FINE STRUCTURE OF FLAT BANDS IN A CHIRAL MODEL OF MAGIC ANGLES

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Abstract. We analyze symmetries of Bloch eigenfunctions at magic angles for the Tarnopolsky–Kruchkov–Vishwanath chiral model of the twisted bilayer graphene (TBG) following the framework introduced by Becker–Embree–Wittsten–Zworski. We show that vanishing of the first Bloch eigenvalue away from the Dirac points implies its vanishing at all momenta, that is the existence of a flat band. We also show how the multiplicity of the flat band is related to the nodal set of the Bloch eigenfunctions. We conclude with two numerical observations about the structure of flat bands.

1. Introduction

In this article we study the chiral version [SGG12, TKV19] of the Bistritzer–MacDonald Hamiltonian [BiMa11] describing twisted bilayer graphene:

$$H(\alpha) := \begin{pmatrix} 0 & D(\alpha)^* \\ D(\alpha) & 0 \end{pmatrix}, \quad D(\alpha) := \begin{pmatrix} 2D_{\bar z} & \alpha U(z) \\ \alpha U(-z) & 2D_{\bar z} \end{pmatrix}, \tag{1.1}$$

where U is a real analytic function on $\mathbb{C} = \mathbb{R}^2$, and

$$\begin{gathered} U(z+\gamma) = e^{i\langle \gamma, K\rangle} U(z), \quad U(\omega z) = \omega U(z), \quad \overline{U(\bar z)} = -U(-z), \quad \omega = e^{2\pi i/3}, \\ \gamma \in \Lambda := \omega\mathbb{Z} \oplus \mathbb{Z}, \quad \omega K \equiv K \not\equiv 0 \mod \Lambda^*, \quad \Lambda^* := \frac{4\pi i}{\sqrt{3}}\Lambda, \quad \langle z, w\rangle := \mathrm{Re}(z\bar w). \end{gathered} \tag{1.2}$$

The most studied case is the Bistritzer–MacDonald potential which in the convention of (1.2) corresponds to

$$U(z) = -\tfrac{4}{3}\pi i \sum_{\ell=0}^{2} \omega^\ell e^{i\langle z, \omega^\ell K\rangle}, \quad K = \tfrac{4}{3}\pi, \tag{1.3}$$

see the Appendix for the translation of the conventions.

Definition. *A value of* α *is called* magical *if the Hamiltonian* $H(\alpha)$ *has a* flat band *at zero energy* (*see* (1.6) *below*). *This is equivalent to* $\mathrm{Spec}_{L^2(\mathbb{C}/3\Lambda)} D(\alpha) = \mathbb{C}$.

In the physics literature – see [TKV19] – α is a dimensionless parameter which, modulo physical constants, is proportional to the angle of twisting of the two sheets of graphene. Hence, large α's correspond to small angles.

We know from [Be*22] that the set of magic α's, $\mathcal{A}$, is a discrete subset of $\mathbb{C}$. In [BHZ22] we proved that for the potential (1.3) $\mathcal{A}$ is in fact infinite. Existence and estimates for the first *real* magic α were obtained by Luskin and Watson [WaLu21] who implemented the method of [TKV19] with computer assistance (see also Remarks following Theorem 4). We also remark that a rigorous derivation of the full Bistritzer–MacDonald model was provided in [CGG22, Wa*22]

Following the physics literature we consider (unlike in [Be*22]) Floquet theory with respect to moiré translations: for $u \in L^2_{\rm loc}(\mathbb{C};\mathbb{C}^2)$ we put

$$\mathscr{L}_\gamma u := \begin{pmatrix} e^{i\langle \gamma, K\rangle} & 0 \\ 0 & e^{-i\langle\gamma,K\rangle} \end{pmatrix} u(z+\gamma), \quad \gamma \in \Lambda, \quad K = \tfrac{4}{3}\pi. \tag{1.4}$$

(Here and elsewhere $\langle z, w\rangle := \operatorname{Re} z\bar{w}$, $z, w \in \mathbb{C}$.) The action is extended diagonally for $\mathbb{C}^4 = \mathbb{C}^2 \times \mathbb{C}^2$ and we use the same notation. We then have have $\mathscr{L}_\gamma D(\alpha) = D(\alpha)\mathscr{L}_\gamma$ and $\mathscr{L}_\gamma H(\alpha) = H(\alpha)\mathscr{L}_\gamma$.

