SHARP POLYNOMIAL BOUNDS ON THE NUMBER OF POLLICOTT-RUELLE RESONANCES

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Abstract. We give a sharp polynomial bound on the number of Pollicott-Ruelle resonances. These resonances, which are complex numbers in the lower half-plane, appear in expansions of correlations for Anosov contact flows. The bounds follow the tradition of upper bounds on the number of scattering resonances and improve a recent bound of Faure–Sjöstrand. The complex scaling method used in scattering theory is replaced by an approach using exponentially weighted spaces introduced by Helffer–Sjöstrand in scattering theory and by Faure–Sjöstrand in the theory of Anosov flows.

1. Introduction and statement of the results

Pollicott-Ruelle resonances appear in correlation expansions for certain chaotic dynamical systems [Po, Ru]. Recently Faure, Roy, and Sjöstrand [FaRoSj, FaSj] explained how some aspects of Anosov dynamics can be analyzed using microlocal methods of scattering theory. As an application of that point of view Faure and Sjöstrand [FaSj] proved the following polynomial upper bound for the number of Pollicott-Ruelle resonances, denoted $\mathrm{Res}(-iV)$, of a contact Anosov flow, $\varphi_t = \exp tV$, on a n-dimensional compact smooth manifold:

$$\#\{\lambda \in \mathrm{Res}(-iV) \ : \ |\operatorname{Re}\lambda - E| \leq \sqrt{E}, \ \ \operatorname{Im}\lambda > -\beta\} = o(E^{n-\frac{1}{2}}), \tag{1.1}$$

for any β. See [FaSj, Theorem 1.8] for a detailed statement.

In this paper we develop their approach further using recent advances in resonance counting [DaDy, NoSjZw, SjZw]. This gives the following improvement of (1.1)

$$\#\{\lambda \in \mathrm{Res}(-iV) \ : \ |\operatorname{Re}\lambda - E| \leq \sqrt{E}, \ \ \operatorname{Im}\lambda > -\beta\} = \mathcal{O}(E^{\frac{n}{2}}). \tag{1.2}$$

This estimate a consequence of an optimal bound holding in smaller energy intervals (1.5).

We briefly review the setting referring to [FaSj, §1.1] and [BaTs] for more details and numerous references to earlier works, in particular in the dynamical systems literature.

Let X be a compact smooth manifold of odd dimension $n \geq 3$, and let $\varphi_t : X \to X$ be an *Anosov flow* on X. We assume that there exists a *contact form* $\alpha \in C^\infty(X, T^*X)$ compatible with that flow. This means that for $E_u(x), E_s(x) \subset T_xX$, stable and unstable subspaces at x, we have

$$\ker(\alpha(x)) = E_u(x) \oplus E_s(x)\,, \quad d\alpha(x)|_{E_s(x)\oplus E_u(x)} \text{ is nondegenerate.} \tag{1.3}$$

If $V \in C^\infty(X, TX)$ is the generator of the flow then $\alpha(V) \neq 0$ and we can modify α so that $\alpha(V) = 1$, the assumption we make. In particular, $\mathcal{L}_V \alpha = 0$.

The volume form on X is given by

$$dx := \alpha \wedge (d\alpha)^{\frac{n-1}{2}}, \quad \mathcal{L}_V dx = 0,$$

and

$$P := \tfrac{h}{i} V, \quad P : L^2(X, dx) \to L^2(X, dx),$$

is a symmetric first order semiclassical differential operator – see [Zw, §14.2]. As shown in [FaSj, Appendix A.1] it is essentially self-adjoint. The addition of the semiclassical parameter, although trivial, makes the final argument more natural.

The Pollicott-Ruelle resonances of P, or φ_t, are defined as eigenvalues of P acting on *exponentially weighted spaces*, H_{tG}, introduced in scattering theory by Helffer–Sjöstrand [HeSj] and in the context of this paper by Faure–Sjöstrand [FaSj] (see also Faure–Roy–Sjöstrand [FaRoSj] for an earlier version for Anosov diffeomorphisms). The construction of these spaces, denoted by H^m in [FaSj], will be reviewed in §3 below. The main point is the following fact given in [FaSj, Theorem 1.4]:

$$P - z : \mathcal{D}_{tG} \to H_{tG} \ \text{ is a Fredholm operator for } \operatorname{Im} z > -th/C, \ t \gg 1,$$
$$\mathcal{D}_{tG} := \{u \in H_{tG} : Pu \in H_{tG}\}, \text{ with } Pu \text{ defined in the sense of distributions.}$$

By Analytic Fredholm theory (see for instance [Zw, Theorem D.4]) the resolvent $(P - z)^{-1} : H_{tG} \to \mathcal{D}_{tG}$ is meromorphic with poles of finite rank, which are called Pollicott-Ruelle resonances. These resonances are independent of t and depend only on quantitative properties of the weight G, see [FaSj, Theorem 1.5]. See §2 for some heuristic ideas behind this construction.

The bound (1.2) is a consequence of a bound in smaller energy intervals given in our main result:

Theorem. *Let X be a compact smooth manifold with an Anosov flow $\varphi_t : X \to X$. Let P be the first order self-adjoint operator such that iP/h is the generator of φ_t, and let $\operatorname{Res}(P)$ be the set of resonances of P. Then for any $C_0 > 0$,*

$$\# \operatorname{Res}(P) \cap D(1, C_0 h) = \mathcal{O}(h^{-\frac{n-1}{2}}), \tag{1.4}$$

where $D(z, r) = \{\zeta : |\zeta - z| < r\}$.

Remarks (i) The bound (1.2) was predicted in remarks after [FaSj, Theorem 1.8] and is an immediate consequence of (1.4). Rescaling $\lambda = z/h$ we rewrite (1.4) as

$$\#\{\lambda \in \operatorname{Res}(-iV) \ : \ |\operatorname{Re}\lambda - E| \leq 1, \ \operatorname{Im}\lambda > -\beta\} = \mathcal{O}(E^{\frac{n-1}{2}}). \tag{1.5}$$

(ii) When $X = S^*M$ where M is a compact surface of constant negative curvature, Pollicott-Ruelle resonances coincide with the zeros of the Smale zeta function – see for instance [Le, §5.2, Figure 1]. Except of a finite number these are then given by

$$\lambda = z - i(k + \tfrac{1}{2}), \quad k \in \mathbb{N}, \quad z^2 \in \operatorname{Spec}(-\Delta_M - \tfrac{1}{4}),$$

where Δ_M is the Laplace-Beltrami operator on M. In that case the spectral asymptotics [Bé],[Ra] give, for $\beta > 0$,

$$\#\{\lambda \in \operatorname{Res}(P) \ : \ |\operatorname{Re}\lambda - E| \leq 1, \ \operatorname{Im}\lambda > -\beta\} = [\beta + \tfrac{1}{2}]\frac{\operatorname{Vol}(M)}{\pi}E + \mathcal{O}\left(\frac{E}{\log E}\right).$$

In this case $n = \dim X = 3$ which shows the optimality of (1.4).

