A FERMI GOLDEN RULE FOR QUANTUM GRAPHS

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ABSTRACT. We present a Fermi golden rule giving rates of decay of states obtained by perturbing embedded eigenvalues of a quantum graph. To illustrate the procedure in a notationally simpler setting we first describe a Fermi golden rule for boundary value problems on surfaces with constant curvature cusps. We also provide a resonance existence result which is uniform on compact sets of energies and metric graphs. The results are illustrated by numerical experiments.

1. Introduction and statement of results

Quantum graphs are a useful model for spectral properties of complex systems. The complexity is captured by the graph but analytic aspects remain one dimensional and hence relatively simple. We refer to the monograph by Berkolaiko–Kuchment [1] for references to the rich literature on the subject.

In this note we are interested in graphs with infinite leads and consequently with continuous spectra. We study dissolution of embedded eigenvalues into the continuum and existence of resonances close to the continuum. Our motivation comes from a recent Physical Review Letter [11] by Gnutzmann–Schanz–Smilansky and from a mathematical study by Exner–Lipovský [10].

We consider an oriented graph with vertices $\{v_j\}_{j=1}^J$, infinite leads $\{e_k\}_{k=1}^K$, $K > 0$, and M finite edges $\{e_m\}_{m=K+1}^{M+K}$. We assume that each finite edge, e_m, has two distinct vertices as its boundary (a non-restrictive no-loop condition) and we write $v \in e_m$ for these two vertices v. An infinite lead has one vertex. The set of (at most two) common vertices of e_m and e_ℓ is denoted by $e_m \cap e_\ell$ and we we denote by $e_m \ni v$ the set of all edges having v as a vertex.

The finite edges are assigned length ℓ_m, $K+1 \leq m \leq M+K$ and we put $\ell_k = \infty$, $1 \leq k \leq K$, for the infinite edges. To obtain a *quantum graph* we define a Hilbert space, is given by

$$L^2 := \bigoplus_{m=1}^{K+M} L^2([0,\ell_m]), \quad L^2 \ni u = (u_1, \cdots u_{M+K}), \quad u_m \in L^2([0,\ell_m]). \tag{1.1}$$

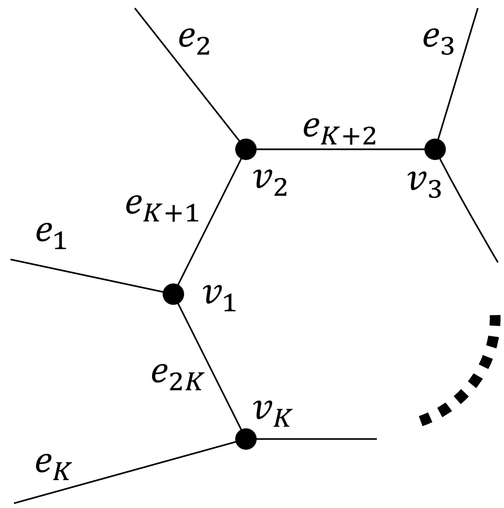

FIGURE 1. A graph given by a cycle $\{e_k\}_{k=K+1}^{2K}$ connected to K infinite leads $\{e_k\}_{k=1}^{K}$ at K vertices: v_k, $e_{K+k}\cap e_{K+k-1} = v_k$, $e_{2K}\cap e_{K+1} = v_1$, $e_k \cap e_{K+k} = v_k$. The lengths of finite edges are given by $\ell_k(t) = e^{-2a_k(t)}\ell_k$, $K+1 \leq k \leq 2K$. If $\ell_k(0)$'s are rationally related then $P(0)$ has eigenvalues, $\lambda(0)$, embedded in the continuous spectrum. If $\lambda(0)$ is simple then $\lambda(0)$ belongs to a smooth family of resonances, $\lambda(t)$, $\operatorname{Im}\lambda(t) \leq 0$. Theorem 1 and Example 1 in §3 show that in this case $\operatorname{Im}\ddot{\lambda} = \lambda^2\sum_{k=1}^{K} |\langle \dot{a}u, e^k(\lambda)\rangle|^2$, where u is the normalized eigenfunction corresponding to u and $e^k(\lambda)$ is the generalized eigenfunction normalized in the *kth* lead – see (1.2).

We then consider the simplest quantum graph Hamiltonian which is unbounded operator P on L^2 defined by $(Pu)_m = -\partial_x^2 u_m$ with

$$\mathcal{D}(P) = \{u : u_m \in H^2([0,\ell_m]),\ u_m(v) = u_\ell(v),\ v \in e_m \cap e_\ell,\ \sum_{e_m \ni v} \partial_\nu u_m(v) = 0\}.$$

Here ∂_ν denotes the outward pointing normal at boundary of e_v:

$$u_m \in H^2([0,\ell_m]), \quad \partial_\nu u_m(0) = -u_m'(0), \quad \partial_\nu u_m(\ell_m) = u_m'(\ell_m).$$

The space $\mathcal{D}_{\rm loc}(P)$ is defined by replacing H^2 by $H^2_{\rm loc}$ when $\ell_m = \infty$.

Quantum graphs with infinite leads fit neatly into the general abstract framework of *black box* scattering [15] and hence we can quote general results [8, Chapter 4] in spectral and scattering theory.

When $K > 0$ then the projection on the continuous spectrum of P is given in terms of generalized eigenfunctions $e^k(\lambda)$, $1 \leq k \leq K$, which for $\lambda \notin \operatorname{Spec}_{\rm pp}(P)$ are characterized as follows:

$$\begin{gathered} e^k(\lambda) \in \mathcal{D}_{\rm loc}(P), \quad (P-\lambda^2)e^k(\lambda) = 0, \\ e^k_m(\lambda, x) = \delta_{mk}e^{-i\lambda x} + s_{mk}(\lambda)e^{i\lambda x}, \quad 1 \leq m \leq K. \end{gathered} \tag{1.2}$$

The family $\lambda \mapsto e^k(\lambda) \in \mathcal{D}_{\rm loc}(P)$ extends holomorphically to a neighbourhood of $\mathbb{R}$ and that defines $e^k(\lambda)$ for all λ. We will in fact be interested in $\lambda \in \operatorname{Spec}_{\rm pp}(P)$. The functions e^k parametrize the continuous spectrum of P – see [8, §4.4] and (3.13) below.

We now consider a family of quantum graphs obtained by varying the lengths ℓ_m, $K+1 \leq m \leq M+K$:

$$\ell_m(t) = e^{-a_m(t)}\ell_m, \quad a_m(0) = 0, \tag{1.3}$$

and the corresponding family of operators, $P(t)$. We consider $a(t)$ as a function which is constant on the edges and denote

$$\dot{a} := \partial_t a(0), \quad (\dot{a}u)_m(x) = \dot{a}_m u_m(x).$$

The works [10] and [11] considered the case in which $P(0)$ has embedded eigenvalues and investigated the resonances of the deformed family $P(t)$ converging to these eigenvalues as $t \to 0$. Here we present a Fermi golden rule type formula (see §2 for references to related mathematical work) which gives an infinitesimal condition for the disappearance of an embedded eigenvalue. It becomes a resonance of P and one can calculate the infinitesimal rate of decay. Resonances are defined as poles of the meromorphic continuation of $\lambda \mapsto (P-\lambda^2)^{-1}$ to $\mathbb{C}$ as an operator $L^2_{\rm comp} \to L^2_{\rm loc}$ (see [9],[8, §4.2] and for a self-contained general argument Proposition 4.1). We denote the set of resonances of P by $\operatorname{Res}(P)$.

Theorem 1. *Suppose that $\lambda^2 > 0$ is a simple eigenvalue of $P = P(0)$ and u is the corresponding normalized eigenfunction. Then for $|t| \leq t_0$ there exists a smooth function $t \mapsto \lambda(t)$ such that $\lambda(t) \in \operatorname{Res}(P(t))$ and*

$$\begin{gathered} \operatorname{Im} \ddot{\lambda} = -\sum_{k=1}^{K} |F_k|^2, \\ F_k := \lambda \langle \dot{a}u, e^k(\lambda)\rangle + \lambda^{-1} \sum_{v} \sum_{e_m \ni v} \tfrac{1}{4} \dot{a}_m \big(3\partial_\nu u_m(v)\overline{e^k(\lambda, v)} - u(v)\partial_\nu \overline{e^k_m(\lambda, v)}\big), \end{gathered} \tag{1.4}$$

where $\langle \bullet, \bullet \rangle$ denotes the inner product on (1.1).

The proof is given in §3 and that section is concluded with two examples: the first gives graphs and eigenvalues for which $F_k = \lambda\langle \dot{a}u, e^k(\lambda)\rangle$ – see Figures 1 and 2. The second example gives a graph and an eigenvalue for which the boundary terms in the formula for F_k are needed – see Fig. 4.

Let us compare (1.4) to the Fermi golden rule in more standard settings of mathematical physics as presented in Reed–Simon [14, §XII.6, Notes to Chapter XII][†]. In that case the we have a Hamitonian $H(t) = H + tV$ such that H has a simple eigenvalue at E_0 embedded in the continuous spectrum of $H(0)$ with the normalized eigenfunction u. Let $\mathbb{P}(E) = \mathbf{1}_{(-\infty,E)\setminus\{E_0\}}(H)$ be a modified spectral projection. Then for small t the operator $H(t)$ has a family of resonances with imaginary parts given by $\Gamma(t)/2$ where

[†]For references to more recent advances see Cornean–Jensen–Nenciu [5].

