Control Theory and High Frequency Eigenfunctions

Forges-les-Eaux

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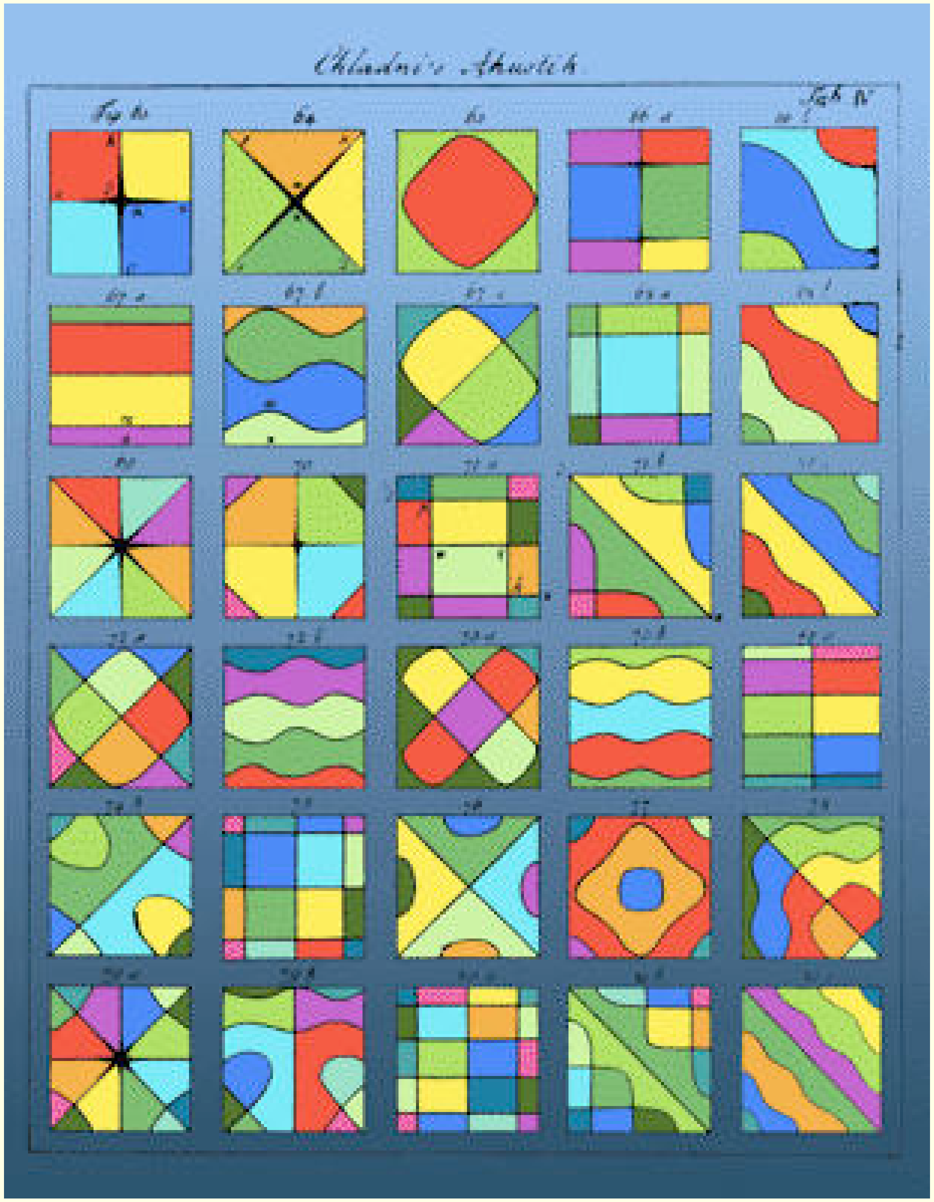
This talk will be concerned with very classical objects:

Eigenfuctions of the Dirichlet (or Neumann) Laplacian on a bounded domain, Ω, in the plane:

$$-\Delta u_j = \lambda_j^2 u_j \,, \quad u_j|_{\partial\Omega} = 0 \,, \quad \int_\Omega |u_j(x)|^2 dx = 1 \,.$$

$$\lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \to \infty \,.$$

This is a theoretical and experimental model for the study of the classical/quantum correspondence.

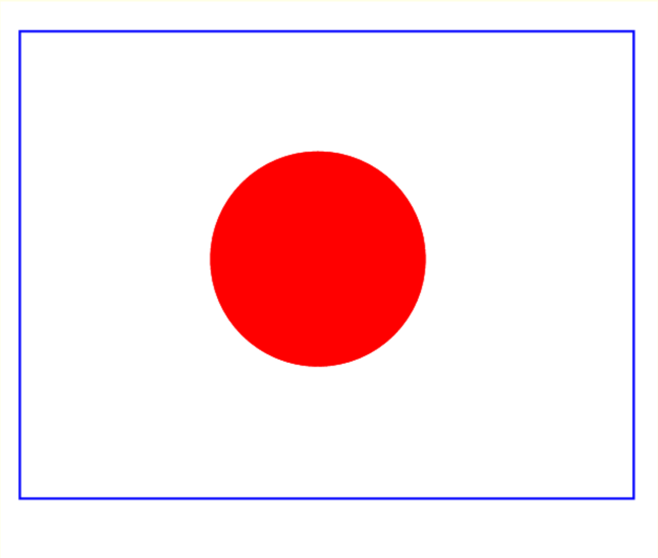


Napoleon asked Chladni what can be said of more complicated domains.

