Internal waves in fluids and spectral theory of 0th order operators

Seminarium algebry operatorów
Wydział Fizyki Uniwersytetu Warszawskiego

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June 7, 2018

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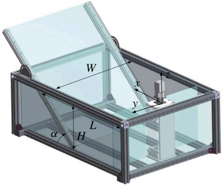
Except for a weakening of assumptions and conclusions the results are due to Colin de Verdière–Saint-Raymond arXiv:1801.05582.

Motivation

(a)



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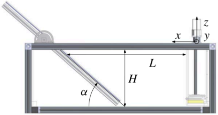

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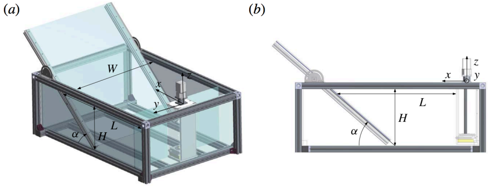
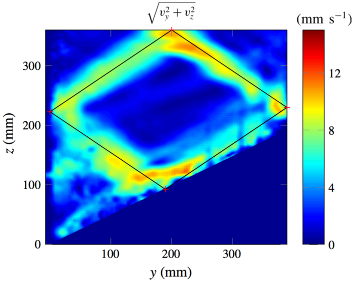
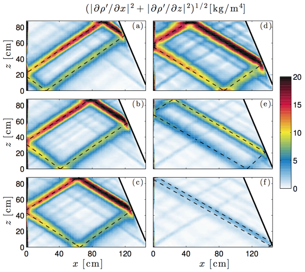




Pillet et al '18

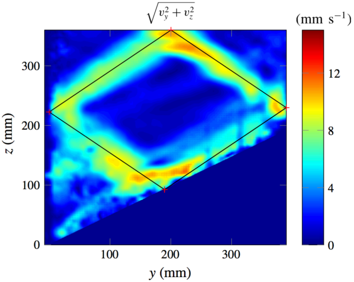


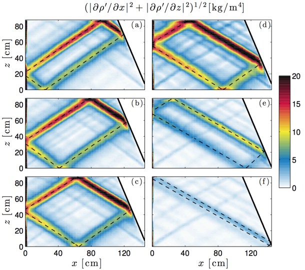
Bouzet '16

Motivation

Boussinesq approximation:

$$\begin{cases} \partial_t \eta + \mathbf{u} \cdot \nabla \rho_0 = 0, \quad \mathrm{div}\,\mathbf{u} = 0, \\ \rho_0 \partial_t \mathbf{u} = -\eta g \mathbf{e}_3 - \nabla P + \mathbf{F} e^{-i\omega_0 t}, \end{cases} \qquad \mathbf{n} \cdot \mathbf{u} = 0.$$





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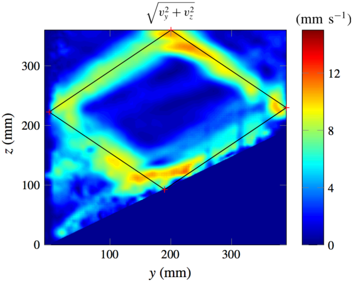

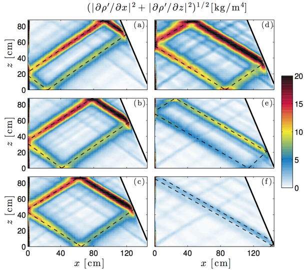

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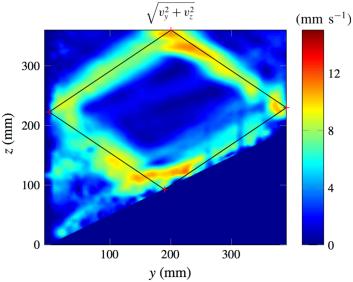

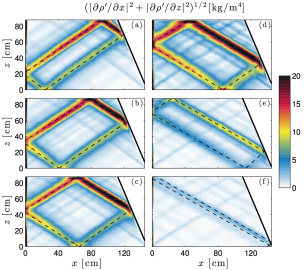

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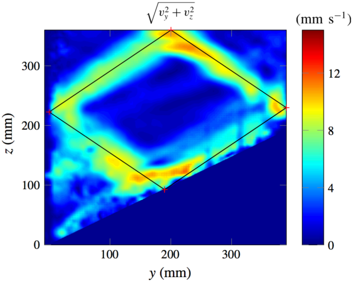

