^{-1}f \end{pmatrix}, \qquad A(x) := \begin{pmatrix} 0 & \lambda \\ \lambda - \lambda^{-1}V & 0 \end{pmatrix}.$$

Now, define $U_s(x) : \mathbb{C}^2 \to \mathbb{C}^2$ such that $D_x U_s(x) = A(x)U_s(x)$, $U_s(s) = \mathrm{Id}$. Then,

$$\partial_x \|U_s(x)u_0\|^2 = \langle i(A - A^*)U_s(x)u_0, U_s(x)u_0\rangle.$$

The eigenvalues of $i(A - A^*)(x)$ are given by

$$\pm\sigma(x) = \pm|2\operatorname{Im}\lambda + i\lambda^{-1}V(x)|$$

Note that, since $\lim_{x\to\infty} V = 0$, and $|\lambda| > 0$, there is $R > 1$ such that for $x > R$ $\sigma(x) < (\gamma + |\operatorname{Im}\lambda|)$. Therefore, for $x > s$,

$$\|U_s(x)u_0\| \leq e^{\frac{1}{2}\int_s^x \sigma(t)dt}\|u_0\| \leq Ce^{\frac{1}{2}(\gamma + |\operatorname{Im}\lambda|)(x-s)}\|u_0\|.$$

Using a similar argument for $x < s$, we have

$$\|U_s(x)u_0\| \leq Ce^{\frac{1}{2}(\gamma + |\operatorname{Im}\lambda|)|x-s|}\|u_0\|$$

Next, observe that

$$w_\lambda(x) = U_s(x)w_\lambda(s) + i\int_s^x U_t(x)\begin{pmatrix} 0 \\ \lambda^{-1}f(t) \end{pmatrix} dt$$

Therefore, for all $s \in \mathbb{R}$,

$$e^{-\gamma|x|}\|w_\lambda(x)\| \leq Ce^{\frac{1}{2}(\gamma + |\operatorname{Im}\lambda|)|x-s| - \gamma|x|}\|w_\lambda(s)\| + C\int_s^x e^{\frac{1}{2}(\gamma + |\operatorname{Im}\lambda|)(|x-t|) - \gamma|x|}|f(t)|dt$$

$$\leq C(e^{\frac{1}{2}(\gamma + |\operatorname{Im}\lambda|)|x-s| - \gamma|x|}\|w_\lambda(s)\| + e^{-\frac{1}{2}(\gamma - |\operatorname{Im}\lambda|)|x|}\|e^{\gamma|\cdot|}f(\cdot)\|_{L^2(s,x)})$$

Therefore averaging in $s \in (2,3)$, we have

$$e^{-2\gamma|x|}\|w_\lambda(x)\|^2 \leq C(e^{-(\gamma - |\operatorname{Im}\lambda|)|x|}\|w_\lambda\|^2_{L^2(2,3)} + e^{-(\gamma - |\operatorname{Im}\lambda|)|x|}\|e^{\gamma|\cdot|}f(\cdot)\|^2_{L^2(1\leq\cdot\leq\max(3,|x|))}).$$

Finally integrating in x, we obtain for $R > 3$,

$$\|e^{-\gamma|x|}w_\lambda\|_{L^2(1,R)} \leq C(\|e^{\gamma|x|}f\|_{L^2(1,R)} + \|w_\lambda\|_{L^2(2,3)}).$$

In particular, for $u \in C_c^\infty([1,\infty))$,

$$\|e^{-\gamma\langle x\rangle}u\|_{H^1(1,R)} \leq C(\|e^{\gamma\langle x\rangle}(P_V - \lambda^2)u\|_{L^2(1,R)} + \|u\|_{H^1(2,3)}. \tag{3.4}$$