The Shnirelman Theorem.

Suppose that the billiard flow on a bounded domain with boundary, Ω, is ergodic.

Here is an example:



The Sinai billiard

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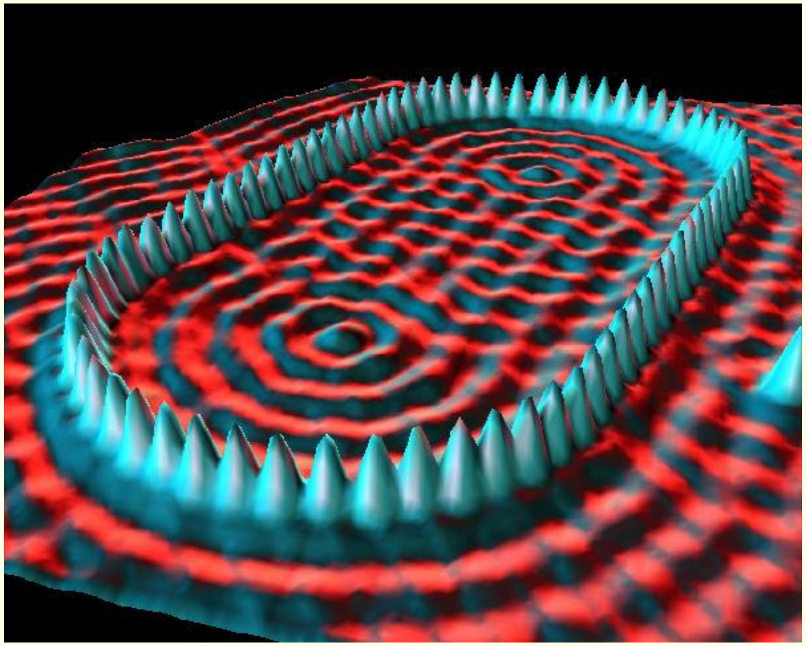


A quantum coral made in the shape of the Bunimovich stadium by Crommie, Eigler et al.

The Shnirelman Theorem.

Suppose that the billiard flow on a bounded domain with boundary, Ω, is ergodic.

Then there exists a sequence $\{j_k\}_{k=1}^{\infty} \subset \mathbf{N}$ of density one,

$$\lim_{N\to\infty} (\max_{j_k \le N} k)/N = 1,$$

such that for any nice open subset V, of Ω,

$$\lim_{k\to\infty} \int_V |u_{j_k}(x)|^2 dx = \frac{\mathrm{Area}(V)}{\mathrm{Area}(\Omega)}\,.$$

This theorem has a long history.

It was announced by Shnirelman 1974, and first proved for closed manifolds by Zelditch and Colin de Verdière 1986,

semiclassical case by Helffer-Robert-Martinez 1989,

for a class of billiards in any dimension (incl. Bunimovich) by Gérard-Leichtnam 1993, and for arbitrary manifolds with piecewise smooth boundaries by Zelditch-Zworski 1996.