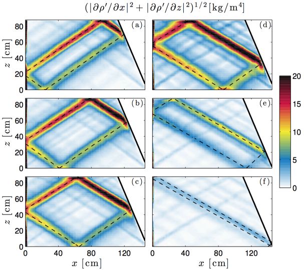

Other related models: rotating fluids Ralston '73

$$\partial_t^2 \Delta_x u = \partial_{x_1}^2 u, \quad u|_{\partial\Omega} = 0$$

$$i\partial_t u - Pu = 0, \quad P = \pm \Delta^{-\frac{1}{2}} \partial_{x_1}$$

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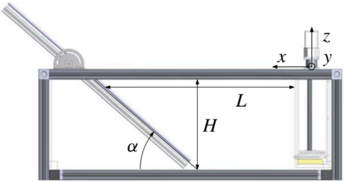

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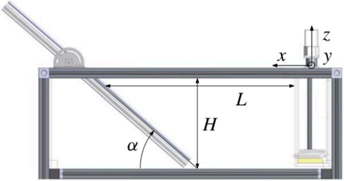

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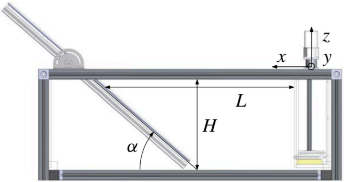

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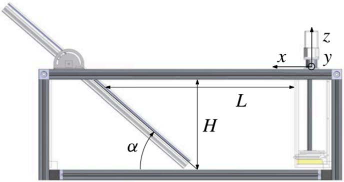

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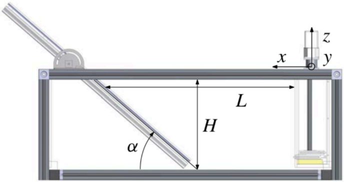

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$$H_p = \partial_\xi p \cdot \partial_x - \partial_x p \cdot \partial_\xi, \quad (x, \xi) \sim (y, \eta) \Leftrightarrow x = y, \ \ \xi = t\eta, \ \ t > 0$$

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$$i\partial_t u - Pu = e^{-i\omega_0 t} f, \;\; P \in \Psi^0(\mathbb{T}^2), \;\; P^* = P, \;\; u|_{t=0} = 0, \;\; f \in C^\infty(\mathbb{T}^2)$$

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(i) $\langle \xi \rangle H_p$ has a finite number of fixed points all of which are hyperbolic;

(ii) $\langle \xi \rangle H_p$ has a finite number of hyperbolic limit cycles;

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(iv) every trajectory different from (i) and (ii) has a unique trajectory (i) or (ii) as its α, ω-limit set.

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(Some comments about fixed points at the end.)

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$$|\partial_{x_2}^k \partial_{\xi_1}^\ell a(x_2,\xi_1)| = \begin{cases} \mathcal{O}(\xi_1^{m-\ell}) & \xi_1 \to +\infty \\ \mathcal{O}(|\xi_1|^{-\infty}) & \xi_1 \to -\infty. \end{cases}$$

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For instance, $w(x) = (x_1 - i0)^{-1}\varphi(x_1,x_2)$, $\varphi \in C^\infty(\mathbb{T}^2)$.

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Theorem *Suppose that* $\omega_0 \notin \mathrm{Spec}_{\mathrm{pp}}(P)$ *and that* u *solves*

$$i\partial_t u - Pu = e^{-i\omega_0 t} f, \quad u|_{t=0} = 0, \quad f \in C^\infty(\mathbb{T}^2).$$

Then,

$$u(t) = e^{-i\omega_0 t} u_\infty + b(t) + \epsilon(t), \quad u_\infty \in I^0(\Lambda_+)$$
$$\|b(t)\|_{L^2} \leq C, \quad \|\epsilon(t)\|_{-\frac{1}{2}-} \to 0, \quad t \to \infty$$

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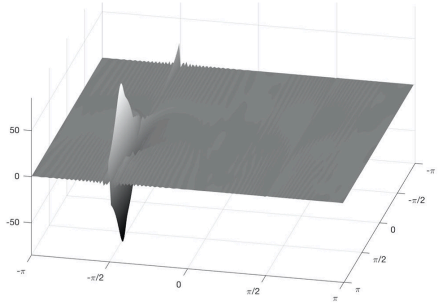