(iii) In a recent preprint [FaTs], Faure and Tsujii consider the case of partially hyperbolic diffeomorphism conserving a smooth contact form and obtain a description of the spectrum in terms of "bands", corresponding to fixed values of k in the example above. The asymptotics given in [FaTs, Theorem 1.19] are in agreement with the upper bound (1.4).

2. Outline of the proof

The proof of the bound (1.4) is based on combining the arguments of [FaSj] with the arguments of [SjZw],[DaDy]. The paper relies heavily on technical results from these earlier works and we provide specific references in the text. Here we will motivate the problem and outline the general idea of the proof of (1.4).

The basic analogy between analysis of flows and semiclassical scattering theory lies in the fact that for a flow $\varphi_t = \exp tV : X \to X$,

$$\varphi_t^* u = e^{itP/h}u, \quad u \in C^\infty(X), \quad \varphi_t \circ \pi = \pi \circ \exp tH_p, \tag{2.1}$$

where $p(x, \xi) = \xi(V_x)$ is the symbol of the differential operator P, $\pi : T^*X \to X$ is the canonical projection, and H_p is the Hamilton vector field of p.

The key object in scattering theory is the trapped set at energy E:

$$K_E = \{(x, \xi) \in p^{-1}(E) : \exp(tH_p)(x, \xi) \not\to \infty, \quad t \to \pm\infty\}. \tag{2.2}$$

Here we note that ∞ means fiber infinity $\xi \to \infty$ in T^*X. It is the only "infinity" in our setting as X is compact.

Just as in scattering theory the spectrum of the unitary operator $\exp(-itP/h)$ is the unit circle $\mathbb{S}^1$, so to expand the correlations

$$\langle \varphi_t^* f, g\rangle = \langle e^{-itP/h} f, g\rangle, \quad f, g \in C^\infty(X),$$

into modes of decay, P has to be considered on a modified space, containing $C^\infty(X)$, and such that $P - z$ becomes a Fredholm operator for $\operatorname{Im} z > -Ah$. Roughly speaking, this may provide an expansion into modes with errors of size e^{-At} as $t \to \infty$ – see

[FaRoSj, Theorem 2] for the case of Anosov diffeomorphism (where $t = n$ is discrete and $\epsilon = e^{-A}$).

Following earlier works of Aguilar-Combes, Balslev-Combes, and Simon, Helffer-Sjöstrand [HeSj] introduced an approach based on an *escape function*, that is a function on T^*X, such that $H_pG \leq 0$ everywhere and $H_pG < 0$ near the infinity of the characteristic set of p (in fact in as large a set as possible). It turns out that the inhomogeneous Sobolev spaces used in the works of Baladi [Ba], Tsujii [Ts08, Ts10a, Ts10b], Blank, Butterley, Gouëzel, Keller, and Liverani [BlKeLi, BuLi, GoLi, Li04, Li05] can be reinterpreted this way. As explained in §3 below,

$$P - z : \mathcal{D}_{tG} \to H_{tG}, \text{ is a Fredholm operator for } \operatorname{Im} z > -th/C,\ t \gg 1,$$
$$H_{tG} := \exp(tG^w(x, hD))L^2(X),$$

where G is an escape function and $\mathcal{D}_{tG}$ is the domain of P in H_{tG}. Working on H_{tG} is equivalent with working with P conjugated by the exponential weight and the basic idea comes from (see (3.11) for a precise statement)

$$e^{tG^w(x,hD)} P e^{-tG^w(x,hD)} \sim P + ith(H_pG)^w(x, hD). \tag{2.3}$$

As shown in [FaSj, §3] negativity of H_pG near infinity implies the Fredholm property of $P - z$ for $\operatorname{Im} z > -th/C$. The eigenvalues of P, which are now complex and lie in $\operatorname{Im} z \leq 0$, are called *Pollicott-Ruelle resonances*, and we denote them by $\operatorname{Res}(P)$.

In scattering theory polynomial bounds on the number of resonances were first obtained by Melrose. Sharp bounds in odd dimensions were given by Melrose [Me] for obstacles and by Zworski [Zw1] for compactly supported potentials; the even dimensional sharp bounds were later obtained by Vodev [Vo].

The seminal work of Sjöstrand [Sj] and numerous mathematics and physics papers that followed (see [DaDy] and [NoSjZw] for references) then indicated that the exponent in the upper bound on the number of resonances near energy, say energy $E = 1$[1] should be related to the dimension of the trapped set K_E. In our case that dimension is in fact integral (see (3.4)):

$$\dim K_E = n + 1 = 2\mu + 1, \quad \mu = \frac{n-1}{2} \in \mathbb{N},$$

For bounds in regions of size h, the result of [SjZw] (following an earlier small neighbourhood bound by Guillopé-Lin-Zworski for Selberg zeta functions of Schottky groups – see [DaDy]) suggests that

$$\# \operatorname{Res}(P) \cap D(1, C_0 h) = \mathcal{O}(h^{-\mu}). \tag{2.4}$$

[1]Since $P = -ihV$ the problem is clearly homogeneous and we can work near any non-zero energy level. The high energy limit corresponds to $h \to 0$.

This is precisely the bound given in (1.4) which by rescaling translates to the bound (1.5). In many situations the interest in (2.4) lies in the fact that μ may not be an integer.

To obtain (2.4) we need to modify G so that, when using (2.3), we can invert the conjugated $P-1$ microlocally on a larger set. More precisely we introduce the additional conjugations in (5.3) below; that is done as in [SjZw] with the modifications presented in §4. The ideas behind that strategy are explained in [SjZw, §2]. To localize in a neighbourhood of size h a second microlocal argument is needed and we followed the functional calculus approach presented in [DaDy].

Eventually, these constructions produce a modified operator $\widetilde{P}_t - z$ given in Main Lemma 5.1 which is invertible for $z \in D(1, C_0 h)$, and which differs from the conjugated operator by an operator $-ithA$, microlocally localized in an $\mathcal{O}(\sqrt{h})$ neighbourhood of K_1 with an additional $\mathcal{O}(h)$ localization in the direction of dp.

The basic semiclassical intuition then dictates that the number of resonances of P in $D(1, C_0 h)$ (which are the same as the eigenvalues of the conjugated operator) is given by the phase space volume occupied by A multiplied by h^{-n}. That volume is estimated by h (due to the energy localization) times the volume of an $\mathcal{O}(\sqrt{h})$ neighbourhood of the smooth set K_1 inside $p^{-1}(1)$. Since K_1 has dimension $2\mu+1$, its codimension inside $p^{-1}(1)$ is given by $2(n-\mu-1)$. This gives the following bound:

$$h^{-n} \times h \times h^{\frac{1}{2}(2(n-\mu-1))} = h^{-\mu},$$

proving (2.4).