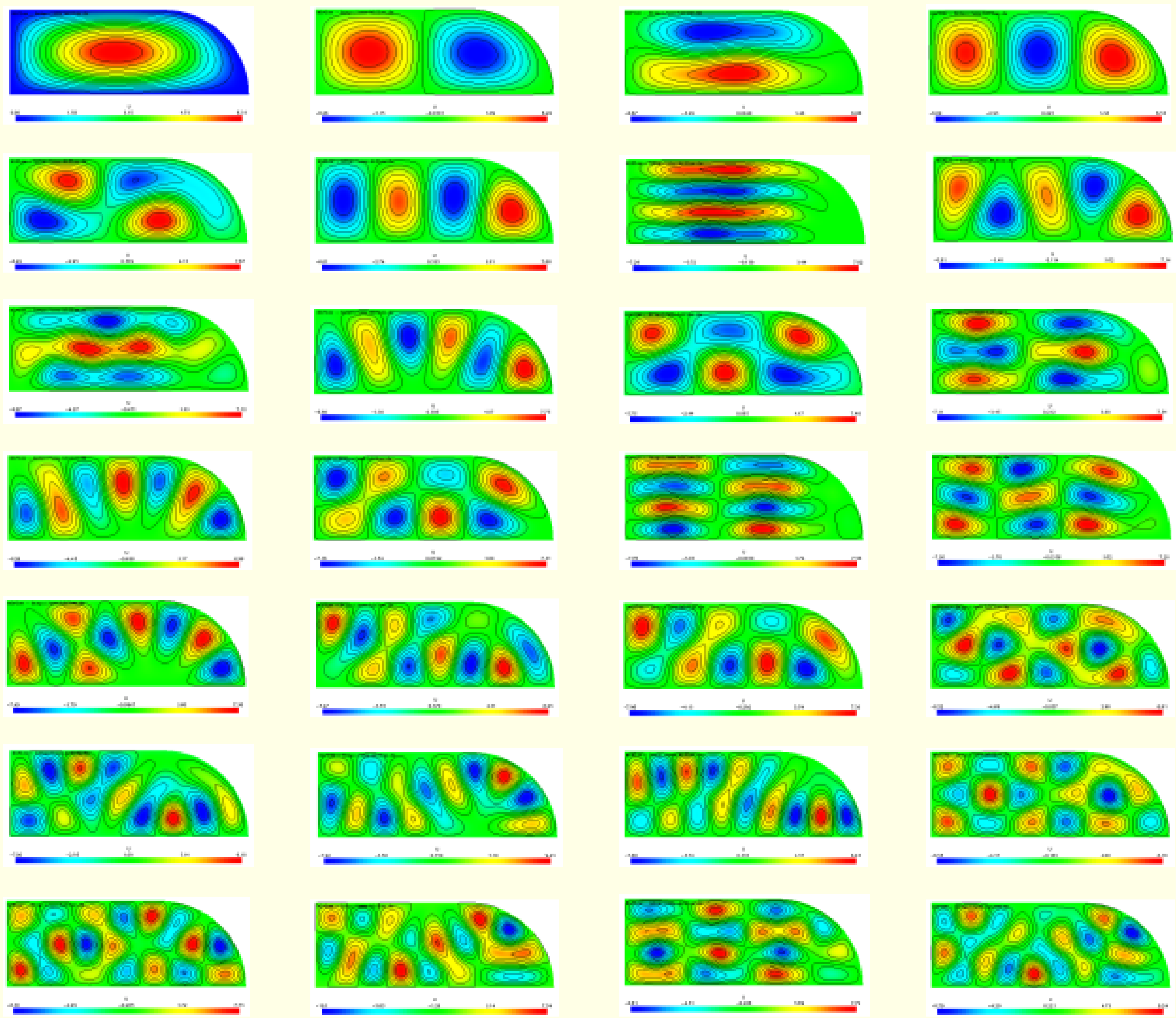
There exist versions for quantum maps with a lot of recent progress on the concentration and non-concentration: De Bièvre, Faure, Nonnenmacher, Rudnick....

The asymptotics

$$\lim_{k\to\infty}\int_V |u_{j_k}(x)|^2 dx = \frac{\mathrm{Area}(V)}{\mathrm{Area}(\Omega)}\,.$$

can be observed.

Here are the experimental images of the first 24 eigenfunctions in a Bunimovich cavity:

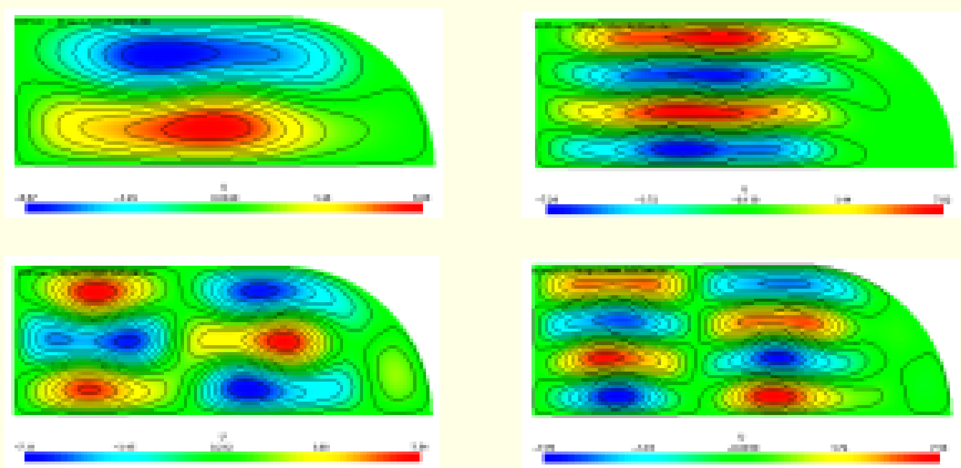


Can there exist exceptional sequences?

That is, can we have a sequence $j_k \to \infty$ and an open set $R \subset \Omega$ such that

$$\lim_{k\to\infty} \int_R |u_{j_k}(x)|^2 dx = 1 > \frac{\text{Area}(R)}{\text{Area}(\Omega)} \, ?$$

We have a candidate sequence:



The open set R is the rectangle obtained by "sawing off" the wings of the table.