Now, let $u \in H^2_{\text{loc}}$ with $e^{\gamma \langle x \rangle}(P_V - \lambda^2)u \in L^2$ and $\chi \in C^\infty_c(\mathbb{R})$ with $\chi \equiv 1$ on $[-1, 1]$. Then, let $u_{n,k} \in C^\infty_c([1, \infty))$ with

$$u_{n,k} \xrightarrow[k \to \infty]{H^2} \chi(n^{-1}x)u$$

Fix $R > 0$. Then, letting $n > R$ and applying (3.4) with $u = u_{n,k}$, we obtain

$$\|e^{-\gamma \langle x \rangle} u_{n,k}\|_{H^1} \leq C(\|e^{\gamma \langle x \rangle}(P_V - \lambda^2)u_{n,k}\|_{L^2(1,R)} + \|u_{n,k}\|_{H^1(2,3)}).$$

Sending $k \to \infty$ and using that $\chi(n^{-1}x) \equiv 1$ on $[1, R)$, we have

$$\|e^{-\gamma \langle x \rangle} u\|_{H^1} \leq C(\|e^{\gamma \langle x \rangle}(P_V - \lambda^2)u\|_{L^2(1,R)} + \|u\|_{H^1(2,3)}).$$

Sending $R \to \infty$ we obtain

$$\|e^{-\gamma \langle x \rangle} u\|_{H^1} \leq C(\|e^{\gamma \langle x \rangle}(P_V - \lambda^2)u\|_{L^2} + \|u\|_{H^1(2,3)}).$$

Finally, to complete the proof of (3.3) observe that

$$\|u\|_{H^2(2,3)} \leq C(\|(P_V - \lambda^2)u\|_{L^2(1,4)} + \|u\|_{L^2(1,4)})$$

and similarly

$$\|e^{-\gamma \langle x \rangle} u\|_{H^2} \leq C\|e^{-\gamma \langle x \rangle} u\|_{H^1} + C\|e^{-\gamma \langle x \rangle} \partial_x^2 u\|_{L^2}$$
$$\leq C\|e^{-\gamma \langle x \rangle} u\|_{H^1} + C\|e^{-\gamma \langle x \rangle}(P_V - \lambda^2)u\|_{L^2}.$$

This completes the proof of (3.3). $\hfill\square$

Proof of Proposition 3. Applying Lemma 4 with $f \in L^2_{\text{comp}}$ and $u = R_{V_2}(\lambda)f$ shows that for $\text{Im}\,\lambda > -\gamma$ the meromorphic family $R_{V_2}(\lambda) : L^2_{\text{comp}} \to L^2_{\text{loc}}$ has a better mapping property:

$$R_{V_2}(\lambda) : L^2_{\text{comp}} \to e^{\gamma \langle x \rangle} L^2.$$

In particular for $\chi \in C^\infty_c(\mathbb{R})$ and $g \in e^{-\gamma \langle x \rangle} L^2(\mathbb{R})$,

$$|\langle R_{V_2}(\lambda)\chi f, g \rangle_{L^2}| \leq (\|e^{\gamma \langle x \rangle}\chi f\|_{L^2} + \|R_{V_2}(\lambda)\chi f\|_{L^2(1,4)})\|g\|_{e^{-\gamma \langle x \rangle} L^2}$$
$$\leq C(\lambda)\|f\|_{L^2}\|g\|_{e^{-\gamma \langle x \rangle} L^2}, \quad \text{Im}\,\lambda > -\gamma \tag{3.5}$$

where $C(\lambda)$ is bounded on compact subsets of $\mathcal{W}_\alpha \setminus \text{Res}(P_{V_2})$ (where Res denotes the set of resonances). On the other hand, for $\text{Im}\,\lambda > 0$,

$$\langle R_{V_2}(-\bar\lambda)\chi f, g \rangle_{L^2} = \langle f, \chi R_{V_2}(-\bar\lambda)^* g \rangle_{L^2} = \langle f, \chi R_{V_2}(\lambda)g \rangle_{L^2}.$$

But then, (3.5) and analytic continuation show that

$$\chi R_{V_2}(\lambda) : e^{-\gamma \langle x \rangle} L^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad \chi \in C^\infty_c(\mathbb{R}).$$