3. Microlocally weighted spaces and discrete spectrum of the generator of the flow

In this section we review, in a slightly modified form, the construction of Faure-Sjöstrand [FaSj] which provides Hilbert spaces on which $P-z$ is a Fredholm operator for $\operatorname{Im} z > -\beta h$.

Following [HeSj],[FaRoSj] the crucial component is the construction of an escape function on T^*X, that is a function G for which $H_p G \leq 0$ everywhere, with strict inequality on a large set.

The decomposition into neutral (one dimensional), stable and unstable subspaces is given by (here $E_0(x)$ is spanned by V)

$$T_x X = E_0(x) \oplus E_s(x) \oplus E_u(x).$$

The dual decomposition is obtained by taking $E_0^*(x)$ to be the annihilator of $E_s(x) \oplus E_u(x)$, $E_u^*(x)$ the annihillator of $E_u(x) \oplus E_0(x)$, and similarly for $E_s^*(x)$. That makes $E_s^*(x)$ dual to $E_u(x)$, $E_u^*(x)$ dual to $E_s(x)$, and $E_0^*(x)$ dual to $E_0(x)$. The fiber of the cotangent bundle decomposes as

$$T_x^*X = E_0^*(x) \oplus E_s^*(x) \oplus E_u^*(x). \tag{3.1}$$

We recall that the distributions $E_s^*(x)$ and $E_u^*(x)$ have only Hölder regularity, but $E_0^*(x)$ and $E_s^*(x) \oplus E_u^*(x)$ are smooth, and that $E_0^* = \mathbb{R}\alpha$ – see (1.3).

Let $|\cdot|$ be any smooth norm on the fibers of T^*X such that the norm of α and the dual norm of V are equal to 1, so that in particular

$$\{|\xi| \leq 1/2\} \cap p^{-1}(1) = \emptyset.$$

Here $p = p(x, \xi)$ is the classical Hamiltonian corresponding to P, i.e. the linear function on the fibers of T^*X defined by V.

The classical flow on T^*X is explicit in terms of φ_t:

$$\exp tH_p(x, \xi) = (\varphi_t(x), (D_x\varphi_t(x)^T)^{-1}\xi) \tag{3.2}$$

It follows from the hyperbolicity of the flow (see [FaSj, (1.13)]) that for some constants $C > 0$ and $\theta > 0$ and for all $t > 0$,

$$\begin{aligned} |\exp tH_p(\rho)| &\leq Ce^{-\theta t}|\rho|, \quad \rho \in E_s^*, \\ |\exp -tH_p(\rho)| &\leq Ce^{-\theta t}|\rho|, \quad \rho \in E_u^*. \end{aligned} \tag{3.3}$$

The *trapped set*, that is the set of (x, ξ) which stay in a compact subset (depending on (x, ξ)) for all $t \in \mathbb{R}$ is given by

$$K = E_0^* = \bigcup_{x \in X} E_0^*(x) \subset T^*X \tag{3.4}$$

and is a smooth submanifold of T^*X, which is symplectic away from the zero section. Indeed, since the decomposition (3.1) is invariant under φ_t, we may apply (3.3) with $\exp(\mp tH_p(\rho))$ in place of ρ to show $K \subset E_0^*$, and $E_0^* \subset K$ follows from $E_0^* = \mathbb{R}\alpha$. The energy slice of the trapped set is defined as

$$K_1 = p^{-1}(1) \cap E_0^*. \tag{3.5}$$

We denote by $S^k(T^*X)$ the standard space of symbols used in [Zw, §14.2] and by $S^{k+}(T^*X)$ the intersection of $S^{k+\epsilon}$ for all $\epsilon > 0$. The class of semiclassical pseudodifferential operators corresponding to $S^k(T^*X)$ is denoted by $\Psi^k(T^*X)$ – see [Zw, §14.2] and [DaDy, §3.1] for a review of the semiclassical notation used below. We also write $A \in \Psi^{k+}$ to denote that $A \in \Psi^{k+\varepsilon}$ for every $\varepsilon > 0$.

Following [FaSj], we construct the weight function G:

Lemma 3.1. *Take any conic neighborhoods U_0, U_0' of E_0^*, with $U_0 \Subset U_0'$ and $U_0' \cap (E_u^* \cup E_s^*) = \emptyset$. Then there exist real-valued functions $m \in S^0(T^*X)$, $f_0 \in S^1(T^*X)$ such that*

(1) *m is positively homogeneous of degree 0 for $|\xi| \geq 1/2$, equal to $-1, 0, 1$ near the intersection of $\{|\xi| \geq 1/2\}$ with E_u^*, E_0^*, E_s^*, respectively, and*

$$H_p m < 0 \text{ near } (U_0' \setminus U_0) \cap \{|\xi| > 1/2\}, \quad H_p m \leq 0 \text{ on } \{|\xi| > 1/2\}; \tag{3.6}$$

(2) *$\langle \xi \rangle^{-1} f_0 \geq c > 0$ for some constant c;*
(3) *the function $G = m \log f_0$ satisfies for some constant c,*

$$H_p G \leq -c < 0 \text{ on } \{|\xi| \geq 1/2\} \setminus U_0, \quad H_p G \leq 0 \text{ on } \{|\xi| \geq 1/2\}. \tag{3.7}$$

Proof. The existence of m follows from [FaSj, Lemma 1.2], where we rescale the parameter ξ to map the region $\{|\xi| \leq R\}$ of [FaSj] into $\{|\xi| \leq 1/2\}$, and use the function $\sqrt{1 + f^2}$ of [FaSj] as f_0. The inequality (3.6) follows directly from the proof of [FaSj, Lemma 1.2], if we choose the neighborhoods $\widetilde{N}_u, \widetilde{N}_0, \widetilde{N}_s$ there so that $U_0' \cap (\widetilde{N}_u \cup \widetilde{N}_s) = \emptyset$ and $\widetilde{N}_0 \subset U_0$. □

Note that [FaSj] needed $H_p G < -C < 0$ on $\{|\xi| \geq 1/2\} \setminus U_0$ for a large constant C; in this paper, we instead multiply G by a large $t > 0$ in the conjugation. The neighborhoods U_0, U_0' will be chosen in §4.2.

The function G satisfies derivative bounds

$$G = \mathcal{O}(\log \langle \xi \rangle), \quad \partial_x^\alpha \partial_\xi^\beta H_p^k G = \mathcal{O}\left(\langle \xi \rangle^{-|\beta|+}\right), \quad |\alpha| + |\beta| + k \geq 1\,. \tag{3.8}$$

In particular, $\partial_x^\alpha \partial_\xi^\beta H_p^k G \in S^{-|\beta|+}$.