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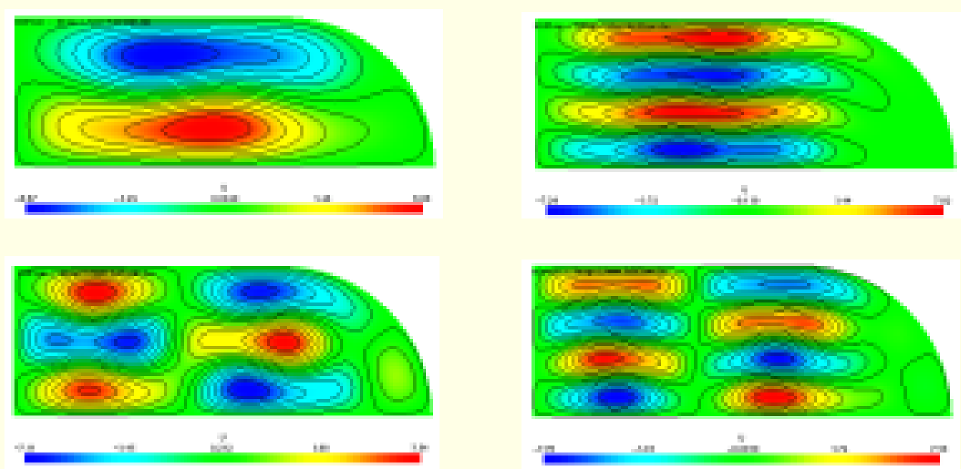


Note that 4 in 24 is a "density zero" sequence!

Theorem 1.

Suppose that u is an eigenfuction of the Laplacian on the Bunimovich stadium. Let V be any open neighbourhood of the "wings".

Then

$$\int_V |u(x)|^2 dx \geq \frac{1}{C_V} > 0 \,.$$

More generally, if

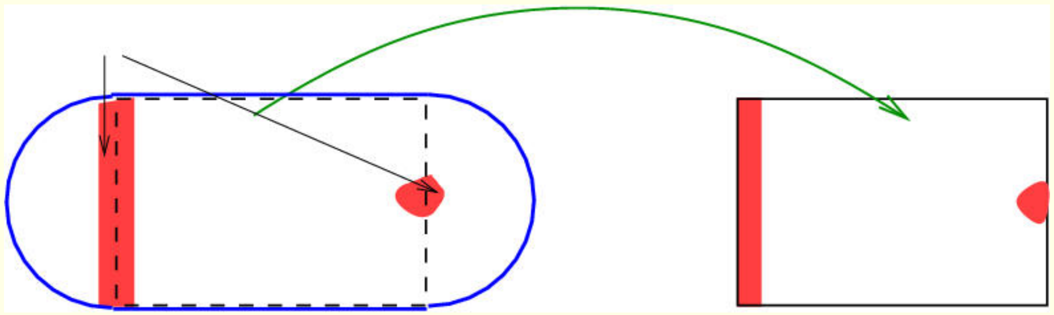
$$(-\Delta - z)v = f \,, \quad v|_{\partial\Omega} = 0 \,,$$

then

$$\int_\Omega |f(x)|^2 dx + \int_V |u(x)|^2 dx \geq \frac{1}{C_V} \int_\Omega |u(x)|^2 dx \,.$$

Remark: By using control theory results à la Bardos-Lebeau-Rauch we can reduce V to a control set.

For instance



The control set V is the red set on the left.

Both this and the relevance of the picture on the right will hopefully become clear in a moment.

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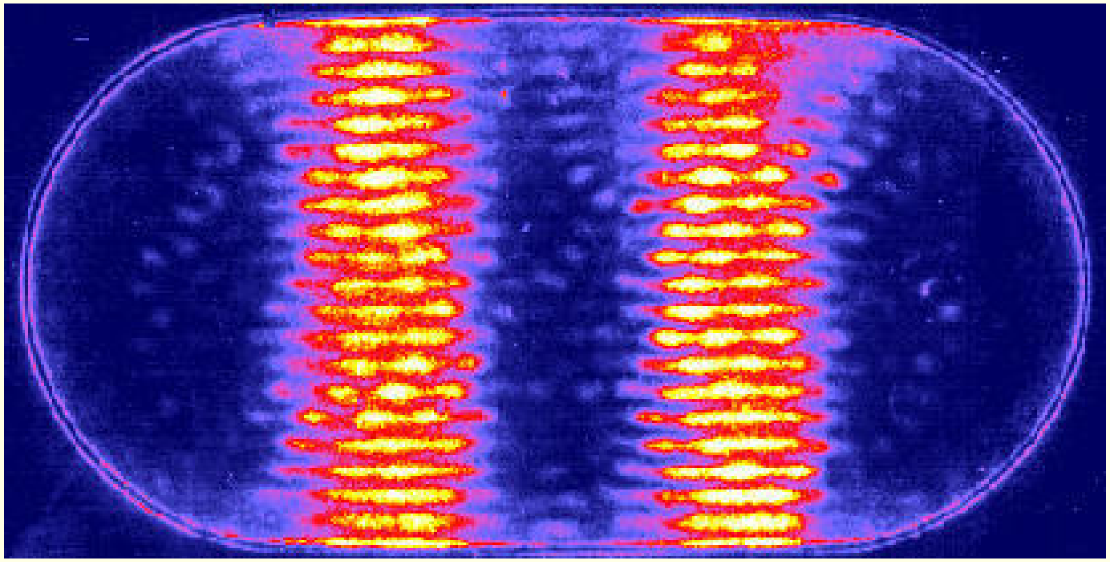

Can we have a concentration in the rectangle?