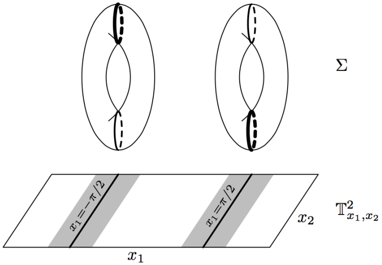

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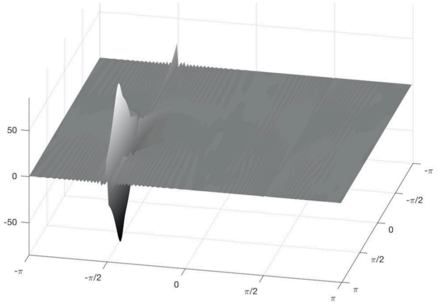

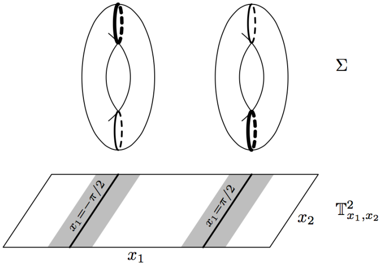

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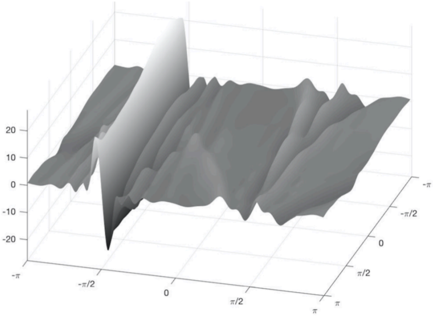

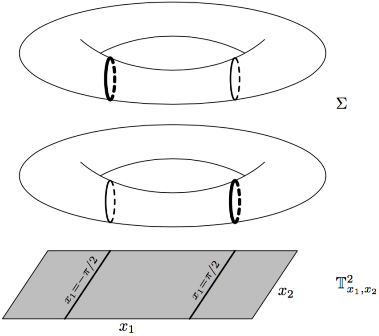

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- $(P - i0)^{-1} f \in I^0(\Lambda_+)$

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The radial estimates were introduced by Melrose '94 for the study of asymptotically Euclidean scattering and have been developed further in various settings.

Some relevant ones:

- scattering by 0th order potentials Hassell–Melrose–Vasy '04
- hyperbolic scattering Vasy '13, Datchev–Dyatlov '13
- general relativity Vasy, Hintz–Vasy '13..., Dyatlov '11–'14
- Lagrangian regularity Haber–Vasy '15
- Anosov flows Dyatlov–Zworski '16, '17
- Axiom A flows Dyatlov–Guillarmou '16, '18

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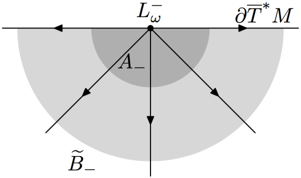

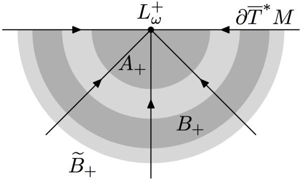

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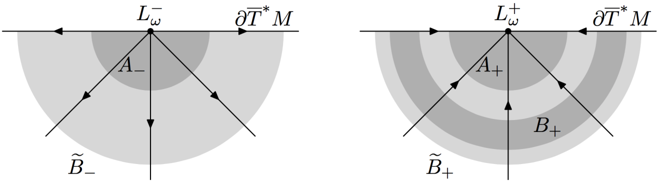

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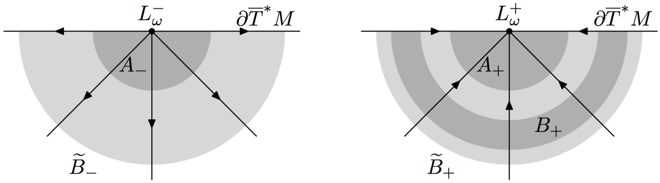

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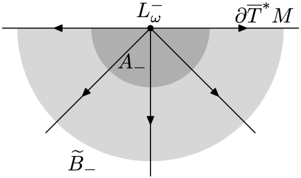

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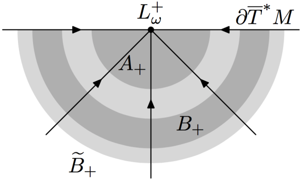

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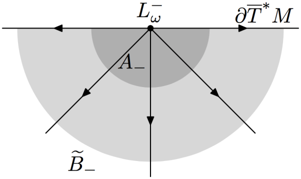

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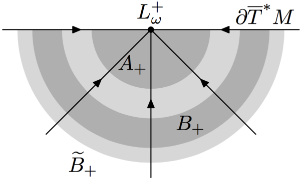

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Hence the length spectrum, $\{\ell_\gamma\}$ (dynamics), determines the genus g (topology).