We now use [Zw, §8.3] (the small modification to take into account the symbol classes S^m and S^{m+} is done as in [Zw, §9.3, §14.2]) and define

$$H_{tG}(X) := \exp(-tG^w(x, hD)) L^2(X, dx). \tag{3.9}$$

Note that this space is topologically isomorphic, with the norm of the isomorphism depending on h, to the nonsemiclassical space used in [FaSj]:

$$H^m(X) := \exp(-tG^w(x, D)) L^2(X, dx).$$

Indeed, for $|\xi| > 1/2$ the difference

$$G(x, \xi) - G(x, h\xi) = m(x, \xi) \log(f_0(x, \xi)/f_0(x, h\xi))$$

and it its derivatives, are bounded uniformly in (x, ξ) for any fixed h, so that equality of the spaces follows from [Zw, Theorem 8.8] applied with h fixed – see also [FaSj, §5.2].

The domain of P acting on H_{tG} is defined as

$$\mathcal{D}_{tG} := \{u \in \mathcal{D}'(X) \ : \ u, Pu \in H_{tG}\}. \tag{3.10}$$

The action of P on H_{tG} is equivalent to the action of the more natural operator P_{tG} on L^2:

$$\begin{aligned} P_{tG} &:= e^{tG^w} P e^{-tG^w} = \exp(t\,\mathrm{ad}_{G^w})P \\ &= \sum_{k=0}^{N} \frac{t^k}{k!}\,\mathrm{ad}_{G^w}^k P + R_N(x,hD), \quad R_N \in h^{N+1}S^{-N+}. \end{aligned} \tag{3.11}$$

One way to see the validity of (3.11) is to note that the operators $e^{\pm tG^w}$ are pseudodifferential operators [Zw, Theorem 8.6] and hence the pseudodifferential calculus applies directly [Zw, Theorem 9.5, Theorem 14.1]. To show that $R_N \in h^{N+1}S^{-N+}$, write R_N as a sum of $2t+1$ many terms of a Taylor series plus an integral remainder which can be analyzed as in, for example, [DaDy, Lemma 7.2].

Using this expansion we can follow arguments in [FaSj, §3] to show

Proposition 3.2. *For P_{tG} defined by (3.11) we have:*

i) $P_{tG} - z : \mathcal{D}(P_{tG}) \to L^2$ is a Fredholm operator of index zero for $\operatorname{Im} z > -th/C$. Here $\mathcal{D}(P_{tG})$ is the domain of P_{tG}.

ii) $P_{tG} - z$ is invertible for $\operatorname{Im} z > Ch$ and C large enough.

4. Construction of the escape function

In this section we modify the escape function of Faure–Sjöstrand near the trapped set. It will be quantized to become the operator F appearing in Main Lemma 5.1 below. We will use symbols depending on two semiclassical parameters $h, \tilde{h}$, see § 5 for details.

4.1. Construction near the trapped set.

We start with an escape function $\hat{f}$ defined in a neighborhood of K_1 and with $H_p\hat{f} \le -c < 0$ away from a $C(h/\tilde{h})^{1/2}$ sized neighborhood of the trapped set K. This is a modification of the construction in [SjZw, §7], based on an earlier construction in [Sj, §5]. The changes come from a different structure of the incoming and outgoing manifold which we now define:

$$\Gamma_\pm = \{(x,\xi) \ : \ \exp(tH_p)(x,\xi) \not\to \infty, \quad t \to \mp\infty\}. \tag{4.1}$$

We note that ∞ refers to the fiber infinity of T^*X. We see that

$$K = \Gamma_+ \cap \Gamma_-,$$

and that by (3.3),

$$\begin{aligned} \Gamma_+ &= \bigcup_{x\in X} \Gamma_{+,x}, \quad \Gamma_- = \bigcup_{x\in X} \Gamma_{-,x}, \\ \Gamma_{+,x} &:= E_0^*(x) \oplus E_u^*(x), \quad \Gamma_{-,x} := E_0^*(x) \oplus E_s^*(x). \end{aligned} \tag{4.2}$$

This provides a continuous but typically non-smooth folliation of $\Gamma_\pm$ by smooth (affine) manifolds. We note that for $(x, \xi) \in K$, the (affine) leaves of the two folliations intersect cleanly with a fixed excess equal to n, the dimension of X,

$$(E_0^*(x) \oplus E_u^*(x)) \cap (E_0^*(x) \oplus E_s^*(x)) = E_0^*(x),$$

see [Hö3, Appendix C.3]. This, rather than the transversality of leaves, assumed in [Sj, §5] and [SjZw, §7] constitutes the only difference in the construction. Nevertheless the basic facts established in [Sj, §5] still hold. To state them we use the notation $f \sim g$ if, for a constant C, $f/C \leq g \leq Cf$.

Lemma 4.1. *Let d be a distance function on a neighbourhood of $K_1 \subset T^*X$. For ρ in a neighbourhood of K_1, we have*

$$d(\rho, K) \sim d(\rho, \Gamma_+) + d(\rho, \Gamma_-). \tag{4.3}$$

Also, there exists a constant C such that for any $\tau \geq 0$ we can find an open neighbourhood Ω_τ of K_1 such that

$$d(\exp(\pm\tau H_p)(\rho), \Gamma_\pm) \leq Ce^{-\tau/C} d(\rho, \Gamma_\pm), \ \ \rho \in \Omega_\tau. \tag{4.4}$$

Proof. The cleanness with a fixed excess (affine spaces always intersect cleanly) shows that for $x \in X$ and ρ close to K_1 we still have a uniform statement,

$$d(\rho, \Gamma_{+,x} \cap \Gamma_{-,x}) \sim d(\rho, \Gamma_{-,x}) + d(\rho, \Gamma_{+,x}).$$

Hence (4.3) follows as in the proof of [Sj, Lemma 5.1].

To obtain (4.4) we choose a euclidean distance d_y on fibers T_y^*X, depending smoothly on y. From the continuity of $x \mapsto \Gamma_{\pm,x}$ we see that for (x, ξ) in a bounded set,

$$d((x, \xi), \Gamma_\pm) \sim d_x(\xi, \Gamma_{\pm,x}). \tag{4.5}$$

We now fix a bounded neighbourhood of K_1, Ω, in which (4.5) is valid uniformly and define

$$\Omega_\tau := \bigcup_{|t| \leq \tau} \exp tH_p(\Omega).$$

Then for $(x, \xi) \in \Omega_\tau$, (3.2) shows that

$$d(\exp(\pm\tau H_p)(x, \xi), \Gamma_\pm) \sim d_{\varphi_{\pm\tau}(x)}((D_x\varphi_{\pm\tau}(x)^T)^{-1}\xi, \Gamma_{\pm,\varphi_{\pm\tau}(x)}),$$

with constants independent of τ.