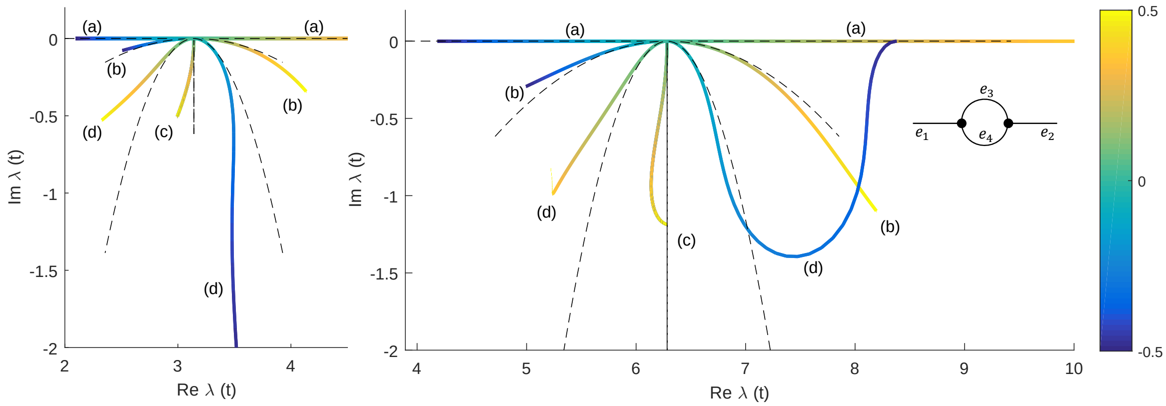

FIGURE 2. A simple graph with embedded eigenvalues, $M = K = 2$. Solid lines and dashed lines indicate the trajectory of $\lambda(t)$ and of the second order approximation $\tilde{\lambda}(t) = \lambda + t\dot{\lambda} + \frac{i}{2}t^2 \operatorname{Im} \ddot{\lambda}$, respectively. (The colour coding indicates the parameter t shown in the colour bar.) We approximate the real part linearly using (3.11) and the imaginary quadratically using (1.4). The four cases are (a): $\ell_3(t) = 1 - t$, $\ell_4(t) = 1 - t$, (b): $\ell_3(t) = 1 - t$, $\ell_4(t) = 1$, (c): $\ell_3(t) = 1 - t$, $\ell_4(t) = 1 + t$, (d): $\ell_3(t) = 1 - t$, $\ell_4(t) = 1 + 2t$.

the width $\Gamma(t)$ function satisfies

$$\frac{\partial^2}{\partial t^2}\Gamma(0) = \pi \frac{\partial}{\partial E}\langle u, V\mathbb{P}(E_0)Vu\rangle.$$

If the continuous spectrum has a nice parametrization by generalized eigenfunctions, $\partial_E \mathbb{P}(E) = \int_{\mathscr{A}} e(E,a) \otimes e(E,a)^* d\mu(a)$, $a \in \mathcal{A}$, this expression becomes

$$\partial_t^2 \Gamma(0) = \pi \int_{\mathcal{A}} |\langle Vu, e(E,a)\rangle|^2 d\mu(a).$$

In the case of quantum graphs $\mathcal{A}$ is a discrete set – see (3.13) below – and the formula is close to our formula (1.4). In [8, Theorem 4.22] a general formula for black box perturbations is given and it applies verbatim to perturbations of quantum graph Hamiltonians *when the domain of the perturbation does not change.* The difference here lies in the fact that the domain changes and that produces additional boundary terms. (We present the results in the simplest case of Kirchhoff boundary conditions.) To explain the method in a similar but notationally simpler setting we first prove the Fermi golden rule in scattering on surfaces with cusp ends and boundaries.

The formula (1.4) gives a condition for the existence a resonance with a nontrivial imaginary part (decay rate) near an embedded eigenvalue of the unperturbed operator: $D(\lambda_0, ct) \cap \operatorname{Res}(P(t)) \neq \emptyset$[‡] for some c and for $|t| \leq t_0$, where the constants c and t_0 depend on λ_0 and $P(t)$. However, it is difficult to estimate the speed with which the

[‡]Here and below $D(\lambda_0, r) = \{\lambda \in \mathbb{C} : |\lambda - \lambda_0| < r\}$.

resonance $\lambda(t)$ moves – that is already visible in comparing Fig. 2 with Fig. 4. (A striking example is given by $P(t) = -\partial_x^2 + tV(x)$ where $V \in C_c^\infty(\mathbb{R})$ and $t \to 0$; infinitely many resonances for $t \neq 0$ [22] disappear and $P(0)$ has only one resonance at 0.) Also, the result is not uniform if we vary λ_0 or the lengths of the edges.

The next theorem adapts the method of Tang–Zworski [20] and Stefanov [17] (see also [8, §7.3]) to obtain existence of resonances near any approximate eigenvalue and in particular near an embedded eigenvalue – see the example following the statement. In particular this applies to the resonances studied in [10] and [11]. The method applies however to very general Hamiltonians – for semiclassical operators on graphs the general black box results of [20] and [17] apply verbatim. The point here is that the constants are uniform even though the dependence on t is slightly weaker.

To formulate the result we define

$$\mathcal{H}_R := \bigoplus_{m=1}^{K} L^2([0,R]) \oplus \bigoplus_{m=K+1}^{K+M} L^2([0,\ell_m]). \tag{1.5}$$

Theorem 2. *Suppose that P is defined above and the lengths, ℓ_m, have the property that $\ell_m \in \mathcal{L}$, $K+1 \leq m \leq M+K$ where $\mathcal{L}$ is a fixed compact subset of the the open half-line.*

Then for any $\mathcal{L} \Subset (0,\infty)$, $I \Subset (0,\infty)$, $R > 0$ and $\gamma < 1$ there exists $\varepsilon_0 > 0$ such that

$$\exists\, u \in \mathcal{H}_R \cap \mathcal{D}(P),\ \lambda_0 \in I \quad \text{such that} \quad \|u\|_{L^2} = 1, \quad \|(P - \lambda_0^2)u\| = \varepsilon < \varepsilon_0 \tag{1.6}$$

implies

$$\operatorname{Res}(P) \cap D(\lambda_0, \varepsilon^\gamma) \neq \emptyset. \tag{1.7}$$

As a simple application of Theorem 1 related to Theorem 2 we present the following **Example.** Suppose that $P(t)$ is the family of operators defined by choosing $\ell_j = \ell_j(t) \in C^1(\mathbb{R})$, and that $\lambda_0 > 0$ is an eigenvalue of $P(0)$. Then for any $\gamma < 1$ there exists t_0 such that for $|t| \leq t_0$

$$\operatorname{Res}(P(t)) \cap D(\lambda_0, t^\gamma) \neq \emptyset. \tag{1.8}$$

Proof. To apply Theorem 2 we need to construct an approximate mode of $P(t)$ using the eigenfunction of $P(0)$. Thus, let u^0 be a normalized eigenfunction of $P(0)$ with eigenvalue λ_0; in particular $u_k^0 \equiv 0$, $1 \leq k \leq K$. Choose $\chi_j \in C^\infty(\mathbb{R})$, $j = 1, 2$, such that $\chi_j \geq 0$, $\chi_0 + \chi_1 = 1$, $\chi_j(s) = 1$ near $|j - s| < \frac{1}{3}$ and define $u_m^-(x) := \chi_0(x/\ell_m)u_m^0(x)$ and $u_m^+(x) := \chi_1(x/\ell_m)u_m^0(x)$, $u^0 = u^+ + u^-$.

We now define a quasimode for $P(t)$, $u = u(t)$ needed in (1.6):

$$u_m(t) = u_m^-(x) + u_m^+(x - \delta_m(t)), \quad \delta_m(t) := \ell_m(t) - \ell_m(0).$$

For t small enough $\operatorname{supp} u_m^- \subset [0, \frac{2}{3})\ell_m(0) \subset [0, \ell_m(t))$ and $\operatorname{supp} u_m^+ \subset (\frac{2}{3}, 1]\ell_m(0) \subset (|\delta_m(t)|, 1]\ell_m(0)$. Hence the values of $u_m(t)$ and $\partial_\nu u_m(t)$ at the vertices are the same as those of u_m^0 and $u_m(t) \in \mathcal{D}(P(t))$. Also, since $(-\partial_x^2 - \lambda_0^2)u_m^0 = 0$ and $\chi_0^{(k)} = -\chi_1^{(k)}$, (and putting $\ell_m = \ell_m(0)$)

$$[(P(t) - \lambda_0^2)u(t)]_m = \ell_m^{-2}(\chi_0''((x - \delta_m(t))/\ell_m)u_m^0(x - \delta_m(t)) - \chi_0''(x/\ell_m)u_m^0(x)) + 2\ell_m^{-1}(\chi_0'((x - \delta_m(t))/\ell_m)\partial_x u_m^0(x - \delta_m(t)) - \chi_0'(x/\ell_m)\partial_x u_m^0(x)).$$

We note that all the terms are supported in $(\frac{1}{3} - |\delta_m(t)|, \frac{2}{3} + |\delta_m(t)|)\ell_m(0)$ and elementary estimates show that $\|(P(t) - \lambda_0^2)u(t)\| \leq Ct$. For instance,

$$\begin{aligned}\|\chi_0''(x)(u_m^0(x - \delta_m(t)) - u_m^0(x))\| &\leq C'|\delta_m(t)| \max_{|x - \frac{1}{2}|\ell_m(0) \leq \frac{1}{6} + |\delta_m(t)} |\partial_x u_m^0(x)| \\ &\leq C'|\delta_m(t)|(\| - \partial_x^2 u_0^m\|_{L^2((\frac{1}{4}, \frac{3}{4})\ell_m(0))} + \|u_0^m\|_{L^2((\frac{1}{4}, \frac{3}{4})\ell_m(0))}) \\ &\leq C''(\lambda_0^2 + 1)t.\end{aligned}$$

From (1.7) we conclude (after decreasing γ and t_0) that (1.8) holds. □

Remarks. 1. A slightly sharper statement than (1.7) can already be obtained from the proof in §4. It is possible that in fact $\text{Res}(P) \cap D(\lambda_0, C_0\varepsilon)$ where C_0 depends on $\mathcal{L}$, R and δ. That is suggested by the fact that the converse to this stronger conclusion is valid – see Proposition 4.5. This improvement would require finer complex analytic arguments. It is interesting to ask if methods more specific to quantum graphs, in place of our general methods, could produce this improvement.

2. By adapting Stefanov's methods [17] one can strengthen the conclusion by adding a statement about multiplicities (see also [8, Exercise 7.1]) but again we opted for a simple presentation.