We can then apply Lemma 4 again to see that (3.2) holds. $\hfill\square$

3.2. **Completion of the proof of Proposition 2.** We first observe that with $V = V_1 + V_2$,

$$(P_V - \lambda^2)R_{V_2}(\lambda) = I + V_1 R_{V_2}(\lambda).$$

Then, for $\operatorname{Im}\lambda \gg 1$, $I + V_1 R_{V_2}(\lambda) : L^2 \to L^2$ is invertible by Neumann series and hence

$$R_V(\lambda) = R_{V_2}(\lambda)(I + V_1 R_{V_2}(\lambda))^{-1}, \qquad \operatorname{Im}\lambda \gg 1.$$

For $\operatorname{Im}\lambda > 0$, $R_{V_2}(\lambda) : L^2 \to H^2$ and hence $e^{-\gamma\langle x\rangle}R_{V_2}(\lambda) : L^2 \to L^2$ is compact. In particular, since $e^{\gamma\langle x\rangle}V_1 \in L^\infty$, $I + V_1 R_{V_2}(\lambda) : L^2 \to L^2$ is a Fredholm operator of index 0 and we have by the analytic Fredholm theorem, (see e,g, [DyZw19, Appendix C.3])

$$R_V(\lambda) = R_{V_2}(\lambda)(I + V_1 R_{V_2}(\lambda))^{-1} : L^2 \to L^2, \qquad \operatorname{Im}\lambda > 0,$$

is a meromorphic family of operators with finite rank poles.

Next, consider $\lambda \in \mathcal{W}_\alpha \cap \{2|\operatorname{Im}\lambda| < \gamma\}$. Then, letting $\gamma' > 0$ such that $2|\operatorname{Im}\lambda| < \gamma' < \gamma$, we have by Proposition 3

$$e^{-\gamma\langle x\rangle}R_{V_2}(\lambda) : e^{-\gamma'\langle x\rangle/2}L^2 \to e^{-(\gamma'+2(\gamma-\gamma'))\langle x\rangle/2}H^2$$

In particular, since $\gamma' < \gamma$,

$$e^{-\gamma\langle x\rangle}R_{V_2}(\lambda) : e^{-\gamma'\langle x\rangle/2}L^2 \to e^{-\gamma'\langle x\rangle/2}L^2$$

is compact and hence, using that $e^{\gamma\langle x\rangle}V_1 \in L^\infty$, we have

$$I + V_1 R_{V_2}(\lambda) : e^{-\gamma'\langle x\rangle/2}L^2 \to e^{-\gamma'\langle x\rangle/2}L^2$$

is Fredholm. Therefore, by the analytic Fredholm theorem and Proposition 3, for all $\lambda \in \mathcal{W}_\alpha$ with $2|\operatorname{Im}\lambda| < \gamma'$,

$$R_V(\lambda) = R_{V_2}(\lambda)(I + V_1 R_{V_2}(\lambda))^{-1} : e^{-\gamma'\langle x\rangle/2}L^2 \to e^{\gamma'\langle x\rangle/2}L^2,$$

is a meromorphic family of operators with finite rank poles. $\qquad\square$

APPENDIX: GEVREY PROPERTIES OF EXPONENTIALS.

To explain the asymptotics appearing in Fig.1, we show (in the notation of (1.4)) that for $\operatorname{Im}z > 0$

$$x \mapsto e^{iz/x} \in \widehat{G}^{2,\sigma(z)}([0,\infty)), \quad \sigma(z) := \frac{|z| - |\operatorname{Re}z|}{2}. \tag{A.1}$$