Trivial quasi-mode concentrating in the interior of the rectangle:

$$(-\Delta - \mu_k)v_k = \mathcal{O}(1)\,, \quad v_k|_{\partial\Omega} = 0\,, \quad \int_R |v_k|^2 \to 1\,.$$

Since in Theorem 1 we had

$$(-\Delta - z)v = f\,, \quad v|_{\partial\Omega} = 0$$

$$\Longrightarrow$$

$$\int_\Omega |f(x)|^2 dx + \int_V |u(x)|^2 dx \geq \frac{1}{C_V} \int_\Omega |u(x)|^2 dx$$

this best possible if we demand that v_k's concentrate in a set smaller than R.

Can we have a concentration in the rectangle?

A highly probable statement:

$$(-\Delta - \mu_k)v_k = o(1)\,, \quad v_k|_{\partial\Omega} = 0\,, \quad \mu_k \to \infty\,,$$

$$\int_\Omega |v_k|^2 = 1\,, \quad \int_R |v_k|^2 \to 1\,.$$

I will offer a dinner in a restaurant in Paris to the first person in this audience who shows me, within five years from now, a proof of this, possibly elementary, result.

Can we have a concentration in the rectangle?

A more dubious statement:

$$(-\Delta - \mu_k) v_k = \mathcal{O}(\mu_k^{-\infty}) \,, \quad v_k|_{\partial\Omega} = 0 \,, \quad \mu_k \to \infty \,,$$

$$\int_\Omega |v_k|^2 = 1 \,, \quad \int_R |v_k|^2 \to 1 \,.$$

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Can we have a concentration in the rectangle?

An impossible statement:

$$(-\Delta - \mu_k)v_k = 0\,, \quad v_k|_{\partial\Omega} = 0\,, \quad \mu_k \to \infty\,,$$

$$\int_\Omega |v_k|^2 = 1\,, \quad \int_R |v_k|^2 \to 1\,.$$

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To encourage everybody I will now show how elementary is the proof of Theorem 1!

We need the following result motivated by control theory:

Proposition.(Burq 1993)

Let $\Delta = \partial_x^2 + \partial_y^2$ be the Laplace operator on the rectangle $R = [0,1]_x \times [0,a]_y$.

Then for any open $\omega \subset R$ of the form $\omega = \omega_x \times [0,a]_y$, there exists C such that for any solution of

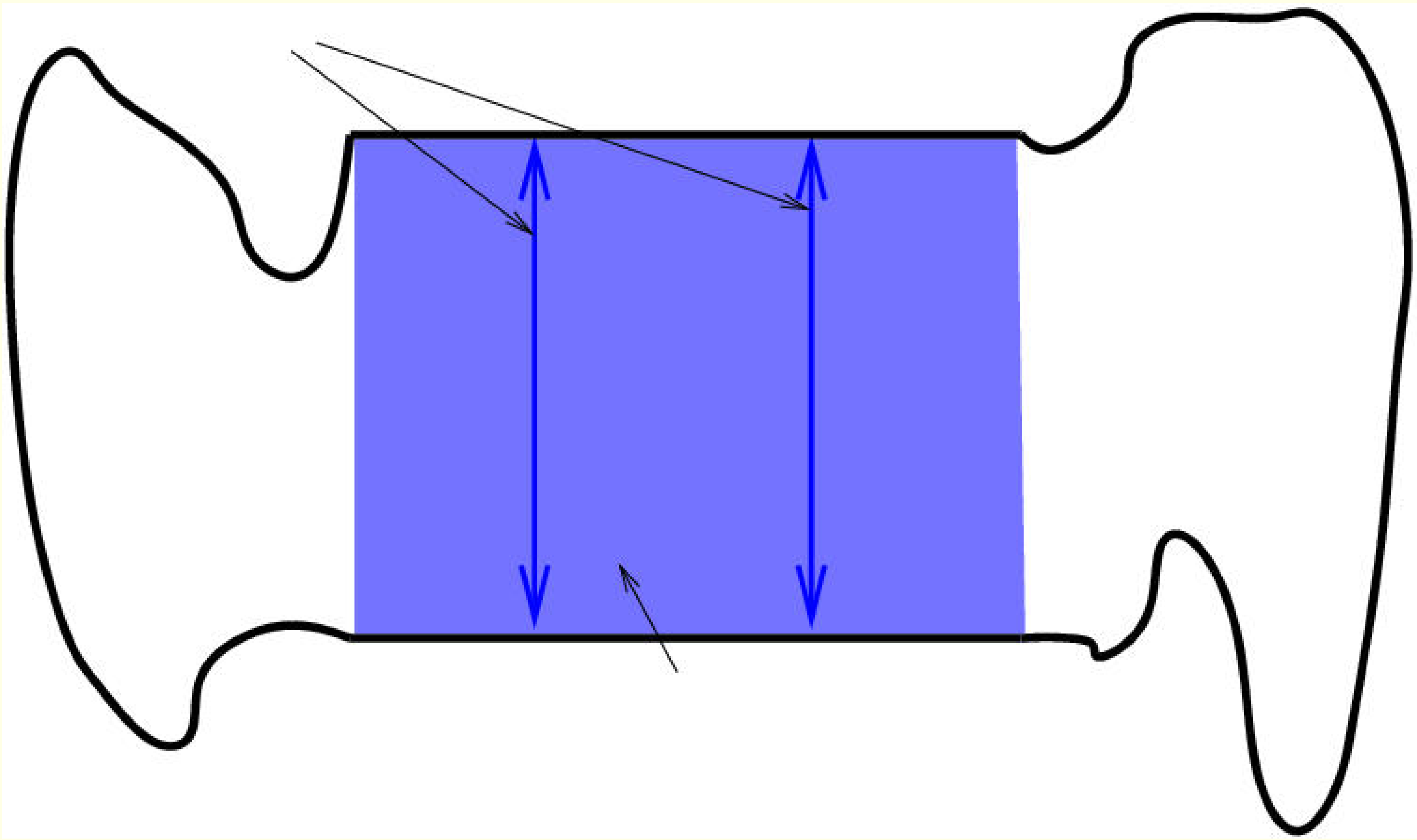
$$(-\Delta - \lambda^2)u = f \quad \text{on } R\,, \quad u|_{\partial R} = 0\,,$$