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2. A Fermi golden rule for boundary value problems: surfaces with cusps

To illustrate the Fermi golden rule in the setting of boundary value problems we consider surfaces, X, with cusps of constant negative curvature. The key point is that the domain of the operator changes and the general results such as [14, Theorem XII.24] or [8, Theorem 4.22] do not apply.

Thus we assume that (X, g) is a Riemannian surface with a smooth boundary and a decomposition (see Fig. 3)

$$\begin{aligned} X = X_1 \cup X_0, \quad \partial X_0 = \partial X_1 \cup \partial X, \quad \partial X \cap \partial X_0 = \emptyset, \\ (X_1, g|_{X_1}) \simeq ([a, \infty)_r \times (\mathbb{R}/\ell\mathbb{Z})_\theta, dr^2 + e^{-2r}d\theta^2). \end{aligned} \tag{2.1}$$

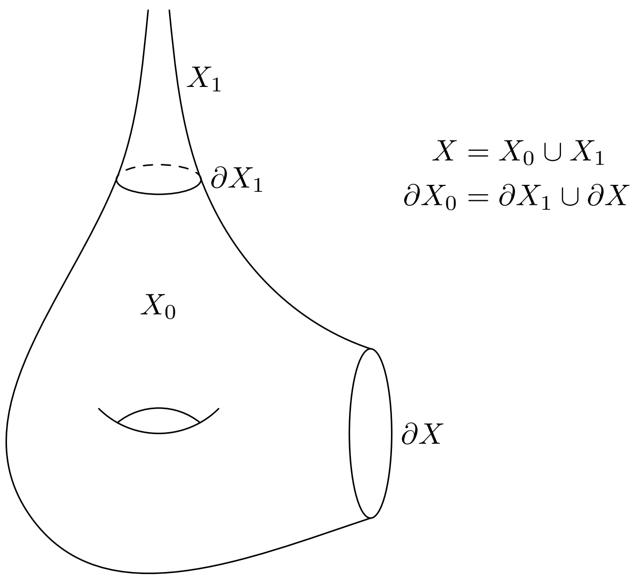

FIGURE 3. A surface with one cusp end and a boundary. Suppose we consider a family of boundary conditions for the Laplacian $-\Delta$: $\partial_\nu w = \gamma(t) w$ at ∂X. The Laplacian has continuous spectrum with a family of generalized eigenfunctions $e(\lambda) \in C^\infty(X)$ – see (2.3). Suppose that for $t = 0$, λ^2 is a *simple embedded* eigenvalue of $-\Delta$ with the boundary condition $\partial_\nu w = \gamma(0) w$, with the normalized eigenfunction given by u. Then $\lambda = \lambda(0)$ belongs to a smooth family of *resonances* of Laplacians with boundary condition $\partial_\nu w = \gamma(t) w$, and $\operatorname{Im} \ddot{\lambda} = -\frac{1}{4\lambda^2} |\langle \dot{\gamma} u, e(\lambda) \rangle_{L^2(\partial X)}|^2$ – see Theorem 3.

We consider the following family of unbounded operators on $L^2(X)$:

$$P(t) = -\Delta_g - \tfrac{1}{4}, \quad \mathcal{D}(P(t)) = \{u \in H^2(X) : \partial_\nu u|_{\partial X} = \gamma(t) u|_{\partial X}\}. \tag{2.2}$$

where $t \mapsto \gamma(t) \in C^\infty(\partial X)$ is a smooth family of functions on ∂X and ∂_ν is the outward pointing normal derivative. The spectrum of the operator P has the following well known decomposition:

$$\operatorname{Spec}(P) = \operatorname{Spec}_{\rm pp}(P) \cup \operatorname{Spec}_{\rm ac}(P), \quad \operatorname{Spec}_{\rm ac}(P) = [0, \infty),$$
$$\operatorname{Spec}_{\rm pp}(P) = \{E_j\}_{j=0}^J, \quad -\tfrac{1}{4} \le E_0 < E_1 \le E_2 \cdots, \quad 0 \le J \le +\infty.$$

(When $J = +\infty$ then $E_j \to \infty$.) The eigenvalues $E_j > 0$ are *embedded* in the continuous spectrum. In addition the resolvent $R(\lambda) := (P - \lambda^2)^{-1} : L^2 \to L^2$, $\operatorname{Im} \lambda > 0$, has a meromorphic continuation to $\lambda \in \mathbb{C}$ as an operator $R(\lambda) : C^\infty_{\rm c}(X) \to C^\infty(X)$. Its poles are called *scattering resonances*. Under generic perturbation of the metric in X_0 all embedded eigenvalues become resonances. For proofs of these well known facts see [4] and also [8, §4.1 (Example 3), §4.2 (Example 3), §4.4.2] for a presentation from the point of view of *black box scattering* [15].

The generalized eigenfunctions, $e(\lambda, x)$, describing the projection onto the continuous spectrum have the following properties:

$$\begin{gathered} (P - \lambda^2) e(\lambda, x) = 0, \quad \frac{1}{\ell} \int_0^\ell e(\lambda, x)|_{X_1} d\theta = e^{\frac{r}{2}} \left(e^{-i\lambda r} + s(\lambda) e^{i\lambda r} \right), \\ (R(\lambda) - R(-\lambda)) f = \tfrac{i}{2\lambda} e(\lambda, x) \langle f, e(\lambda, \bullet) \rangle, \quad \lambda \in \mathbb{R}, \quad f \in C^\infty_{\rm c}(X), \end{gathered} \tag{2.3}$$

see [8, Theorem 4.20]. With these preliminaries in place we can now prove

Theorem 3. *Suppose that the operators $P(t)$ are defined by (2.2) and that $\lambda > 0$ is a* simple *eigenvalue of $P(0)$ and $(P(0) - \lambda^2)u = 0$, $\|u\|_{L^2} = 1$.*

Then there exists a smooth function $t \mapsto \lambda(t)$, $|t| < t_0$, such that $\lambda(0) = \lambda$, $\lambda(t)$ is a scattering resonance of $P(t)$ and

$$\operatorname{Im} \ddot{\lambda} = -\frac{1}{4\lambda^2} \left| \langle \dot{\gamma} u, e \rangle_{L^2(\partial X)} \right|^2, \quad e(x) = e(\lambda, x), \tag{2.4}$$

where $e(\lambda, x)$ is given in (2.3), $\dot{f} := \partial_t f|_{t=0}$ and $L^2(\partial X)$ is defined using the metric induced by g.

1. In the case of scattering on constant curvature surfaces with cusps the Fermi golden rule was explicitly stated by Phillips–Sarnak – see [13] and for a recent discussion [12]. For a presentation from the black box point of view see [8, §4.4.2].

2. The proof generalizes immediately to the case of several cusps (which is analogous to a quantum graph with several leads), $(X_k, g|_{X_k}) \simeq ([a_k, \infty) \times \mathbb{R}/\ell_k\mathbb{Z}, dr^2 + e^{-2r} d\theta^2$, $1 \leq k \leq K$. In that case the generalized eigenfunction are normalized using

$$\frac{1}{\ell_m} \int_0^{\ell_m} e^k(\lambda, x)|_{X_m} d\theta = e^{\frac{r}{2}} \left(\delta_{km} e^{-i\lambda r} + s_{km}(\lambda) e^{i\lambda r} \right).$$

The Fermi golden rule for the boundary value problem (2.2) is given by

$$\operatorname{Im} \ddot{\lambda} = -\frac{1}{4\lambda^2} \sum_{k=1}^{K} \left| \langle \dot{\gamma} u, e^k \rangle_{L^2(\partial X)} \right|^2, \quad e^k(x) = e^k(\lambda, x). \tag{2.5}$$

Proof. For notational simplicity we assume that $\gamma(0) \equiv 0$, that is that $P(0)$ is the Neumann Laplacian on X. We will also omit the parameter t when that is not likely to cause confusion. It is also convenient to use $z = \lambda^2$ and to write $\langle \bullet, \bullet \rangle$ for the $L^2(X, d\,\mathrm{vol}_g)$ inner product and $\langle \bullet, \bullet \rangle_{L^2(\partial X)}$ for the inner product on $L^2(\partial)$ with the measure induced by the metric g.

We first define the following orthogonal projection:

$$\begin{gathered} \mathbb{1}_{r \geq R}\, u := \frac{1}{\ell} \int_0^{\ell} u|_{X_1 \cap \{r \geq R\}}\, d\theta, \qquad \mathbb{1}_{r \geq R} : L^2(X) \to L^2([R, \infty), e^{-r} dr), \quad R > a, \\ \mathbb{1}_{r \leq R} := I - \mathbb{1}_{r \geq R}, \quad \mathcal{H}_R := \mathbb{1}_{r \leq R}\, L^2(X). \end{gathered} \tag{2.6}$$

The smoothness of scattering resonances arising from a smooth perturbation of a simple resonance follows from smooth dependence of the continuation of $(P(t) - \lambda^2)^{-1}$ (see Proposition 4.3 below for a general argument). Let $t \mapsto u(t)$, $u(0) = u$ denote a smooth family of resonant states:

$$\begin{gathered} (P(t) - z(t))u(t) = 0, \quad \frac{1}{\ell} \int_0^{\ell} u(t)|_{X_1} d\theta = a(t) e^{\frac{r}{2}} e^{i\lambda(t) r}, \\ a(0) = 0, \quad \operatorname{Im} \lambda(t) \leq 0, \quad \lambda(0)^2 = z(0). \end{gathered} \tag{2.7}$$

The second equation in (2.7) means that $u(t)$ is *outgoing* – see [8, §4.4].