To prove this we need the following characterization of $G^{2,\sigma}(\mathbb{R}) \cap C_c^\infty(\mathbb{R})$: (see [HöII, Lemma 12.7.4] for a similar argument): for $u \in C_c^\infty(\mathbb{R})$

$$\exists C \; |\widehat{u}(\xi)| \leq C\langle\xi\rangle^{-1/4}e^{-2|\sigma\xi|^{\frac{1}{2}}} \;\Rightarrow\; u \in G^{2,\sigma}(\mathbb{R}) \;\Rightarrow\; \exists C \; |\widehat{u}(\xi)| \leq C\langle\xi\rangle^{\frac{1}{2}}e^{-2|\sigma\xi|^{\frac{1}{2}}}. \tag{A.2}$$

Proof of (A.2). We start by estimating $D_x^n u$ under the assumption on $\widehat{u}$:

$$|D_x^n u(x)| \le \int_{\mathbb{R}} |\xi|^n |\widehat{u}(\xi)| d\xi \le C(4\sigma)^{-n} \int_{\mathbb{R}} ((4\sigma|\xi|)^{\frac{1}{2}})^{2n} \langle \xi \rangle^{-1/4} e^{-(4\sigma|\xi|)^{\frac{1}{2}}} d\xi$$

$$= 4C(4\sigma)^{-n-1} \int_0^\infty t^{2n+1} \langle \tfrac{t^2}{4\sigma} \rangle^{-1/4} e^{-t} dt$$

$$\le 4C(4\sigma)^{-n-3/4} \int_0^\infty t^{2n+\frac{1}{2}} e^{-t} dt$$

$$= 4C\sqrt{\pi}(4\sigma)^{-n-3/4} \frac{(4n+2)!}{4^{2n+1}(2n+1)!(n!)^2}(n!)^2 \; \le C'\sigma^{-n}(n!)^2,$$

where in the last inequality we used Stirling's approximation.

On the other hand, if $u \in G^{2,\sigma} \cap C_c^\infty$ then, for $|\xi| > 1$,

$$|\widehat{u}(\xi)| = \left| \int_{\mathbb{R}} D_x^n u(x) \xi^{-n} e^{-ix\xi} dx \right| \le C|\xi|^{-n} \sup |D_x^n u| \le C'|\xi|^{-n} \sigma^{-n}(n!)^2$$

$$\le C'' n \left(\frac{n}{|\sigma\xi|^{\frac{1}{2}}} \right)^{2n} e^{-2n} \le C'''|\xi|^{\frac{1}{2}} e^{-2|\sigma\xi|^{\frac{1}{2}}},$$

where again we used Stirling's formula and to obtain the last inequality we chose $n = [|\sigma\xi|^{\frac{1}{2}}]$. $\qquad\square$

Proof of (A.1). We start by computing the asymptotics of the (distributional) Fourier transform of $w(x) := e^{iz/x}$ for a *fixed* z with $0 < \arg z < \pi$. For $\xi \ge 1$ we deform the contour to $x = -e^{i\varphi_+} y$ with $\varphi_+ := \frac{\arg(z)+\pi}{2}$ to obtain

$$\widehat{w}(\xi) = \lim_{\epsilon \to 0+} \int_0^\infty e^{i(z/x - x(\xi - i\epsilon))} dx = \lim_{\epsilon \to 0+} -e^{i\varphi_+} \int_0^\infty e^{ie^{i\varphi_+}(|z|/x + x(\xi - i\epsilon))} dx$$

$$= -e^{i\varphi_+} |z|^{\frac{1}{2}} \xi^{-\frac{1}{2}} \int_0^\infty e^{i\xi^{\frac{1}{2}}|z|^{\frac{1}{2}} e^{i\varphi_+}(1/t+t)} dt.$$

Since $0 < \arg z < \pi$, $\pi/2 < \varphi_+ < \pi$, we obtain $\cos(\varphi_+ + \pi/2) = -\cos(\arg(z)/2) < 0$. We can now apply the method of steepest descent to obtain

$$\widehat{w}(\xi) = c_+ \xi^{-\frac{3}{4}} |z|^{\frac{1}{4}} e^{ie^{i\varphi_+} + 2|z|^{\frac{1}{2}} \xi^{\frac{1}{2}}} (1 + O(|\xi|^{-\frac{1}{2}})), \quad \xi \ge 1.$$