we have

$$\|u\|_{L^2(R)} \leq C \left(\|f\|_{L^2_y H^{-1}_x(R)} + \|u\|_{L^2(\omega)} \right)$$

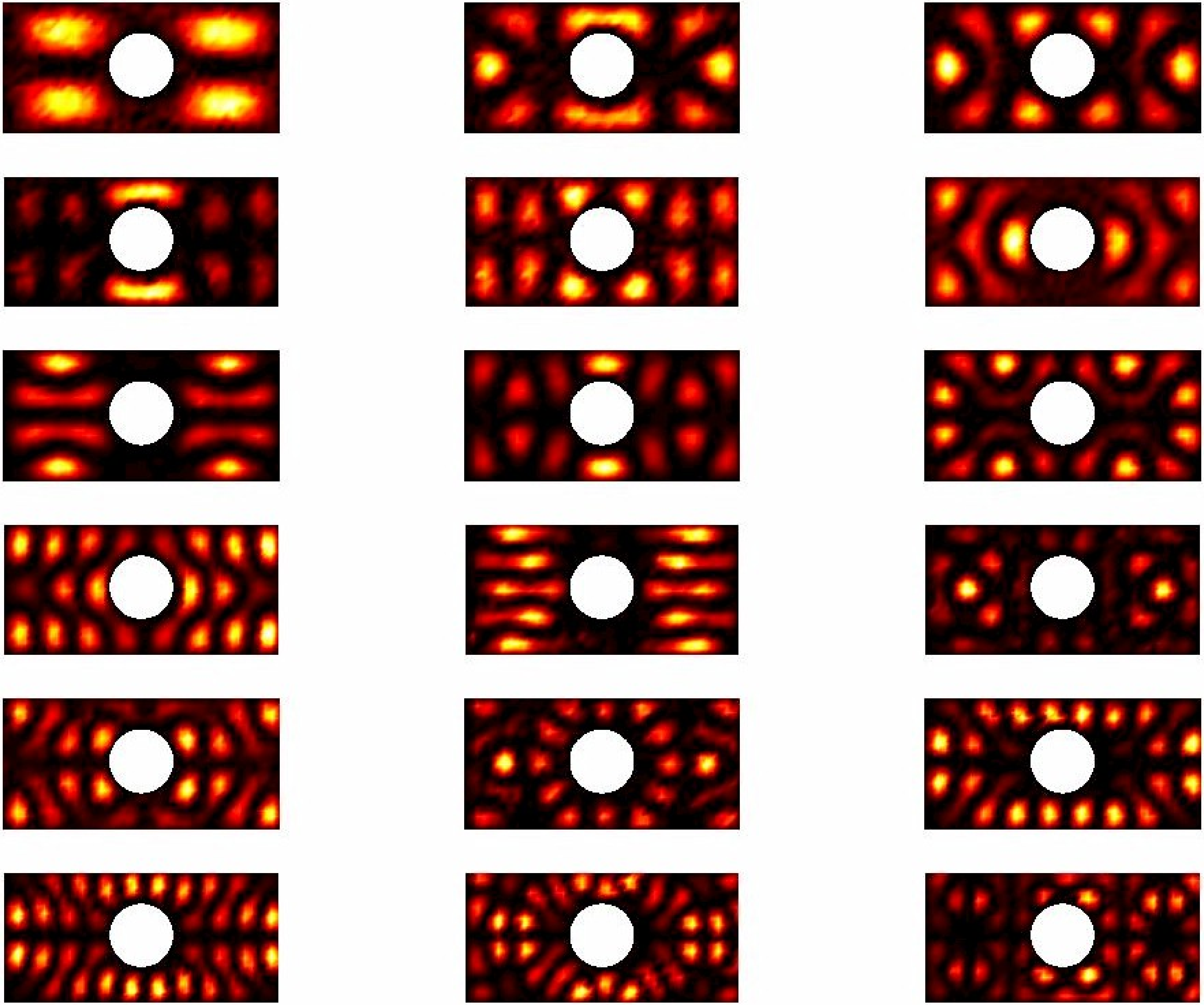
Partially rectangular billiards

No concentration possible on single bouncing ball orbits

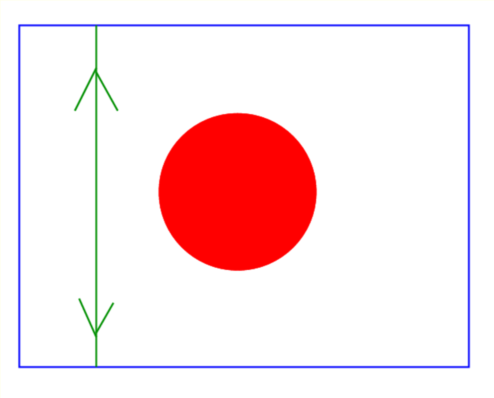


We expect, in some cases, concentration in the entire rectangle.

What about the Sinai billiard?



Bouncing ball trajectories:

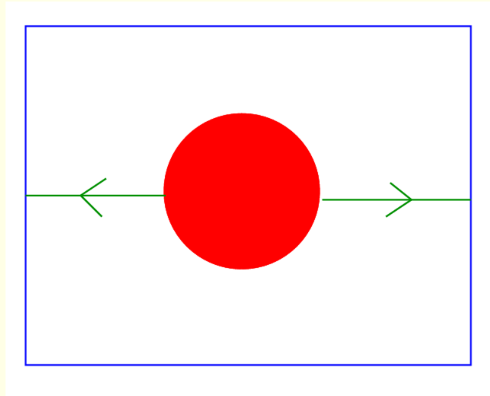


The same argument as before shows that no concentration is possible on a rectangle which does not touch the obstacle.

A refinement shows that for any neighbourhood, V, of the obstacle we have

$$\int_V |u(x)|^2 dx \geq \frac{1}{C_V} > 0 \, .$$

Hyperbolic trajectories:



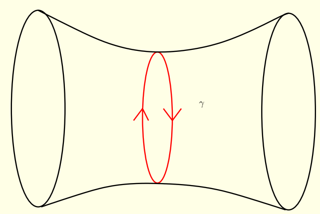