The self-adjointness of $P(t)$ and integration by parts for the zero mode in the cusp show that for $u = u(t)$ and $P = P(t)$,

$$\begin{aligned} 0 &= \operatorname{Im}\langle (P-z)u, \mathbb{1}_{r\leq R}\, u\rangle \\ &= -\operatorname{Im} \partial_r(\mathbb{1}_{r\geq R}\, u)(R)\overline{\mathbb{1}_{r\geq R}\, u}(R) - \operatorname{Im} z \| \mathbb{1}_{r\leq R}\, u\|^2_{L^2(X)}. \end{aligned} \tag{2.8}$$

(See [8, (4.4.17)] for a detailed presentation in the general black box setting.) Since $\operatorname{Im} \dot z = 0$ (as $\operatorname{Im} z(t) \leq 0$, see also (2.12) below) and since $\mathbb{1}_{r\geq R}\, u(0) = 0$, we have, at $t = 0$, $\operatorname{Im} \ddot z = -2 \operatorname{Im} \partial_r(\mathbb{1}_{r\geq R}\, \dot u)(R)\overline{\mathbb{1}_{r\geq R}\, \dot u}$. We would like to argue as in (2.8) but in reverse. However, as $\dot u$ will not typically be in $\mathcal{D}(P)$ we now obtain boundary terms:

$$\operatorname{Im} \ddot z = 2 \operatorname{Im}\langle (P-z)\dot u, \mathbb{1}_{r\leq R}\, \dot u\rangle + 2 \operatorname{Im}\langle \partial_\nu \dot u, \dot u\rangle_{L^2(\partial X)}. \tag{2.9}$$

We now need an expression for $\dot u$. Since $(P(t) - z(t))u(t) = 0$, $\partial_\nu u|_{\partial X} = \gamma u|_{\partial X}$, we have (at $t = 0$),

$$(P-z)\dot u = \dot z u, \quad \partial_\nu \dot u|_{\partial X} = \dot\gamma u|_{\partial X}. \tag{2.10}$$

In addition, differentiation of the second condition in (2.7) shows that $\dot u$ is outgoing.

Without loss of generality we can assume that $u = u(0)$ is real valued. Choose $g \in \bar C^\infty(X, \mathbb{R})$ (real valued, compactly supported and smooth up to the boundary) such that $\partial_\nu g|_{\partial X} = \dot\gamma u|_{\partial X}$. We claim that

$$\langle \dot z u - (P-z)g, u\rangle = 0. \tag{2.11}$$

In fact, Green's formula shows that the left hand side of (2.11) is equal to $\dot z + \int_{\partial X} \dot\gamma u^2$. On the other hand, using the fact that $\mathbb{1}_{r\leq R}\, u(0) = u(0)$,

$$\begin{aligned} 0 &= -\frac{d}{dt}\langle (P(t)-z(t))u(t), \mathbb{1}_{r\leq R}\, u(t)\rangle|_{t=0} = \langle \dot z u - (P-z)\dot u, u\rangle \\ &= \dot z + \int_{\partial X} \partial_\nu \dot u u = \dot z + \int_{\partial X} \dot\gamma u^2. \end{aligned} \tag{2.12}$$

In view of (2.11), $v := g + R(\lambda)(\dot z u - (P-z)g)$, $\lambda^2 = z$, $\lambda > 0$, is well defined, outgoing (see (2.7)) and solves the boundary value problem (2.10) satisfied by $\dot u$. Since the eigenvalue at z is simple that means that $\dot u - v$ is a multiple of u (see [8, Theorem 4.18] though in this one dimensional case this is particularly simple). Hence

$$\dot u = \alpha u + g + R(\lambda)(\dot z u - (P-z)g). \tag{2.13}$$

With this formula in place we return to (2.9). First we note that the first term on the right hand side vanishes:

$$\begin{aligned} \operatorname{Im}\langle (P-z)\dot u, \mathbb{1}_{r\leq R}\, \dot u\rangle &= \operatorname{Im}\langle \dot z u, \dot u\rangle = \dot z \operatorname{Im}\langle u, \alpha u + g + R(\lambda)(\dot z u - (P-z)g)\rangle \\ &= \dot z \operatorname{Im} \alpha + \dot z \operatorname{Im}\langle u, R(\lambda)(\dot z u - (P-z)g)\rangle \\ &= \dot z \operatorname{Im} \alpha. \end{aligned} \tag{2.14}$$

Here we used the fact that u and g were chosen to be real. The last identity followed from (2.11). To analyse the second term on the right hand side of (2.9) we recall some properties of the Schwartz kernel of the resolvent:

$$R(\lambda)(x,y) = R(\lambda)(y,x) = \overline{R(-\bar{\lambda})(x,y)}, \quad \lambda \in \mathbb{C}. \tag{2.15}$$

(The first property follows from considering $\lambda = ik$, $k \gg 1$, and using the fact that $\overline{Pu} = P\bar{u}$, and the second from considering $\operatorname{Im}\lambda \gg 1$, $z = \lambda^2$, and noting that $((P-z)^{-1})^* = (P-\bar{z})^{-1}$.) Using (2.9),(2.10),(2.14),(2.13),(2.15),(2.12) and the fact that u and g are real, we now see that (with $\lambda = \sqrt{z} > 0$)

$$\begin{aligned} \operatorname{Im}\ddot{z} &= 2\dot{z}\operatorname{Im}\alpha + 2\operatorname{Im}\langle\dot{\gamma}u, \dot{u}\rangle_{L^2(\partial X)} \\ &= 2\dot{z}\operatorname{Im}\alpha + 2\operatorname{Im}\alpha\langle\dot{\gamma}u, u\rangle + 2\operatorname{Im}\langle\dot{\gamma}u, [R(\lambda)(\dot{z}u - (P-z)g)]|_{\partial X}\rangle_{L^2(\partial X)} \\ &= \tfrac{1}{i}\langle\dot{\gamma}u, [(R(\lambda) - R(-\lambda))(\dot{z}u - (P-z)g)]|_{\partial X}\rangle_{L^2(\partial X)}. \end{aligned} \tag{2.16}$$

Since $(R(\lambda) - R(-\lambda))u = 0$ we have now use (2.3) to see that

$$\begin{aligned} [(R(\lambda) - R(-\lambda))(\dot{z}u - (P-z)g)]|_{\partial X} &= -\tfrac{i}{2\lambda}e(\lambda)|_{\partial X}\int_X \overline{e(\lambda)}(P-z)g \\ &= -\tfrac{i}{2\lambda}e(\lambda)|_{\partial X}\int_{\partial X}(\partial_\nu\bar{e}(\lambda)g - \partial_\nu g\overline{e(\lambda)}) \\ &= \tfrac{i}{2\lambda}e(\lambda)|_{\partial X}\langle\dot{\gamma}u, e\rangle_{L^2(\partial X)}. \end{aligned}$$

Inserting this into (2.16) gives (2.4) completing the proof. $\square$

3. Proof of Theorem 1

We follow the same strategy as in the proof of Theorem 3 but with some notational complexity due to the graph structure.

Let $H^2 := \bigoplus_{m=1}^{M+K} H^2([0,\ell_m])$. Then for $u, v \in H^2$, $(\partial_x^k u)_m := \partial_x^k u_m$,

$$\begin{aligned} -\langle\partial_x^2 f, g\rangle_{L^2} &= \langle\partial_x f, \partial_x g\rangle_{L^2} - \sum_v \sum_{e_m \ni v} \partial_\nu f_m(v)\bar{g}_m(v) \\ &= -\langle f, \partial_x^2 g\rangle_{L^2} + \sum_v \sum_{e_m \ni v} \big(f_m(v)\partial_\nu\bar{g}_m(v) - \partial_\nu f_m(v)\bar{g}_m(v)\big). \end{aligned} \tag{3.1}$$

We note here that the sum over vertices can be written as a sum over edges:

$$\sum_v \sum_{e_m \ni v} \partial_\nu f_m(v)\bar{g}_m(v) = \sum_{m=1}^{M+K} \sum_{v \in \partial e_m} \partial_\nu f_m(v)\bar{g}_m(v). \tag{3.2}$$

Just as in §2 the domain of the deformed operators will change but we make a modification which will keep the Hilbert space on which $\widetilde{P}(t)$ (we change the notation from §1 and will use $P(t)$ for a unitarily equivalent operator) acts fixed by changing the lengths in (1.3). For that let

$$L^2_t := \bigoplus_{m=1}^{M+K} L^2([0, e^{-a_m(t)}\ell_m]), \quad L^2 := L^2_0, \quad U(t) : L^2_t \to L^2,$$
$$[U(t)u]_m(y) := e^{-a_m(t)/2} u_m(e^{-a_m(t)}y), \quad U(t)^{-1} = U(t)^*.$$

Let $\widetilde{P}(t)$ be defined in L^2_t by $(\widetilde{P}(t)u)_m = -\partial_x^2 u_m$,

$$\mathcal{D}(\widetilde{P}(t)) = \{u : u_m \in H^2([0, e^{-a_j(t)}\ell_m]),\ u_m(v) = u_\ell(v),\ v \in e_m \cap e_\ell,\ \sum_{e_m \ni v} \partial_\nu u_m(v) = 0\}.$$

That is just the family of Neumann Laplace operators on the graph with the lengths $e^{-a_j(t)}\ell_j$.

On L^2 we define a new family of operators: $P(t) := U(t)\widetilde{P}(t)U(t)^*$. It is explicitly given by $[P(t)u]_m = -e^{2a_m(t)}\partial_x^2 u_m$,

$$\begin{aligned} \mathcal{D}(P(t)) = \{u \in H^2 : e^{a_m(t)/2}u_m(v) = e^{a_\ell(t)/2}u_\ell(v),\ v \in e_m \cap e_\ell, \\ \sum_{e_m \ni v} e^{3a_m(t)/2}\partial_\nu u_m(v) = 0\}. \end{aligned} \tag{3.3}$$

Using Proposition 4.3 from the next section we see that for small t there exists a smooth family $t \mapsto u(t) \in H^2_{\rm loc}$ such that

$$\begin{aligned} (P(t) - z(t))u(t) = 0, \quad u_k(t, x) = a(t)e^{i\lambda(t)x}, \quad 1 \le k \le K, \\ \operatorname{Im}\lambda(t) \le 0, \quad \lambda(0)^2 = z, \quad \lambda(0) > 0. \end{aligned} \tag{3.4}$$

We defined $\mathcal{H}_R$ by (1.5) and denote by $\mathbb{1}_{x\le R}$ the orthogonal projection $L^2 \to \mathcal{H}_R$.