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Standard arguments in scattering theory (cf. Melrose '94) show that the limit

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Lemma (Dyatlov–Zworski '18; related to Haber–Vasy '15)
Suppose that

$$(P - \omega)u \in C^\infty, \quad \mathrm{WF}(u) \subset \Lambda_+(\omega), \quad u \in H^{-\frac{1}{2}-}.$$

Then $u \in I^0(\Lambda_+(\omega))$.
Moreover, if $u(\omega) = (P - \omega - i0)^{-1} f$, $f \in C^\infty$, *then*

$$u(\omega) \in C^\infty((-\delta, \delta)_\omega; I^0(\Lambda_+(\omega)))$$

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The general set up:

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Suppose that, locally,

$$\Lambda_\omega = \{(x, \xi) : x = \partial_\xi F(\omega, \xi)\}$$

where $\xi \mapsto F(\omega, \xi)$ is a family of homogeneous functions of order one. Then for some $c > 0$,

$$\partial_\omega F(\omega, \xi) < -c|\xi|, \quad \xi \in \Gamma_0.$$

Theorem *Suppose that* $0 \notin \mathrm{Spec}_{\mathrm{pp}}(P)$ *and that* u *solves*

$$i\partial_t u - Pu = f, \quad u|_{t=0} = 0, \quad f \in C^\infty(\mathbb{T}^2).$$

Then, $u(t) = u_\infty + b(t) + \epsilon(t)$*, where* $u_\infty \in I^0(\Lambda_+)$*,* $\|b(t)\|_{L^2} \leq C$ *and* $\|\epsilon(t)\|_{-\frac{1}{2}-} \to 0$*, as* $t \to \infty$*.*

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Proof: From spectral theorem and Stone's formula

$$\begin{aligned}
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&= \frac{1}{2\pi}\int_0^t e^{-is\omega}\left((P-\omega-i0)^{-1} - (P-\omega+i0)^{-1}\right) f \chi(\omega) d\omega \\
&\quad + b_1(t), \qquad \|b_1(t)\|_{L^2} \leq C, \quad \chi = 1 \text{ near } 0 \\
&= \frac{1}{2\pi}\int_0^t e^{-is\omega}(u_+(\omega) - u_-(\omega))d\omega + b_1(t), \; u_\pm(\omega) \in I^0(\Lambda_\pm(\omega)) \\
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Suppose that $\varepsilon\partial_\omega F(0,\xi)<0$. *Then for* $w(\omega)$ *supported near* 0,

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The geometric lemma provides the sign condition! QED

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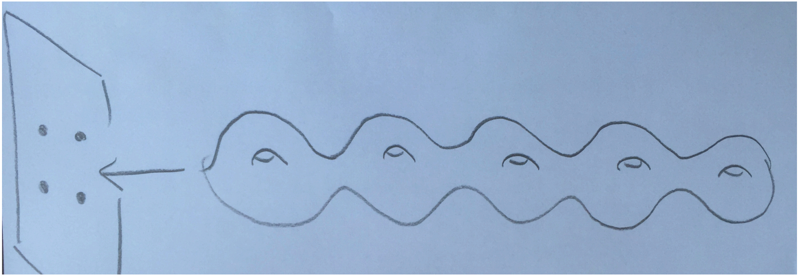

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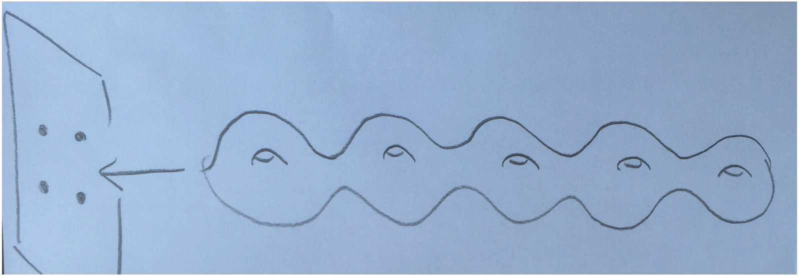

$$u(t) = u_\infty + b(t) + \epsilon(t), \quad \mathrm{WF}(u) \subset \Lambda_+ \cup \Gamma_+,$$

$$u_\infty \in H^{-\frac{1}{2}-}, \quad \|b(t)\|_{L^2} \le C, \quad \|\epsilon(t)\|_{H^{-\frac{3}{2}-}} \to 0, \quad t \to \infty$$