Theorem 2. Let V be a neighbourhood of the hyperbolic trajectory above. Then

$$\int_{\Omega \setminus V} |u(x)|^2 dx \geq \frac{c}{\log \lambda}, \quad c > 0 .$$

The proof is based on ideas from scattering in the presence of one trapped hyperbolic orbit: Ikawa, Gérard, Sjöstrand, ... , J.F.Bony-Michel.

The bound $\int_{\Omega\setminus V} |u(x)|^2 dx \geq c/\log\lambda$ is in some sense optimal.

Colin de Verdière-Parisse 1994 considered a truncated hyperbolic cylinder:



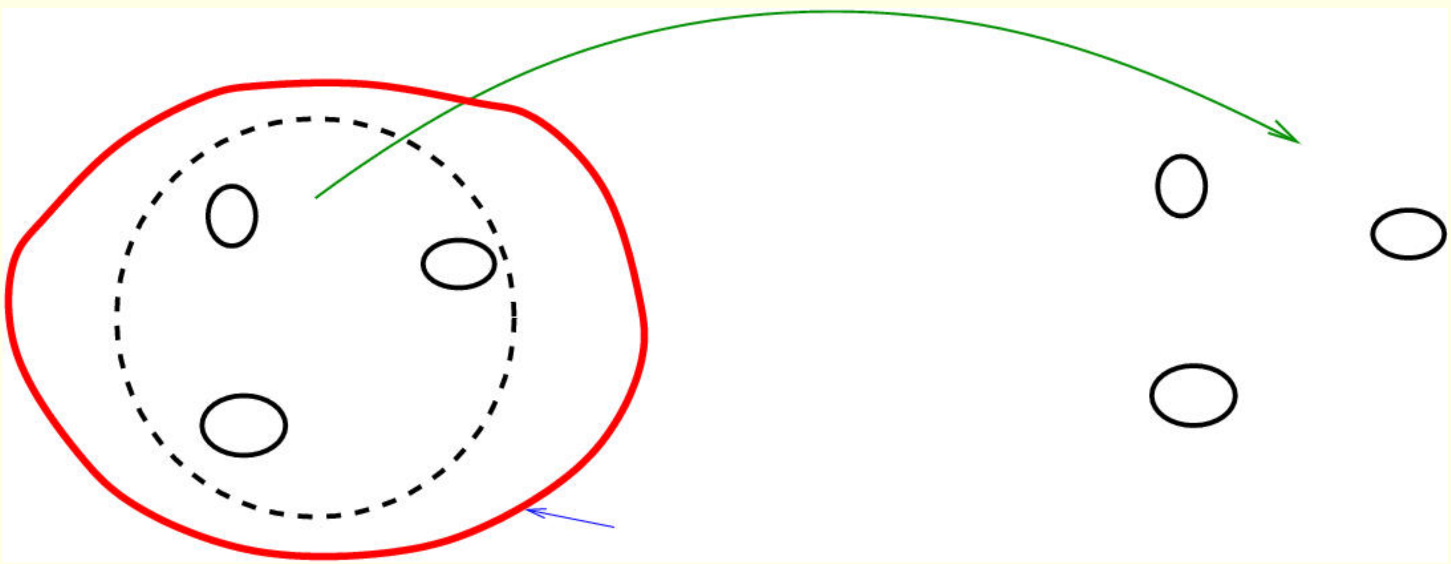
They showed the lower bound for surfaces containing this type of "neck" and for the truncated cylinder itself showed that the weak concentration is possible. The methods were based on the reduction to one dimension (cf. Paul-Uribe, Fujiie-Ramond).