Writing $P = P(t)$, $u = u(t)$, $z = z(t)$ we see, as in (2.8), that

$$0 = \operatorname{Im}\langle (P - z)u, \mathbb{1}_{x\le R}\, u\rangle = -\operatorname{Im}\sum_{m=1}^{K}\partial_x u_m(R)\bar{u}_m(R) - \operatorname{Im} z\|u\|^2_{\mathcal{H}_R}. \tag{3.5}$$

We recall that e_m, $1 \le m \le K$ are the infinite edges with unique boundaries. Hence, using (3.1), at $t = 0$,

$$\begin{aligned} \operatorname{Im}\ddot{z} &= 2\operatorname{Im}\sum_{m=1}^{K}\partial_x \dot{u}_m(R)\overline{\dot{u}}_m(R) \\ &= 2\operatorname{Im}\langle (P - z)\dot{u}, \mathbb{1}_{x\le R}\,\dot{u}\rangle + 2\operatorname{Im}\sum_{v}\sum_{e_m \ni v}\partial_\nu \dot{u}_m(v)\overline{\dot{u}}_m(v). \end{aligned} \tag{3.6}$$

We now look at the equation satisfied by $\dot{u}$ at $t = 0$:

$$\frac{d}{dt}(P(t) - z(t))u(t) = 2\dot{a}(-\partial_x^2 u) - \dot{z}u + (P - z)\dot{u} = (2\dot{a}z - \dot{z})u + (P - z)\dot{u}. \tag{3.7}$$

Hence,

$$(-\partial_x^2 - z)\dot u_m = (\dot z - 2z\dot a_m)u_m, \quad \sum_{e_m\ni v}\partial_\nu \dot u_m(v) = -\tfrac32\sum_{e_m\ni v}\dot a_m\partial_\nu u_m(v), \\ \dot u_m(v) - \dot u_\ell(v) = \tfrac12(\dot a_\ell - \dot a_m)u(v), \quad v\in e_m\cap e_\ell. \tag{3.8}$$

We used here the fact that $u(v) := u_m(v)$ does not depend on m. The second condition can be formulated as $\dot u_m(v) = w(v) - \frac12\dot a_m(v)u(v)$, where $w := \partial_t(e^{a(t)/2}u(t))|_{t=0}$ is continuous on the graph.

To find an expression for $\dot u$ (similar to (2.13)) we first find

$$g\in\bigoplus_{m=1}^{K} C^\infty_{\rm c}([0,\infty))\oplus\bigoplus_{m=K+1}^{M+K} C^\infty([0,\ell_m]),$$

such that

$$\sum_{e_m\ni v}\partial_\nu g_m(v) = -\tfrac32\sum_{e_m\ni v}\dot a_m\partial_\nu u_m(v), \quad g_m(v) - g_\ell(v) = \tfrac12(\dot a_\ell - \dot a_m)u(v). \tag{3.9}$$

We can assume without loss of generality that both g and u are real valued.

In analogy to (2.11) we claim that

$$\langle(\dot z - 2z\dot a)u - (P-z)g, u\rangle = 0. \tag{3.10}$$

In fact, using (3.1), (3.7) and (3.8) we obtain

$$\begin{aligned} 0 &= -\frac{d}{dt}\langle(P(t)-z(t))u(t), \mathbb{1}_{x\le R}\,u(t)\rangle|_{t=0} = \langle\dot z u - 2z\dot a u - (P-z)\dot u, u\rangle \\ &= \dot z - 2z\langle\dot a u, u\rangle + \sum_v\sum_{e_m\ni v}(\partial_\nu\dot u_m(v)u(v) - \dot u_m(v)\partial_\nu u_m(v)) \\ &= \dot z - 2z\langle\dot a u, u\rangle + \sum_v\sum_{e_m\ni v}(-\tfrac32\dot a_m\partial_\nu u_m(v)u(v) - (w(v) - \tfrac12\dot a_m u(v))\partial_\nu u_m(v)) \\ &= \dot z - 2z\langle\dot a u, u\rangle - \sum_v\sum_{e_m\ni v}\dot a_m\partial_\nu u_m(v)u(v). \end{aligned} \tag{3.11}$$

(We used the continuity of u and the Neumann condition $\sum_{e_m\ni v}\partial_\nu u_m(v) = 0$.) Since g and u satisfy the same boundary conditions (3.8) and (3.9) (3.10) follows from (3.11).

As in the derivation of (2.13) we now see that for some $\alpha\in\mathbb{C}$ we have

$$\dot u = \alpha u + g + R(\lambda)(\dot z u - 2z\dot a u - (P-z)g). \tag{3.12}$$

With this in place we return to (3.6). The first term on the right hand side is

$$\begin{aligned} 2\operatorname{Im}\langle(P-z)\dot u, \mathbb{1}_{x\le R}\,\dot u\rangle &= 2\operatorname{Im}\langle\dot z u - 2z\dot a u, \dot u\rangle \\ &= 2\operatorname{Im}\langle\dot z u - 2z\dot a u, \alpha u + g + R(\lambda)(\dot z u - 2z\dot a u - (P-z)g)\rangle \\ &= 2\operatorname{Im}\alpha(\dot z - 2z\langle\dot a u, u\rangle) \\ &\qquad - 4z\operatorname{Im}\langle\dot a u, R(\lambda)(\dot z u - 2z\dot a u - (P-z)g)\rangle. \end{aligned}$$

(We used here the simplifying assumption that g and u are real valued.)

As in (2.16) we conclude that

$$4z\,\mathrm{Im}\langle \dot a u, R(\lambda)(-\dot z u + 2z\dot a u + (P-z)g)\rangle = \tfrac{2z}{i}\langle \dot a u, [(R(\lambda) - R(-\lambda)](2z\dot a u + (P-z)g)\rangle$$

Now, as in (2.3), [8, Theorem 4.20] shows that

$$(R(\lambda) - R(-\lambda))f = \frac{i}{2\lambda}\sum_{k=1}^{K} e^k(\lambda, x)\langle f, e^k(\lambda, \bullet)\rangle, \quad \lambda \in \mathbb{R}, \quad f \in \mathcal{H}_R, \tag{3.13}$$

which means that (with $z = \lambda^2$ and $e^k = e^k(\lambda)$)

$$\begin{aligned} &\tfrac{2z}{i}\langle \dot a u, [(R(\lambda) - R(-\lambda)](2z\dot a u + (P-z)g)\rangle = \\ &\quad -\lambda\sum_{k=1}^{K}\langle \dot a u, e^k\rangle\overline{\langle 2\lambda^2\dot a u + (P-z)g, e^k\rangle} = \\ &\quad -2\lambda^3\sum_{k=1}^{K}|\langle \dot a u, e^k\rangle|^2 - \lambda\sum_{k=1}^{K}\langle \dot a u, e^k\rangle\langle e^k, (P-z)g\rangle. \end{aligned}$$

The second term on the right hand side is now rewritten using (3.1) and the boundary conditions (3.9):

$$\begin{aligned} &\lambda\sum_{k=1}^{K}\langle \dot a u, e^k\rangle\left(\sum_{v}\sum_{e_m\ni v}(\partial_\nu e^k_m(v)g_m(v) - \partial_\nu g_m(v)e^k(v))\right) = \\ &\quad \lambda\sum_{k=1}^{K}\langle \dot a u, e^k\rangle\left(\sum_{v}\sum_{e_m\ni v}\tfrac12\dot a_m(-\partial_\nu e^k_m(v)u(v) + 3\partial_\nu u_m(v)e^k(v))\right). \end{aligned}$$

We conclude that

$$\begin{aligned} 2\,\mathrm{Im}\langle (P-z)\dot u, \mathbb{1}_{x\le R}\,\dot u\rangle &= 2\,\mathrm{Im}\,\alpha(\dot z - 2z\langle \dot a u, u\rangle) - 2\lambda^3\sum_{k=1}^{K}|\langle \dot a u, e^k\rangle|^2 \\ &\quad - 2\lambda\sum_{k=1}^{K}\langle \dot a u, e^k\rangle\left(\sum_{v}\sum_{e_m\ni v}\tfrac14\dot a_m(3\partial_\nu u_m(v)e^k(v) - \partial_\nu e^k_m(v)u(v))\right). \end{aligned} \tag{3.14}$$

A similar analysis of the second term on the right hand side of (2.9) shows that

$$\begin{aligned} 2\,\mathrm{Im}\sum_{v}\sum_{e_m\ni v}\partial_\nu \dot u_m(v)\bar{\dot u}_m(v) &= \mathrm{Im}\,\alpha\left(2\sum_{v}\sum_{e_m\ni v}\dot a_m\partial_\nu u_m(v)u(v)\right) \\ &\quad - 2\lambda^{-1}\sum_{k=1}^{K}\left|\sum_{v}\sum_{e_m\ni v}\tfrac14\dot a_m(\partial_\nu e^k_m(v)u(v) - 3\partial_\nu u_m(v)e^k(v))\right|^2 \\ &\quad - 2\lambda\sum_{k=1}^{K}\overline{\langle \dot a u, e^k\rangle}\left(\sum_{v}\sum_{e_m\ni v}\tfrac14\dot a_m(3\partial_\nu u_m(v)\bar e^k(v) - \partial_\nu \bar e^k_m(v)u(v))\right). \end{aligned} \tag{3.15}$$

Inserting (3.14),(3.15) into (3.6), using (3.11) and $\operatorname{Im} \ddot{z} = 2\lambda \operatorname{Im} \ddot{\lambda}$ gives (1.4). ☐

Example 1. Consider a connected graph with M bonds and K leads. Suppose that an embedded eigenvalue λ is simple and satisfies

$$\lambda \ell_m \in \pi \mathbb{Z}, \ m = K+1, \cdots, M+K. \tag{3.16}$$

Then

$$\operatorname{Im} \ddot{\lambda} = -\sum_{k=1}^{K} \left| \lambda \langle \dot{a} u, e^k(\lambda) \rangle \right|^2 . \tag{3.17}$$