Hence with C independent of τ, and $(x, \xi) \in \Omega_\tau$, we have

$$d_{\varphi_\tau(x)}((D_x\varphi_\tau(x)^T)^{-1}\xi, \Gamma_{+,\varphi_\tau(x)}) \leq Ce^{-\tau/C} d_x(\xi, \Gamma_{+,x}), \quad (x, \xi) \in \Omega_\tau. \tag{4.6}$$

(We state this for +, the other case being analogous.) We write the unique decomposition $\xi = \xi_u + \xi_s + \xi_0$, $\xi_\bullet \in E^*_\bullet(x)$, so that by the invariance of the subspaces $E^*_\bullet$,

$$(D_x\varphi_\tau(x)^T)^{-1}\xi = (D_x\varphi_\tau(x)^T)^{-1}\xi_u + (D_x\varphi_\tau(x)^T)^{-1}\xi_s + (D_x\varphi_\tau(x)^T)^{-1}\xi_0,$$
$$(D_x\varphi_\tau(x)^T)^{-1}\xi_\bullet \in E^*_\bullet(\varphi_\tau(x)).$$

This means that

$$d_{\varphi_\tau(x)}((D_x\varphi_\tau(x)^T)^{-1}\xi, \Gamma_{+,\varphi_\tau(x)}) \sim \|(D_x\varphi_\tau(x)^T)^{-1}\xi_s\|, \quad d_x(\xi, \Gamma_{+,x}) \sim \|\xi_s\|.$$

The estimate (4.6) then follows from the Anosov property of the flow (3.2),(3.3). □

We now proceed as in [SjZw, §7] and obtain regularizations, $\widehat{\varphi}_\pm$, of $d(\bullet, \Gamma_\pm)^2$ – see [SjZw, Proposition 7.4]. The next lemma states the properties of the resulting escape functions obtained using [SjZw, Lemma 7.6] applied with $\epsilon = (h/\tilde{h})^{\frac{1}{2}}$:

Lemma 4.2. *There exists a conic neighborhood U_0' of E_0^* and a real-valued function*

$$\hat{f}(x, \xi; h, \tilde{h}) \in C^\infty(U_0' \cap \{1/2 < |\xi| < 2\})$$

such that:

(1) *$\hat{f}$ satisfies the derivative bounds*

$$\hat{f} = \mathcal{O}(\log(1/h)), \quad \partial^\alpha_{x,\xi} H_p^k \hat{f} = \mathcal{O}((h/\tilde{h})^{-|\alpha|/2}), \ |\alpha| + k \geq 1; \tag{4.7}$$

(2) *there exists a constant $C_{\hat{f}}$ such that*

$$H_p \hat{f}(x, \xi) \leq -C_{\hat{f}}^{-1} < 0 \text{ for } d((x, \xi), K) \geq C_{\hat{f}}(h/\tilde{h})^{1/2}. \tag{4.8}$$

4.2. **A global escape function.** We recall that our goal is to construct a function f such that for the escape function, G, given in Lemma 3.1 $H_p(G + f)$ is as negative as possible.

For that we cut $\hat{f}$ off and modify it to get an escape function defined on the whole T^*X. Let U_0' be the conic neighborhood of E_0^* from Lemma 4.2 and shrink it so that

$$U_0' \cap (E_u^* \cup E_s^*) = \emptyset, \quad U_0' \cap p^{-1}(1) \subset \{1/2 < |\xi| < 2\}.$$

The second statement is possible since $E_0^*(x) \cap p^{-1}(1) = \alpha(x)$ and $|\alpha| = 1$. Take any conic neighborhood U_0 of E_0^* such that $U_0 \Subset U_0'$ and a nonnegative function

$$\chi_{\hat{f}} \in C_0^\infty(U_0' \cap \{1/2 < |\xi| < 2\}), \quad \chi_{\hat{f}} = 1 \text{ near } U_0 \cap p^{-1}(1).$$

Let $m \in S^0(X)$ be the function constructed in Lemma 3.1. We choose a constant $M > 0$ large enough so that the function

$$f := \chi_{\hat{f}} \hat{f} + M \log(1/h) m \tag{4.9}$$

satisfies

$$H_p f(x,\xi) \le -c < 0 \text{ for } (x,\xi) \in U_0' \cap p^{-1}(1) \text{ with } d((x,\xi),K) \ge C_{\hat{f}}(h/\tilde{h})^{1/2},$$
$$H_p f \le 0 \text{ near } p^{-1}(1).$$

This is possible since $H_p(\chi_{\hat{f}}\hat{f}) \le -C_{\hat{f}}^{-1} < 0$ when $(x,\xi) \in U_0 \cap p^{-1}(1)$ and $d((x,\xi),K) \ge C_{\hat{f}}(h/\tilde{h})^{1/2}$; $H_p(\chi_{\hat{f}}\hat{f}) = \mathcal{O}(\log(1/h))$ everywhere by (4.7); $\operatorname{supp}\chi_{\hat{f}} \subset U_0'$; and $H_p m \le -c < 0$ on $(U_0' \setminus U_0) \cap p^{-1}(1)$ by (3.6).

Lemma 4.3. *There exists a nonnegative function $\tilde{a}$ supported $\mathcal{O}((h/\tilde{h})^{1/2})$ close to K and such that for G given in Lemma 3.1 and f given by (4.9),*

$$H_p(G+f) - \tilde{a} \le -c < 0 \text{ on } p^{-1}(1), \quad \partial^\alpha \tilde{a} = \mathcal{O}((h/\tilde{h})^{-|\alpha|/2}). \tag{4.10}$$

Equation (4.10) is the key component of the positive commutator argument in §5.2. By (3.7) and the properties of f, it suffices to verify (4.10) in an $\mathcal{O}((h/\tilde{h})^{1/2})$ sized neighborhood of K_1, where $H_p(G+f) = H_p\hat{f}$ (since $m = 0$ near K_1).

Proof of Lemma 4.3. To construct $\tilde{a}$, take a nonnegative function $\theta \in C^\infty(\mathbb{R})$ such that $\operatorname{supp}\theta \subset (-\infty, C_{\hat{f}}^{-1})$ and $\theta(\lambda) + \lambda = 1$ for $\lambda \le C_{\hat{f}}^{-1}/2$. Then by (4.8), the function $\theta(-H_p\hat{f})$ is supported $\mathcal{O}((h/\tilde{h})^{1/2})$ close to K. Now, take any nonnegative $\chi_a \in C_0^\infty(T^*X)$ such that $\chi_{\hat{f}} = 1$ near $\operatorname{supp}\chi_a$, but $\chi_a = 1$ near $U_0 \cap p^{-1}(1)$, and define

$$\tilde{a} := \theta(-H_p\hat{f})\chi_a \in C_0^\infty(T^*X). \tag{4.11}$$

Then (4.10) follows since on $U_0 \cap p^{-1}(1)$,

$$H_p\hat{f} - \tilde{a} = H_p\hat{f} - \theta(-H_p\hat{f}) \le -C_{\hat{f}}^{-1}/2.$$

□

5. Upper bound on the number of resonances

In this section, we prove the bound (1.4) on the number of Pollicott-Ruelle resonances.