Theorem 2 works for any closed (real) hyperbolic orbit which does not intersect the boundary.

General point of view:

Studying an effect of a black box in a closed system by putting that black box in an open system (or a system with an absorbing barrier).

Reversal of the black box strategy of Sjöstrand-Zworski 1991 in the study of resonances.



A different approach to Burq 1993.

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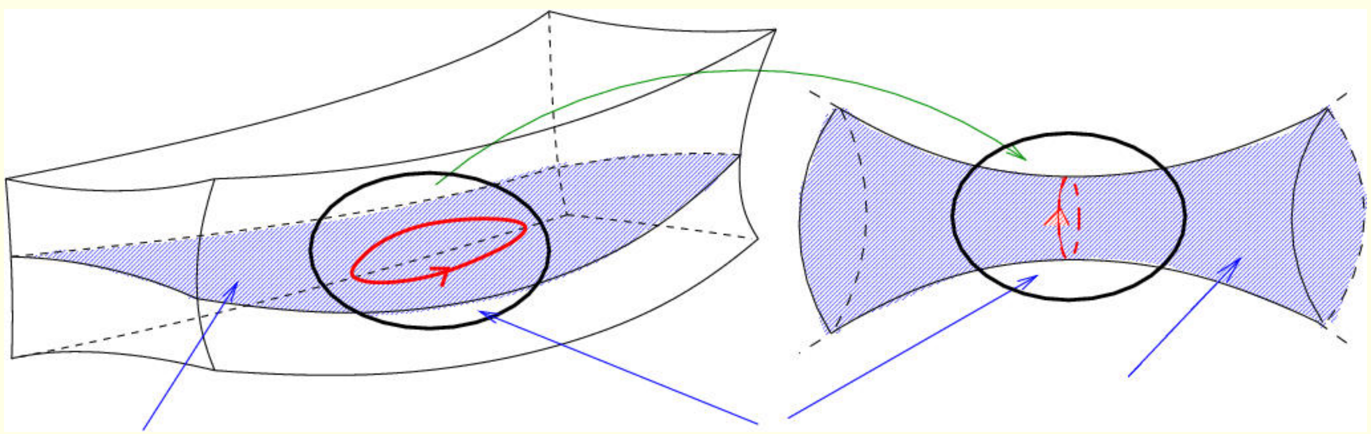

A hyperbolic orbit.

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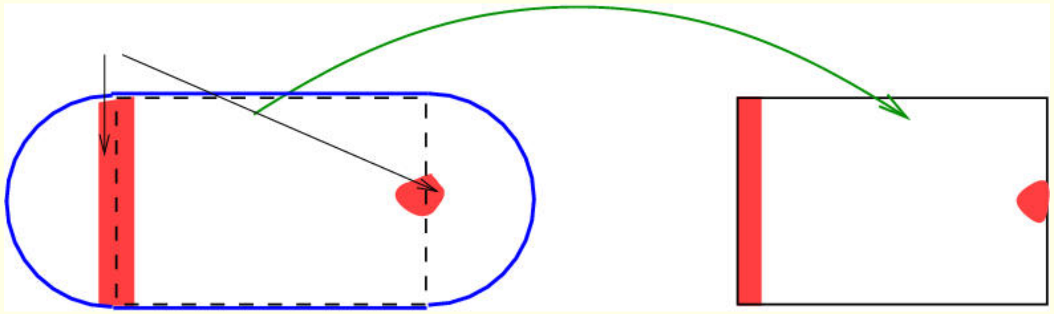

Rectangle as a black box for the Bunimovich stadium.

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References:

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