Proof. $u_m(x) = C_m \sin(\lambda x)$ where e_m and a lead are meeting at a vertex. Since the graph is connected, $u_m(x) = C_m \sin(\lambda x)$ for $K+1 \leq m \leq M+K$. Let $n_m = \frac{\lambda \ell_m}{\pi}$ and let

$$e^k_m(\lambda, x) = A_{mk} \sin(\lambda x) + B_{mk} \cos(\lambda x).$$

Then $u_m(0) = u_m(\ell_m) = 0$ and

$$\partial_\nu u_m(0) = (-1)^{n_m+1} \partial_\nu u_m(\ell_m), \ e^k_m(\lambda, \ell_m) = (-1)^{n_m} e^k_m(\lambda, 0).$$

We can use this and (3.2) to reduce F_k in (1.4) to

$$\begin{aligned} F_k &= \lambda \langle \dot{a} u, e^k(\lambda) \rangle + \lambda^{-1} \sum_{m=K+1}^{M+K} \tfrac{3}{4} \dot{a}_m (\partial_\nu u_m(0) \overline{e^k(\lambda, 0)} + \partial_\nu u_m(\ell_m) \overline{e^k(\lambda, \ell_m)}) \\ &= \lambda \langle \dot{a} u, e^k(\lambda) \rangle. \end{aligned}$$

Theorem 1 then gives (3.17). ☐

Example 2. Let us consider a graph with $M = 5, K = 2$ and four vertices: see in Fig. 4. Let $\ell_m(0) = 1, \ 3 \leq m \leq 7$. Then the sequence of embedded eigenvalues λ is given as $S_1 \cup S_2$ where

$$S_1 = \pi \mathbb{Z}, \quad S_2 = \left\{ \lambda \ : \ \tan \lambda + 2 \tan \tfrac{\lambda}{2} = 0, \quad \lambda \notin \tfrac{\pi}{2} \mathbb{Z} \right\}.$$

If $\lambda \in S_1$, then (3.16) is satisfied. If $\lambda \in S_2$, however, we have (with v_1 and v_2 corresponding to $x = 0$ for e_3, e_6 and e_4, e_5 respectively, and v_4 to $x = 0$ for e_7)

$$u_3(x) = C \sin(\lambda x), \quad u_4(x) = C \sin(\lambda x), \quad u_5(x) = -C \sin(\lambda x),$$

$$u_6(x) = -C \sin(\lambda x), \quad u_7(x) = C \frac{\sin \lambda}{\sin \frac{\lambda}{2}} \sin \left(\lambda \left(x - \tfrac{1}{2} \right) \right),$$

where $C > 0$ is the normalization constant. Note that

$$u(v_3) = C \sin \lambda \neq 0, \quad u(v_4) = -C \sin \lambda \neq 0.$$

So we do not have the simple formula (3.17) in this case.

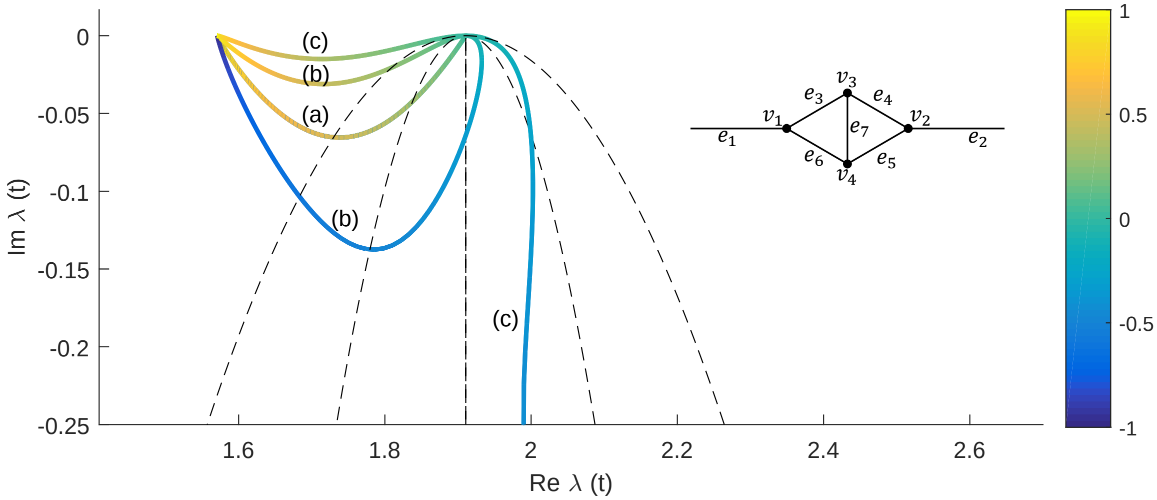

FIGURE 4. The graph from Example 2: in this case boundary terms in our Fermi golden rule appear at some embedded eigenvalues such as λ_0 which is the smallest solution of $\tan \lambda + 2 \tan \frac{\lambda}{2} = 0$, $\lambda_0 \approx 1.9106$. We consider the following variation of length: $\ell_3 = 1 - t, \ell_4 = 1 + t, \ell_5 = 1 - t, \ell_6 = 1 + t$, and (a): $\ell_7 = 1$ (b): $\ell_7 = 1 + t/2$ (c): $\ell_7 = 1 + t$. As pointed out by the referee, the approximation $\lambda(t) \simeq \lambda(0) + t\dot{\lambda} + \frac{i}{2} t^2 \operatorname{Im} \ddot{\lambda}$ is not as accurate as in Fig.2 since now $\operatorname{Re} \ddot{\lambda} \neq 0$. The main point is to illustrate the appearance of the boundary contributions to F_k in (1.4).

4. Proof of Theorem 2

The proof adapts to the setting of quantum graphs and of quasimodes u satisfying (1.6) the arguments of [20]. They have origins in the classical work of Carleman [3] on completeness of eigenfunctions for classes non-self-adjoint operators, see also [17] and [19].

We start with general results which are a version of the arguments of [8, §7.2]. In particular they apply without modification to quantum graphs with general Hamiltonians and general boundary conditions. We note that for metric graphs considered here much more precise estimates are obtained by Davies–Pushnitski [6] and Davies–Exner–Lipovský [7] but since we want uniformity we present an argument illustrating the black box point of view.

Proposition 4.1. *Suppose that P satisfies the assumptions of Theorem 2 and $\Omega_1 \Subset \Omega_2 \Subset \mathbb{C}$, where Ω_j are open sets.*

Then there exist constants C_1 depending only on Ω_2 and $\mathcal{L}$, and C_2 depending on Ω_1, Ω_2, R and $\mathcal{L}$ such that

$$
\begin{gathered}
|\operatorname{Res}(P) \cap \Omega_2| \leq C_1, \\
\| \mathbb{1}_{r\leq R}\, R(\lambda)\, \mathbb{1}_{r\leq R} \|_{L^2\to L^2} \leq C_2 \prod_{\zeta\in \operatorname{Res}(P)\cap\Omega_2} |\lambda-\zeta|^{-1}, \quad \lambda\in\Omega_1,
\end{gathered} \tag{4.1}
$$

where the elements of $\operatorname{Res}(P)$ *are included according to their multiplicities.*

Proof. Let $R_0(\lambda) : \bigoplus_{k=1}^K L^2_{\rm comp}(e_k) \to \bigoplus H^2_{\rm loc} \cap H^1_{0,{\rm loc}}(e_k)$, be defined as the diagonal operator acting on each component as $R_0^0(\lambda)$, the Dirichlet resolvent on $L^2_{\rm comp}([0,\infty))$ continued *analytically* to all of $\mathbb{C}$:

$$
R_0^0(\lambda) f(x) = \int_0^\infty \frac{e^{i\lambda(x+y)} - e^{i\lambda|x-y|}}{2i\lambda} f(y)dy.
$$

To describe $\mathbb{1}_{r\leq R}\, R(\lambda)\, \mathbb{1}_{r\leq R}$ we follow the general argument of [15] (see also [8, §4.2,4.3]). For that we choose $\chi_j \in C^\infty_{\rm c}$, $j = 0, \cdots, 3$ to be equal to 1 on all edges and to satisfy

$$
\chi_j|_{e_k} \in C^\infty_{\rm c}([0,2R)), \quad \chi_0|_{e_k}(x) = 1, \ x\leq R, \quad \chi_j|_{e_k}(x) = 1, \ x\in \operatorname{supp}\chi_{j-1}|_{e_k},
$$

for $k = 1, \cdots, K$. For λ_0 with $\operatorname{Im}\lambda_0 > 0$, we define

$$
Q(\lambda,\lambda_0) := (1-\chi_0)R_0(\lambda)(1-\chi_1) + \chi_2 R(\lambda_0)\chi_1, \quad Q(\lambda,\lambda_0) : L^2_{\rm comp} \to \mathcal{D}_{\rm loc}(P).
$$

Then

$$
\begin{gathered}
(P-\lambda^2)Q(\lambda,\lambda_0) = I + K(\lambda,\lambda_0), \\
K_0(\lambda,\lambda_0) := -[P,\chi_0]R_0(\lambda)(1-\chi_1) + (\lambda_0^2-\lambda^2)\chi_2 R(\lambda_0)\chi_1 + [P,\chi_2]R(\lambda_0)\chi_1.
\end{gathered}
$$

We now choose $\lambda_0 = e^{\pi i/4}\mu$, $\mu \gg 1$. Then

$$
I + K_0(\lambda_0,\lambda_0) \ \text{ and } \ I + K_0(\lambda_0,\lambda_0)\chi_3 \ \text{ are invertible on } \ L^2, \tag{4.2}
$$

$K(\lambda,\lambda_0)\chi_3$ is compact, and

$$
R(\lambda) = Q(\lambda,\lambda_0)(I + K_0(\lambda,\lambda_0)\chi_3)^{-1}(I - K_0(\lambda,\lambda_0)(1-\chi_3)), \tag{4.3}
$$

where $\lambda \mapsto (I + K_0(\lambda,\lambda_0)\chi_3)^{-1}$ is a meromorphic family of operators. We now put

$$
K(\lambda,\lambda_0) := K_0(\lambda,\lambda_0)\chi_3
$$

and conclude that

$$
\mathbb{1}_{r\leq R}\, R(\lambda)\, \mathbb{1}_{r\leq R} = \mathbb{1}_{r\leq R}\, Q(\lambda,\lambda_0)\chi_3(I + K(\lambda,\lambda_0))^{-1}\, \mathbb{1}_{r\leq R}, \tag{4.4}
$$

and the set of resonances is given by the poles of $(I + K(\lambda,\lambda_0))^{-1}$. (See [8, §4.2] and in particular [8, (4.2.19)].)