5.1. **Reduction to a weighted estimate.**

We start by showing how (1.4) follows from the estimate (5.1) given in the following lemma.

Lemma 5.1. *(Main lemma) There exist families of bounded operators*[2] $\widehat{F}, F_1, A$ *on* $L^2(X)$*, depending on two parameters* $h, \tilde{h}$ *(where we choose* $\tilde{h}$ *small enough and* h

[2]To combine the notation of [FaSj] and [DaDy], we denote by G and $\hat{f}$ their respective escape functions and by G^w and $\widehat{F}$ the corresponding pseudodifferential operators.

small enough depending on $\tilde{h}$), such that for any fixed constant C_0 and $t > 0$ large enough, the modified conjugated operator

$$\widetilde{P}_t := e^{t\widehat{F}} e^{tF_1} P_{tG} e^{-tF_1} e^{-t\widehat{F}} - ithA$$

satisfies the estimate (with C independent of $h, \tilde{h}$) for any $u \in C^\infty(X)$

$$\begin{gathered}\|u\|_{L^2} \leq \frac{C}{\max(h, \operatorname{Im} z)} \|(\widetilde{P}_t - z)u\|_{L^2}, \\ \text{for } \ |\operatorname{Re} z - 1| \leq C_0 h, \quad -C_0 h \leq \operatorname{Im} z \leq 1.\end{gathered} \tag{5.1}$$

Moreover, we can write $A = A_R + A_E$, where for some constant $C(\tilde{h})$ depending on $\tilde{h}$,

$$\begin{gathered}\|A_R\|_{L^2 \to L^2} = \mathcal{O}(1), \quad \|A_E\|_{L^2 \to L^2} = \mathcal{O}(\tilde{h}), \\ \operatorname{rank} A_R \leq C(\tilde{h}) h^{-\frac{n-1}{2}}.\end{gathered} \tag{5.2}$$

Note that in [DaDy] we required the estimate (5.1) for the $H_h^{-1/2} \to H_h^{1/2}$ norm instead of the $L^2 \to L^2$ norm; this is because the Laplacian considered there is a differential operator of order 2, while our differential operator P has order 1.

Assume that (5.1) holds. Since $e^{\pm t\widehat{F}}, e^{\pm tF_1}$ are bounded on L^2, the operator

$$P_t := e^{t\widehat{F}} e^{tF_1} P_{tG} e^{-tF_1} e^{-t\widehat{F}} \tag{5.3}$$

satisfies part (i) of Proposition 3.2, and its eigenvalues in $D(1, C_0 h)$ are precisely the Pollicott-Ruelle resonances. The operator A will be compactly microlocalized in the sense of [DaDy, §3.1] and in particular compact $L^2 \to L^2$; therefore, adding it will not change the Fredholm property of P_t. By [FaSj, Lemma A.1], the bound (5.1) implies

$$\|(\widetilde{P}_t - z)^{-1}\|_{L^2 \to L^2} \leq \frac{C}{\max(h, \operatorname{Im} z)}, \quad |\operatorname{Re} z - 1| \leq C_0 h, \ -C_0 h \leq \operatorname{Im} z \leq 1.$$

The estimate (1.4) is now proved as in [DaDy, §2], using Jensen's inequality.

We now construct the operators $\widehat{F}, F_1, A$ of Lemma 5.1. We will use the class $\Psi^{\mathrm{comp}}_{1/2}(X)$ of pseudodifferential operators whose symbols have compact essential support and satisfy the bound

$$\sup |\partial^\alpha_{x,\xi} a| = \mathcal{O}((h/\tilde{h})^{-|\alpha|/2}).$$

We refer to [SjZw, §3.3] and [DaDy, §5.1] for the motivation for this class of symbols and the properties of corresponding operators. We take

$$\widehat{F} := (\chi_{\hat{f}} \hat{f})^w, \quad F_1 := M \log(1/h) m^w,$$

so that the operator $\widehat{F} + F_1$ has the symbol f from (4.9). Recalling the derivative bounds (4.7), we see that

$$\widehat{F} \in \log(1/h) \Psi^{\mathrm{comp}}_{1/2}(X), \quad F_1 \in \log(1/h) \Psi^0(X).$$

Finally, we put

$$A := \chi((\tilde h/h)\widehat P)\widetilde A, \tag{5.4}$$

where:

- $\widetilde A = \tilde a^w$, with $\tilde a$ defined in (4.11). By (4.7) and (4.11), we have $\widetilde A \in \Psi^{1/2}(X)$;
- $\chi \in C_0^\infty(\mathbb{R})$ is equal to 1 near zero;
- $\widehat P$ is any symmetric pseudodifferential operator in $\Psi^1(X)$ with principal symbol $\hat p(x,\xi)$ elliptic in the class S^1 for $|\xi|$ large enough and $\hat p = p-1$ on $U_0'\cap\{1/2 < |\xi| < 2\} \supset \operatorname{supp}\tilde a$;
- $\chi((\tilde h/h)\widehat P)$ is defined by means of functional calculus of self-adjoint operators on $L^2(X)$ (see [DaDy, §5.2] for properties of such operators).

Under these conditions, (5.2) follows from [DaDy, Lemma 6.1]. The key observation here is that $\tilde a$ is supported in an $\mathcal{O}((h/\tilde h)^{1/2})$ sized neighborhood of K; the latter is an $n+1$ dimensional smooth manifold invariant under the flow $\exp(tH_p)$ and thus under $\exp(tH_{\hat p})$ near $\operatorname{supp}\tilde a$; therefore, for each $R>0$ (see [DaDy, §7.4])

$$\begin{gathered}\operatorname{Vol}_{\hat p^{-1}(0)}\{\exp(tH_{\hat p})(x,\xi)\mid |t|\le R,\\ (x,\xi)\in(\operatorname{supp}\tilde a\cap\hat p^{-1}(0)) + B_{\hat p^{-1}(0)}(R(h/\tilde h)^{1/2})\} \le C(h/\tilde h)^{\frac{n-1}{2}}.\end{gathered}$$

5.2. **Proof of Main Lemma 5.1.** In this section, we assume that $|\operatorname{Re} z - 1| \le C_0 h$ and $-C_0 h \le \operatorname{Im} z \le 1$, and $u \in C^\infty(X)$; we will prove the estimate (5.1). See the outline of the proof of Theorem 2 in [DaDy, Introduction] for an explanation of the positive commutator argument used here.