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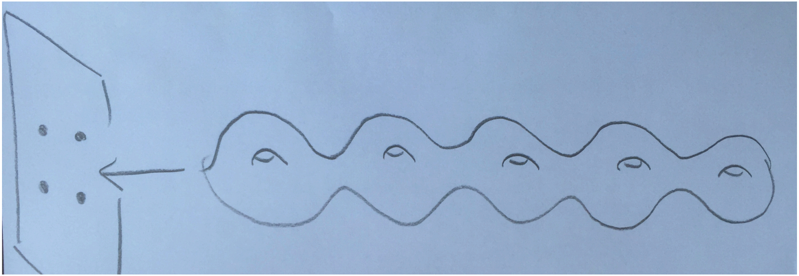

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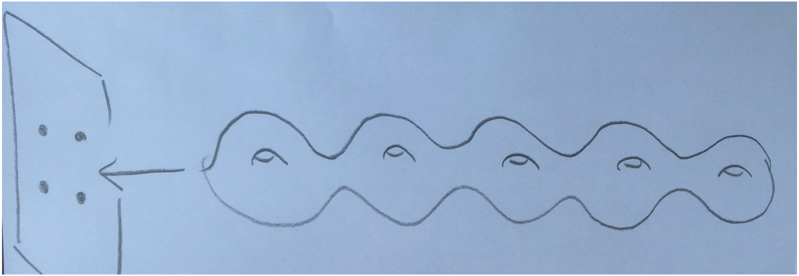

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In the Morse–Smale case, Colin de Verdière '18 used a hybrid of Mourre and radial estimates to show that $\|\epsilon(t)\|_{H^{-\frac{1}{2}-}} \to 0$.

More general geometries

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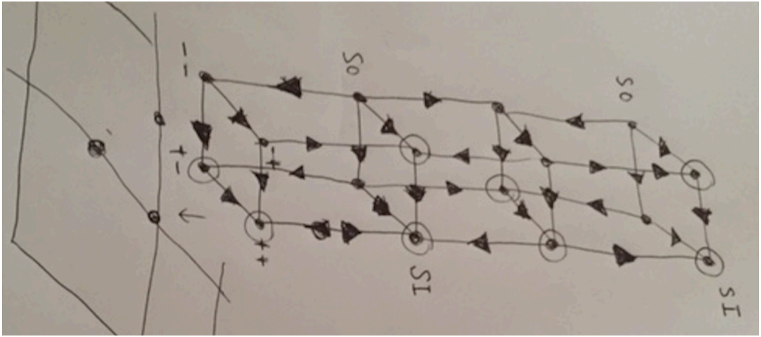

$$\pi(\Lambda_+) = \{x_1 = x_2 = -\tfrac{1}{2}\pi\} \cup \{x_1 = -\tfrac{\pi}{2}, x_2 = \tfrac{\pi}{2}\} \cup \{x_1 = \tfrac{\pi}{2}, x_2 = -\tfrac{\pi}{2}\}$$

$$\pi(\Gamma_+) = \{x_1 = \tfrac{\pi}{2}\} \cup \{x_2 = \tfrac{\pi}{2}\}$$

Finally,

Finally, a word from our sponsor...

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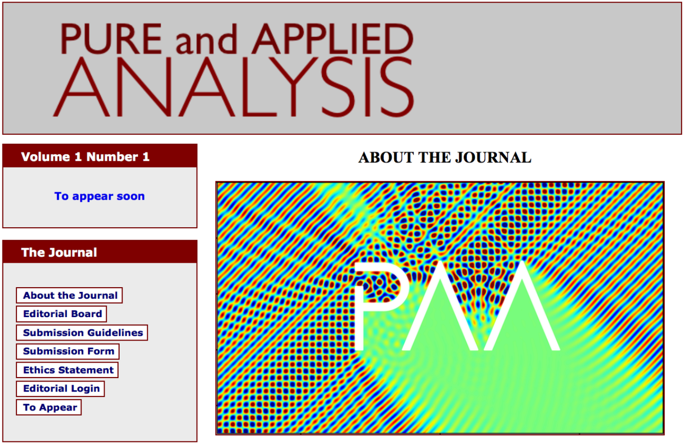

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Thank you for your attention!