We now claim that $K(\lambda,\lambda_0)$ is of trace class for $\lambda \in \mathbb{C}$ and that for a any compact subset $\Omega \Subset \mathbb{C}$ there exists a constant C_3 depending only on Ω, $\mathcal{L}$ and λ_0 such that

$$
\|K(\lambda,\lambda_0)\|_{\rm tr} \leq C_3. \tag{4.5}
$$

To see this, let $\widetilde{P}$ be the operator of $\mathcal{H}_{3R}$ where we put, say the Neumann boundary condition at $3R$ on each infinite lead. Let $\widetilde{P}_{\min}, \widetilde{P}_{\max}$ be the same operators but on metric graphs were all the length $\ell_j \in \mathcal{L}$, $K+1 \leq j \leq K+M$ were replaced by $\ell_{\min} := \min \mathcal{L}$ and $\ell_{\max} := \max \mathcal{L}$ respectively. These operators have discrete spectra and the ordered eigenvalues of these operators satisfy

$$\lambda_p(\widetilde{P}_{\max}) \leq \lambda_p(\widetilde{P}) \leq \lambda_p(\widetilde{P}_{\min}). \tag{4.6}$$

This is a consequence of the following lemma:

Lemma 4.2. *Suppose that the unbounded operator $\widetilde{P}_k(t) : \mathcal{H}_\rho \to \mathcal{H}_\rho$, $\rho > 0$, with edge length given by*

$$\ell_k(t) = \rho,\ 1 \leq k \leq K, \quad \ell_m(t) = e^{-\delta_{mk} t} \ell_m, \quad K+1 \leq m \leq M+K,$$

and

$$\mathcal{D}(\widetilde{P}_k(t)) = \{u : u_m \in H^2([0, \ell_m(t)]),\ u_m(v) = u_\ell(v),\ v \in e_m \cap e_\ell,\ \sum_{e_m \ni v} \partial_\nu u_m(v) = 0\}.$$

If $0 = \mu_0(t) \leq \mu_1(t) \leq \mu_2(t) \cdots$, is the ordered sequence of eigenvalues of $P_k(t)$, then $\mu_p(t)$ is a non-decreasing function of t.

Proof. From [2, Theorem 3.10] we know that if μ is an eigenvalue of $P(s)$ of multiplicity N then we can choose analytic functions $\mu^n(t) \in \mathbb{R}$, $u^n(t) \in \mathcal{D}(P(s))$, such that $\mu^n(s) = \mu$, and for small $t-s$, $P(t)u^n(t) = \mu^n(t)u^n(t)$, and $\{u^n(t)\}_{n=1}^N$ is an orthonormal setting spanning $\mathbb{1}_{|P(t)-\mu| \leq \varepsilon} L^2$, for $\varepsilon > 0$ small enough. The lemma follows from showing that $\partial_t \mu^n(s) \geq 0$ for any n.

Without loss of generality we can assume that $s = 0$. We can then use the same calculation as in (3.11) with $z = \mu^n(0)$, $a_m(t) = \delta_{km} t$ and $u = u^n(0)$. That gives

$$\mu_p'(0) = 2\mu_p(0) \langle u, u \rangle_{L^2(e_k)} + \sum_{v \in \partial e_k} \partial_\nu u_k(v) u_k(v).$$

Since $u_k(x) = a \sin \sqrt{\mu_p} x + b \cos \sqrt{\mu_p} x$, for some $a, b \in \mathbb{R}$, a calculation shows that

$$\mu_p'(0) = \mu_p(0) \ell_k (a^2 + b^2) \geq 0,$$

completing the proof. ∎

The inequality (4.6) follows from the lemma as we can change the length of the edges in succession. The Weyl law for $\widetilde{P}$ (see [1]) and the fact that $\widetilde{P}\chi_3 = P\chi_3$ (where χ_3 denotes the multiplication operator), now shows that for any operator $A : L^2 \to \mathcal{D}(P)$,

$$\|\chi_3 A \chi_3\|_{\mathrm{tr}} \leq C_4 \|P \chi_3 A \chi_3\| + C_4 \|\chi_3 A \chi_3\|,$$

where the constant C_4 depends only on $\mathcal{L}$. From this we deduce (4.5) and $\| \bullet \| = \| \bullet \|_{L^2 \to L^2}$. For instance,

$$\begin{aligned}\|[P, \chi_2] R(\lambda_0) \chi_1\|_{\rm tr} &\leq C_4 \|P [P, \chi_2] R(\lambda_0) \chi_1\| + C_4 \|[P, \chi_2] R(\lambda_0) \chi_1\| \\ &= C_4 \|[P, [P, \chi_2]] R(\lambda_0) \chi_1\| + C_4 (1 + |\lambda_0|^2) \|[P, \chi_2] R(\lambda_0) \chi_1\| \\ &\leq C_5.\end{aligned}$$

Here we used the facts that $\chi_2 \equiv 1$ on the support of χ_1, hence $[P, \chi_2]\chi_1 = 0$, and that $[P, [P, \chi_2]]$ $[P, \chi_2]$ are second and first order operators respectively and that $R(\lambda_0)$ maps L^2 to $\mathcal{D}(P)$. The other terms in $K(\lambda, \lambda_0)$ are estimated similarly and that gives (4.5). (Finer estimates for large λ are possible – see [8, §4.3, §7.2] and [20]– but we concentrate here on uniformity near a given energy.)

Now, let $\Omega_3 = \{\lambda : |\lambda - \lambda_0| < R$ where R is large enough so that $\Omega_2 \subset \Omega_3$. It follows that for a constant C_3 depending only on Ω_3 and $\mathcal{L}$, (and hence only on Ω_2), we have

$$|\det(I + K(\lambda, \lambda_0)| \leq e^{C_3}. \tag{4.7}$$

(For basic facts about determinants see for instance [8, §B.5].) Writing

$$(I + K(\lambda_0, \lambda_0))^{-1} = (I - (I + K(\lambda_0, \lambda_0))^{-1} K(\lambda_0, \lambda_0))$$

we obtain

$$\begin{aligned}|\det(I + K(\lambda_0, \lambda_0))|^{-1} &= |\det(I + K(\lambda_0, \lambda_0)^{-1}| \\ &\leq \exp\left(\|(I + K(\lambda_0, \lambda_0))^{-1}\| \|K(\lambda_0, \lambda_0)\|_{\rm tr}\right) \leq e^{C_4},\end{aligned}$$

that is

$$|\det(I + K(\lambda_0, \lambda_0))| \geq e^{-C_4}, \tag{4.8}$$

where C_4 depends only on λ_0 and $\mathcal{L}$. The Jensen formula (see for instance [21, §3.61]) then gives a bound on the number of zeros of $\det(I + K(\lambda, \lambda_0))$ in Ω_3. That proves the first bound in (4.1).

We can write

$$\det(I + K(\lambda_0, \lambda)) = e^{g(\lambda)} \prod_{\zeta \in \mathrm{Res}(P) \cap \Omega_3} (\lambda - \zeta),$$

where $g(\lambda)$ is holomorphic in Ω_3. From the upper bound (4.7) and the lower bound (4.8) we conclude that $|g(\lambda)| \leq C_5$ in a smaller disc containing Ω_2, with C_5 depending only on the previous constants. (For instance we can use the Borel–Carathéodory inequality – see [21, §5.5].) Hence

$$|\det(I + K(\lambda_0, \lambda)| \geq e^{-C_6} \prod_{\zeta \in \mathrm{Res}(P) \cap \Omega_2} |\lambda - \zeta|, \quad \lambda \in \Omega_1,$$

To deduce the the second bound in (4.1) from this we use the inequality

$$\|(I + A)^{-1}\| \leq \frac{\det(I + |A|)}{|\det(I + A)|}$$

which gives

$$\begin{aligned}\| \mathbb{1}_{r\leq R} R(\lambda) \mathbb{1}_{r\leq R} \| &= \| \mathbb{1}_{r\leq R} Q(\lambda,\lambda_0)\chi_3(I+K(\lambda,\lambda_0))^{-1} \mathbb{1}_{r\leq R} \| \\ &\leq \| \mathbb{1}_{r\leq R} Q(\lambda,\lambda_0)\chi_3 \| |\det(I+|K(\lambda,\lambda_0))| |\det(I+K(\lambda,\lambda_0))|^{-1} \\ &\leq C_7 e^{\|K(\lambda,\lambda_0)\|_{\rm tr}} \prod_{\zeta\in \mathrm{Res}(P)\cap\Omega_2} |\lambda-\zeta|^{-1},\end{aligned}$$

for $\lambda \in \Omega_1$ and C_7 depending only on Ω_j's $\mathcal{L}$, and R. This completes the proof. □

Before proving Theorem 2 we will use the construction of the meromorphic continuation in the proof of Proposition 4.1 to give a general condition for smoothness of a family of resonances (see also [16]):

$$(P(t)-\lambda_0^2)^{-1} \in C^\infty((-t_0,t_0);\mathcal{L}(L^2,L^2)), \quad \operatorname{Im}\lambda_0 > 0. \tag{4.9}$$

That is the only property used in the proof of

Proposition 4.3. *Let $P(t)$ be the family of unbounded operators on L^2 (of a fixed metric graph) defined by (3.3). Let $R(\lambda,t)$ be the resolvent of $P(t)$ meromorphically continued to $\mathbb{C}$. Suppose that γ is a smooth Jordan curve such that $R(\lambda,t)$ has no poles on γ for $|t|<t_0$. Then for $\chi_j \in C^\infty_{\rm c}$, $j=1,2$,*

$$\int_\gamma \chi_1 R(\zeta,t)\chi_2 d\zeta \in C^\infty((-t_0,t_0);\mathcal{L}(L^2,L^2)). \tag{4.10}$$

In particular, if λ_0 is a simple pole of $R(\lambda,0)$ then there exist smooth families $t\mapsto\lambda(t)$ and $t\mapsto u(t)\in\mathcal{D}_{\rm loc}(P(t))$ such that $\lambda(0)=\lambda_0$, $\lambda(t)\in\mathrm{Res}(P(t))$ and $u(t)$ is a resonant state of $P(t)$ corresponding to $\lambda(t)$.