We start by writing an expansion for the operator P_t from (5.3). By (3.11),

$$P_{tG} = P + t[G^w, P] + \mathcal{O}_t(h^2)_{\Psi^{-1+}}.$$

Similarly,

$$e^{tF_1}P_{tG}e^{-tF_1} = P + t[G^w + F_1, P] + \mathcal{O}_t(h^{2-})_{\Psi^{-1+}}.$$

Finally, using the Bony–Chemin Theorem [BoCh, Théorème 6.4],[Zw, Theorem 8.6] as in [DaDy, Lemma 7.2], we have

$$P_t = P + t[G^w + F_1 + \widehat F, P] + \mathcal{O}_t(h\tilde h)_{L^2\to L^2}. \tag{5.5}$$

In particular,

$$P_t = P + \mathcal{O}_t(h)_{\Psi^{0+}} + \mathcal{O}_t(h\log(1/h))_{L^2\to L^2}. \tag{5.6}$$

Next, we get rid of the $\chi((\tilde h/h)\widehat P)$ part of the operator A; namely, we claim that (5.1) follows from the estimate

$$\|u\|_{L^2} \le \frac{C}{\max(h,\operatorname{Im} z)}\|(P_t - ith\widetilde A - z)u\|_{L^2}. \tag{5.7}$$

For that, write $1-\chi(\lambda)=\lambda\psi(\lambda)$ with $\psi\in C^\infty(\mathbb{R})$ bounded; then

$$\|(A-\widetilde{A})u\|_{L^2}\leq C(\tilde{h}/h)\|\widehat{P}\widetilde{A}u\|_{L^2}.$$

By (4.7) and (4.11), $H_p\tilde{a}=\mathcal{O}(1)_{S^{\mathrm{comp}}_{1/2}}$; therefore, by part 7 of [DaDy, Lemma 5.2] we have $[\widehat{P},\widetilde{A}]=\mathcal{O}(h)_{L^2\to L^2}$, and

$$\|(A-\widetilde{A})u\|_{L^2}\leq C(\tilde{h}/h)\|\widetilde{A}\widehat{P}u\|_{L^2}+\mathcal{O}(\tilde{h})\|u\|_{L^2}.$$

Since $\widehat{P}=P-1+\mathcal{O}(h)$ near $\mathrm{WF}_h(\widetilde{A})$, and by our assumptions on z we find

$$\|(A-\widetilde{A})u\|_{L^2}\leq C(\tilde{h}/h)\|\widetilde{A}(P-z)u\|_{L^2}+\mathcal{O}(\tilde{h}\max(1,h^{-1}\operatorname{Im}z))\|u\|_{L^2}.$$

Next, $\mathrm{WF}_h(F_1)\cap\mathrm{WF}_h(\widetilde{A})=\emptyset$ since $\mathrm{WF}_h(\widetilde{A})\subset K$ and $m=0$ in a conic neighborhood of K by Lemma 3.1. Also, by (4.7),

$$H_p(\chi_{\hat{f}}\hat{f})=H_p\hat{f}=\mathcal{O}(1)_{S_{1/2}}\ \text{ near }\mathrm{WF}_h(\widetilde{A})\subset\{\chi_{\hat{f}}=1\}.$$

From part 7 of [DaDy, Lemma 5.2], we have $[P,\widehat{F}]=\mathcal{O}(h)$ near $\mathrm{WF}_h(\widetilde{A})$. Then by (5.5), we have $\widetilde{A}(\widetilde{P}_t-P)=\mathcal{O}_t(h)_{L^2\to L^2}$ and thus

$$\|(A-\widetilde{A})u\|_{L^2}\leq C(\tilde{h}/h)\|(\widetilde{P}_t-z)u\|_{L^2}+\mathcal{O}(\tilde{h}\max(1,h^{-1}\operatorname{Im}z))\|u\|_{L^2}.$$

Combining this with (5.7), we get

$$\|u\|_{L^2}\leq\frac{C}{\max(h,\operatorname{Im}z)}\|(\widetilde{P}_t-z)u\|_{L^2}+\mathcal{O}(\tilde{h})\|u\|_{L^2},$$

which implies (5.1) if $\tilde{h}$ is small enough.

To prove (5.7), we restrict to a neighborhood of the energy surface as follows. Recalling (4.10), the fact that the function on the left-hand side of (4.10) is $\mathcal{O}(\log(1/h))_{S^{0+}}$, and the only unbounded part of this function as $|\xi|\to\infty$ is $H_pG\leq 0$, we see that there exists $B_E\in\Psi^0$ such that $p-1$ is elliptic on $\mathrm{WF}_h(B_E)$ and

$$H_p(G+f)-\tilde{a}-\log(1/h)|\sigma(B_E)|^2\leq -c<0\text{ everywhere on }T^*X. \tag{5.8}$$

By the elliptic estimate (see for instance [DaDy, (3.3)])

$$\|B_Eu\|_{L^2}\leq C\|(P-z)u\|_{H_h^{-1}}+\mathcal{O}(h^\infty)\|u\|_{L^2}.$$

By (5.6), we have

$$\|B_Eu\|_{L^2}\leq C\|(P_t-ith\widetilde{A}-z)u\|_{L^2}+\mathcal{O}(h\log(1/h))\|u\|_{L^2}. \tag{5.9}$$

We then claim that (5.7) follows from

$$\operatorname{Im}\langle(P_t-ith(\widetilde{A}+\log(1/h)B_E^*B_E))u,u\rangle_{L^2}\leq(-c_1th+\mathcal{O}_t(h\tilde{h}))\|u\|^2_{L^2}, \tag{5.10}$$

where the constant $c_1>0$ is independent of t. Indeed, if t is large enough depending on C_0 and $\tilde{h}$ is small enough depending on t, then (5.10) implies

$$\operatorname{Im}\langle(P_t-ith\widetilde{A}-z)u,u\rangle_{L^2}\leq -C_t^{-1}\max(h,\operatorname{Im}z)\|u\|^2_{L^2}+th\log(1/h)\|B_Eu\|^2_{L^2}.$$

Combining this with (5.9), we get (5.7), which is the claim in Lemma 5.1.

5.3. **Proof of (5.10).** Here we depart slightly from the strategy of [DaDy] and replace a microlocal partition of unity argument of [DaDy, §7] by global positive commutator estimates.