The cases of $-\xi \ge 1$ is handled similarly except that we deform to $x = e^{i\varphi_-}$ with $\varphi_- = \arg(z)/2$ to obtain

$$\widehat{w}(\xi) = c_- (-\xi)^{-\frac{3}{4}} |z|^{\frac{1}{4}} e^{ie^{i\varphi_-} - 2|z|^{\frac{1}{2}} (-\xi)^{\frac{1}{2}}} (1 + O(|\xi|^{-\frac{1}{2}})), \quad \xi \le -1.$$

We now choose $\chi \in C_c^\infty(\mathbb{R})$ with $\chi \in \gamma^2(\mathbb{R})$ (see (1.4)), $\chi(x) \equiv 1$ for $|x| \le 1$, and consider $u(x) := \chi(x) w(x)$. From (A.2) we see that for any $\gamma > 0$ $|\widehat{\chi}(\xi)| \le$

$C_\gamma e^{-\gamma|\xi|^{\frac{1}{2}}}$. Hence, for $\xi \gg 1$, and $\gamma \gg |z|$, and with $\sigma_+(z) := \cos^2(\arg(z)/2)|z|$, $\sigma_-(z) := \sin^2(\arg(z)/2)|z|$, we have

$$\left| c_+^{-1}|z|^{-\frac{1}{4}} \xi^{\frac{3}{4}} e^{-ie^{i\varphi}+2|z|^{\frac{1}{2}}\xi^{\frac{1}{2}}} \widehat{u}(\xi) - \chi(0) \right|$$

$$= \left| \frac{1}{2\pi} \int (c_+^{-1}|z|^{-\frac{1}{4}} \xi^{\frac{3}{4}} e^{-ie^{i\varphi}+2|z|^{\frac{1}{2}}\xi^{\frac{1}{2}}} \widehat{w}(\xi-\zeta) - 1)\widehat{\chi}(\zeta)d\zeta \right|$$

$$\leq C_\gamma \int_{-\xi/2}^{\xi/2} \frac{|\xi|^{\frac{3}{4}} - |\xi-\zeta|^{\frac{3}{4}}}{|\xi-\zeta|^{\frac{3}{4}}} e^{-2|\sigma_+(z)(\xi-\zeta)|^{\frac{1}{2}}+2|\sigma_+(z)\xi|^{\frac{1}{2}}-\gamma|\zeta|^{\frac{1}{2}}}d\zeta + O(e^{-c\gamma|\xi|^{\frac{1}{2}}})$$

$$\leq C_\gamma |\xi| \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2|\xi|^{\frac{1}{2}}(|\sigma_+(z)|^{\frac{1}{2}}(\gamma|r|^{\frac{1}{2}}+(1-r)^{\frac{1}{2}}-1))}|r|dr + O(e^{-c\gamma|\xi|^{\frac{1}{2}}}) = O(\xi^{-\frac{1}{2}}).$$

Similarly, for $-\xi \gg 1$, and $\gamma \gg \sigma_-(z)$,

$$\left| c_-^{-1}|z|^{-\frac{1}{4}} |\xi|^{\frac{3}{4}} e^{-ie^{-i\varphi}-2|z|^{\frac{1}{2}}|\xi|^{\frac{1}{2}}} \widehat{u}(\xi) - \chi(0) \right| = O(|\xi|^{-\frac{1}{2}}).$$

In particular, with we have $\widehat{u}(\xi) = c_\pm |z|^{\frac{1}{4}} |\xi|^{-\frac{3}{4}} e^{-2|\sigma_\pm(z)\xi|^{\frac{1}{2}}}(\chi(0) + o(1))$, as $\pm\xi \to \infty$. Since $\min(\sigma+(z), \sigma_-(z)) = \sigma(x) := (|z| - |\operatorname{Re} z|)/2$, this and (A.2) give (A.1). $\square$

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