Proof. The proof of (4.10) under the condition (4.9) follows from (4.4) and the definitions of $Q(\lambda,\lambda_0)$ and $K(\lambda,\lambda_0)$. From that the conclusion about the deformation of a simple resonance is immediate – see [8, Theorems 4.7,4.9].

It remains to establish (4.9). Suppose $f\in L^2$ and define $u(t) := R(\lambda_0,t)f \in L^2$. Formally, $\dot u := \partial_t u(t)$ satisfies (3.8) with $\dot z = 0$ and $z=\lambda_0^2$. We can find a smooth family $g(t)\in L^2$ satisfying (3.9) with $u=u(t)$. We then have $\partial_t u(t) = g + R(\lambda_0,t)(-2\partial_t a(t)u(t) - G(t))$, where $G_m := (-e^{-2a(t)}\partial_x^2 - \lambda_0^2)g_m(t)$. By considering difference quotients a similar argument shows that $u(t)\in L^2$ is differentiable. The argument can be iterated showing that $u(t)\in C^\infty((-t_0,t_0),L^2)$ and that proves (4.9). □

We now give

Proof of Theorem 2. We proceed by contradiction by assuming that, for $0<\delta\ll\rho\ll 1$ to be chosen,

$$\mathrm{Res}(P)\cap(\Omega(\rho,\delta)+D(0,\delta)) = \emptyset, \quad \Omega(\rho,\delta) := [\lambda_0-\rho,\lambda_0+\rho] - i[0,\delta]$$

does not contain any resonances. Choosing pre-compact open sets, independent of ε, ρ and δ, $\Omega(\rho,\delta) + D(0,\delta) \Subset \Omega_1 \Subset \Omega_2$ we apply Proposition 4.1 to see that for

$$\| \mathbb{1}_{r\leq R} R(\lambda) \mathbb{1}_{r\leq R} \| \leq C_2 \delta^{-C_1}, \quad \lambda \in \Omega(\rho,\delta). \tag{4.11}$$

On the other hand, the resolvent estimate in the physical half-plane $\operatorname{Im}\lambda > 0$ and the fact that $\lambda_0 \in I \Subset (0,\infty)$, give

$$\| \mathbb{1}_{r\leq R} R(\lambda) \mathbb{1}_{r\leq R} \| \leq C_3 / \operatorname{Im}\lambda, \quad \operatorname{Im}\lambda > 0, \quad |\operatorname{Re}\lambda - \operatorname{Re}\lambda_0| < \rho. \tag{4.12}$$

To derive a contradiction we use the following simple lemma:

Lemma 4.4. *Suppose that $f(z)$ is holomorphic in a neighbourhood of $\Omega := [-\rho,\rho] + i[-\delta_-,\delta_+]$, $\delta_\pm > 0$. Suppose that, for $M > 1$, $M_\pm > 0$, and $0 < \delta_+ \leq \delta_- < 1$,*

$$|f(z)| \leq M_\pm, \quad \operatorname{Im} z = \pm\delta_+, \quad |\operatorname{Re} z| \leq \rho, \qquad |f(z)| \leq M, \quad z \in \Omega. \tag{4.13}$$

and that $\rho^2 > (1 + 2\log M)\delta_-^2$. Then

$$|f(0)| \leq e M_+^\theta M_-^{1-\theta}, \quad \theta := \frac{\delta_-}{\delta_+ + \delta_-}. \tag{4.14}$$

Proof. We consider the following subharmonic function defined in a neighbourhood of Ω. To define it we put $m_\pm = \log M_\pm$, $m = \log M > 0$, $z = x + iy$, and

$$u(z) := \log|f(x+iy)| - \frac{\delta_- m_+ + \delta_+ m_- + y(m_+ - m_-)}{\delta_+ + \delta_-} - Kx^2 + Ky^2,$$

where $K := 2m/(\rho^2 - \delta_-^2)$. Then for $\operatorname{Im} z = \pm\delta_\pm$, $|\operatorname{Re} z| \leq \rho$, $u(z) \leq \delta_-^2 K \leq 1$ since we assumed $\rho^2 > (1+2m)\delta_-^2$). When $|\operatorname{Re} z| = \rho$ then $u(z) \leq 2m - K(\rho^2 - \delta_-^2) \leq 0$. The maximum principle for subharmonic functions shows that $\log|f(0)| - \theta m_+ - (1-\theta)m_- \leq 1$ and that concludes the proof. ☐

We apply this lemma to $f(z) := \langle \mathbb{1}_{r\leq R} R(z+\lambda_0) \mathbb{1}_{r\leq R} \varphi, \psi \rangle$, $\varphi, \psi \in L^2$, with $M_+ = C_3/\delta_+$, $M = M_- = C_2\delta^{-C_1}$. If we show that

$$|f(0)| \ll \frac{1}{\varepsilon} \|\varphi\| \|\psi\|, \tag{4.15}$$

we obtain a contradiction to (1.6) by putting $\varphi = (P - \lambda_0^2)u$ and $\psi = u$ and using the support property of u (the outgoing resolvent is the right inverse of $P - \lambda_0^2$ on compactly supported function):

$$1 = \langle R(\lambda_0)(P-\lambda_0^2)u, u\rangle = \langle \mathbb{1}_{r\leq R} R(\lambda_0) \mathbb{1}_{r\leq R} (P-\lambda_0^2)u, u \rangle \ll \frac{1}{\varepsilon}\varepsilon \ll 1.$$

For $\gamma < 1$ choose $\gamma < \gamma_1 < \gamma_2 < \gamma_3 < 1$ and put

$$\rho = \varepsilon^{\gamma_1}, \quad \delta_- = \varepsilon^{\gamma_2}, \quad \delta_+ = \varepsilon^{\gamma_3}.$$

Then (4.14) implies (4.15) and that completes the proof. ☐

For completeness we also include the following proposition which would be a converse to Theorem 2 for $\gamma = 1$. The more subtle higher dimensional version in the semiclassical setting was given by Stefanov [18].

Proposition 4.5. *Suppose that P satisfies the assumptions of Theorem 2 and let $R > 0, \delta > 0$. There exists a constant C_0 depending only of R, δ and $\mathcal{L}$ such that for any $0 < \varepsilon < \delta/2$,*

$$\begin{aligned} & D(\lambda_0, \varepsilon) \cap \operatorname{Res}(P) \neq \emptyset, \ \lambda_0 > \delta \implies \\ & \qquad \exists\, u \in \mathcal{H}_R \cap \mathcal{D}_P, \ \|u\| = 1, \quad \|(P - \lambda_0^2)u\| \leq C_0 \varepsilon (\lambda_0 + \varepsilon). \end{aligned} \tag{4.16}$$

Proof. Suppose that λ a resonance of P with $|\lambda - \lambda_0| < \varepsilon$ and let v be the corresponding resonant state. Then in each infinite lead, $v_m(x) = a_m e^{i\lambda x}$, $1 \leq m \leq K$. As in (3.5),

$$\begin{aligned} \operatorname{Im}(\lambda^2)\|v\|_{\mathcal{H}_0}^2 &= -\operatorname{Im} \sum_{m=1}^{K} \partial_x v_m(0) \bar{v}_m(0) = -\operatorname{Im} \sum_{m=1}^{K} i\lambda |a_m|^2 \\ &= -\operatorname{Re} \lambda \sum_{m=1}^{K} |a_m|^2. \end{aligned}$$

Since $\operatorname{Re} \lambda \neq 0$, it follows that $\sum_{m=1}^{K} |a_m|^2 = 2|\operatorname{Im} \lambda| \|v\|_{\mathcal{H}_0}^2 \leq 2\varepsilon \|v\|_{\mathcal{H}_0}^2$.

Suppose $r < R/2$ and $\chi \in C_c^\infty([0, 2))$ is equal to 1 on $[0, 1]$. We then define $\widetilde{u} \in \mathcal{H}_R \cap \mathcal{D}_P$ by

$$\widetilde{u}_m(x) := \begin{cases} \chi(x/r) v_m(x), & 1 \leq m \leq K \\ v_m(x) & K + 1 \leq m \leq K + M. \end{cases}$$

Now,

$$\|\widetilde{u}\|^2 = \|v\|_{\mathcal{H}_0}^2 + \sum_{m=1}^{K} |a_m|^2 \int_{\mathbb{R}} e^{2|\operatorname{Im} \lambda| x} \chi(x/r)^2 dx = \|v\|_{\mathcal{H}_0}^2 (1 + \mathcal{O}(\varepsilon r e^{2\varepsilon r})),$$

and hence,

$$\begin{aligned} \|(P - \lambda_0^2)\widetilde{u}\|^2 &= |\lambda^2 - \lambda_0^2|^2 \|\widetilde{u}\|^2 + \|[P, \chi(\bullet/r)]\widetilde{u}\|^2 \\ &\leq (2\varepsilon(\lambda_0 + \varepsilon))^2 \|\widetilde{u}\|^2 + C \sum_{m=1}^{K} |a_m|^2 (r^{-2} + (\lambda_0 + \varepsilon)^2) e^{2\varepsilon r} \\ &\leq C_{r,\delta} \varepsilon^2 (\lambda_0 + \varepsilon)^2 \|v\|_{\mathcal{H}_0}^2. \end{aligned}$$

We conclude that we can take $u := \tilde{u}/\|\tilde{u}\|$ as the quasimode. ☐

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