By (5.5) we reduce (5.10) to

$$\operatorname{Im}\langle (P - t([P, G^w + F_1 + \widehat{F}] + ih\widetilde{A} + ih\log(1/h)B_E^* B_E))u, u\rangle_{L^2} \leq -c_1 th\|u\|_{L^2}^2.$$

Since P is self-adjoint on $L^2(X)$, this would follow from

$$\operatorname{Re}\langle Qu, u\rangle_{L^2} \geq -C\tilde{h}\|u\|_{L^2}^2, \tag{5.11}$$

where

$$Q := -ih^{-1}[P, G^w + F_1 + \widehat{F}] + \widetilde{A} - 2c_1 + \log(1/h)B_E^* B_E,$$

and

$$Q \in \Psi^{0+} + \log(1/h)\Psi^0 + \log(1/h)\Psi^{\mathrm{comp}}_{1/2}.$$

Its principal symbol is given by

$$q := -H_p(G + f) + \tilde{a} - 2c_1 + \log(1/h)|\sigma(B_E)|^2.$$

Note that Q is equal to any quantization of q plus a remainder that is $\mathcal{O}(\tilde{h})_{L^2\to L^2}$. (See part 3 of [DaDy, Lemma 5.4] for the term involving $\widehat{F}$.) Therefore, we can replace Q by any quantization of q in (5.11). Using (5.8), choose c_1 small enough so that

$$q \geq c_1 > 0 \text{ everywhere.}$$

Formally speaking, (5.11) is a version of the sharp Gårding inequality, however the symbol involved is exotic and grows like $\mathcal{O}(\log(1/h))$, therefore we have to break it into pieces using a partition of unity. Note that, with a correct choice of B_E, by (3.6) we have

$$\begin{aligned}\operatorname{supp}(\chi_{\hat{f}}) \cap \operatorname{supp}(1 - \chi_{\hat{f}}) &\subset (U_0' \cap \{1/2 < |\xi| < 2\}) \setminus (U_0 \cap p^{-1}(1))\\ &\subset \{H_p m > 0\} \cup (T^*X \setminus p^{-1}(1)) \subset \{H_p m > 0\} \cup \{\sigma(B_E) \neq 0\}.\end{aligned}$$

Therefore, we can write $T^*X = \Omega_0 \cup \Omega_1 \cup \Omega_2$, where Ω_j are open and

$$\begin{aligned}\chi_{\hat{f}} &= 1 \text{ near } \Omega_0,\\ -MH_p m + |\sigma(B_E)|^2 &\geq c > 0 \text{ on } \Omega_1,\\ \chi_{\hat{f}} &= 0 \text{ near } \Omega_2.\end{aligned}$$

We also make Ω_0 and Ω_1 bounded sets. Now, take a partition of unity

$$1 = \chi_0 + \chi_1 + \chi_2, \quad \chi_j \in C^\infty(T^*X; [0, 1]), \quad \operatorname{supp}\chi_j \subset \Omega_j.$$

We use this partition to decompose q into a sum of three symbols, except that the term $-\chi_{\hat f} H_p \hat f + \tilde a$ will be put entirely into the part corresponding to Ω_0. Namely, put

$$\begin{gathered} q_0 := \chi_0 q + (1-\chi_0)(-\chi_{\hat f} H_p \hat f + \tilde a) + \chi_1 |\sigma(B_E)|^2, \\ q_1 := \chi_1 (q + \chi_{\hat f} H_p \hat f - \tilde a - |\sigma(B_E)|^2), \\ q_2 := \chi_2 q. \end{gathered}$$

Since $\chi_2(\chi_{\hat f} H_p \hat f - \tilde a) = 0$, we have

$$q = q_0 + q_1 + q_2.$$

Since $q_2 \in \log(1/h) S^{0+}$, the sharp Gårding inequality [Zw, Theorem 9.11] implies

$$\langle q_2^w u, u\rangle \geq -Ch\log(1/h)\|u\|_{L^2}^2. \tag{5.12}$$

Next, we consider the term corresponding to $q_1 \in \log(1/h) S^{\mathrm{comp}}_{1/2}$, which we write as

$$q_1 = \chi_1(\log(1/h)(-MH_p m + |\sigma(B_E)|^2) - \hat f H_p \chi_{\hat f} - H_p G - 2c_1 - |\sigma(B_E)|^2).$$

Since $-MH_p m + |\sigma(B_E)|^2 > 0$ on Ω_1, we can increase M and $|\sigma(B_E)|$ to make $q_1 \geq c\log(1/h)\chi_1$. We will show that

$$\langle q_1^w u, u\rangle \geq -Ch\log(1/h)\|u\|_{L^2}^2. \tag{5.13}$$

Note that $q_1 + \chi_1 \hat f H_p \chi_{\hat f} \in \log(1/h) S^0$. To exploit this, put

$$\Omega = T^*X \setminus (\Omega_0 \cup \Omega_2),$$

so that Ω is a compact set contained in Ω_1. Since $\chi_1 = 1$ on Ω, we find $q_1 \geq c\log(1/h) > 0$ there. Therefore, there exists $\chi_\Omega \in C_0^\infty(T^*X)$ such that $\chi_\Omega \neq 0$ near Ω, but $q_1 \geq \log(1/h)|\chi_\Omega|^2$ everywhere. We now apply the sharp Gårding inequality for the $\Psi_{1/2}$ calculus, which follows from the usual sharp Gårding inequality by the standard rescaling (see for example the proof of [SjZw, Lemma 3.5]), to the symbol

$$q_1 - \log(1/h)|\chi_\Omega|^2 \in \log(1/h) S^{\mathrm{comp}}_{1/2}.$$

Since the only exotic term in q_1 is $-\chi_1 \hat f H_p \chi_{\hat f}$, supported in Ω, we have

$$\langle q_1^w u, u\rangle - \log(1/h)\|\chi_\Omega^w u\|_{L^2}^2 \geq -Ch\log(1/h)\|u\|_{L^2}^2 - C\tilde h\log(1/h)\|\chi_\Omega^w u\|_{L^2}^2.$$

For $\tilde h$ small enough, this yields (5.13). Now, we write q_0 as a sum of two terms, one non-exotic and one compactly microlocalized:

$$\begin{gathered} q_0 = q_0' + q_0'', \\ q_0' = \chi_0(-H_p G - M\log(1/h) H_p m + (\log(1/h) - 1)|\sigma(B_E)|^2), \\ q_0'' = -\chi_{\hat f} H_p \hat f + \tilde a - 2\chi_0 c_1 + (1-\chi_2)|\sigma(B_E)|^2. \end{gathered}$$

Then $q_0' \in \log(1/h)S^0$ and $q_0' \geq 0$ everywhere (increasing $|\sigma(B_E)|$ if necessary to handle the set $\{|\xi| \leq 1/2\}$); by sharp Gårding inequality [Zw, Theorem 4.32] applied to the symbol $q_0'/\log(1/h)$,

$$\langle (q_0')^w u, u\rangle \geq -Ch\log(1/h)\|u\|_{L^2}^2. \tag{5.14}$$

Next, $q_0'' \in S_{1/2}^{\text{comp}}$ and, if we choose the function χ_a from the definition (4.11) of $\tilde{a}$ so that $\chi_a = 1$ near $\operatorname{supp}\chi_0$, and take c_1 small enough, we have $q_0'' \geq 0$ everywhere. Again using the sharp Gårding inequality for $\Psi_{1/2}$ calculus, we find

$$\langle (q_0'')^w u, u\rangle \geq -C\tilde{h}\|u\|_{L^2}^2. \tag{5.15}$$

Adding together (5.12), (5.13), (5.14), and (5.15), we get (5.11).

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