It is then natural to look at the spectrum of $H(\alpha)$ satisfying the following boundary conditions:

$$\begin{gathered} H(\alpha)u = Eu, \quad u \in H^1_k(\mathbb{C}/\Lambda, \mathbb{C}^4), \quad H^s_k(\mathbb{C},\mathbb{C}^4) := L^2_k(\mathbb{C};\mathbb{C}^4) \cap H^1_{\rm loc}(\mathbb{C};\mathbb{C}^4), \\ L^2_k(\mathbb{C}/\Lambda, \mathbb{C}^4) := \{ u = L^2_{\rm loc}(\mathbb{C};\mathbb{C}^4) : \mathscr{L}_\gamma u = e^{i\langle k,\gamma\rangle} u \}. \end{gathered} \tag{1.5}$$

The spectrum is discrete and symmetric with respect to the origin and we index it as follows (with $\mathbb{Z}^* := \mathbb{Z} \setminus \{0\}$)

$$\begin{gathered} \{E_j(\alpha,k)\}_{j\in\mathbb{Z}^*}, \quad E_j(\alpha,k) = -E_{-j}(\alpha,k), \\ 0 \le E_1(\alpha,k) \le E_2(\alpha,k) \le \cdots, \quad E_1(\alpha,K) = E_1(\alpha,-K) = 0, \end{gathered} \tag{1.6}$$

see §2.2 for more details. The points $K, -K$ are called the *Dirac points* and are typically denoted by K and K' in the physics literature. (See the appendix to see different K and K' when different representation of Λ is used.)

The definition of the set of magical α's can now be rephrased as follows

$$\mathcal{A} := \{\alpha \in \mathbb{C} : \forall\, k \in \mathbb{C}, \quad E_1(\alpha,k) \equiv 0\} \tag{1.7}$$

Our first theorem states that if the Bloch eigenvalue vanishes away from the Dirac points then it vanishes identically, that is the band is flat:

Theorem 1. *Suppose $\alpha \in \mathbb{C}$ and $E_1(\alpha,k)$ is defined using (1.5),(1.6) for $H(\alpha)$ given by (1.1) with U satisfying (1.2). Then*

$$\exists\ k \notin \{-K,K\} + \Lambda^* \quad E_1(\alpha,k) = 0 \iff \forall\ k \in \mathbb{C} \quad E_1(\alpha,k) = 0. \tag{1.8}$$

In other words, zero energy band is flat if and only if the Bloch eigenvalue is 0 at some $k \notin \{-K,K\} + \Lambda^*$, which is the lattice of conic points (see Figure 2).

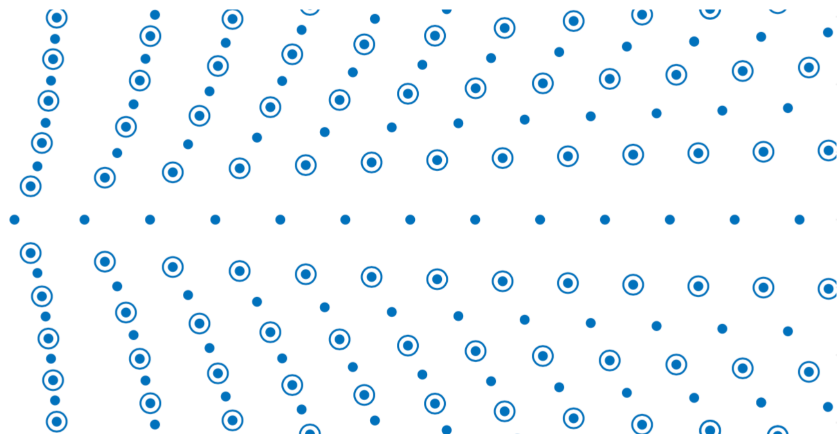

FIGURE 1. The multiplicity of the flat band for complex values of α can be double as illustrated here. When the potential is replaced by $U_\theta(z) = \cos\theta U(z) + \sin\theta \sum_{k=0}^2 \omega^k e^{\bar z\omega^k - z\bar\omega^k}$ the symmetries (A.1) (we are using coordinates of [Be*22] – see Appendix A) are preserved but the dynamics of α's is interesting when μ varies. A movie showing $\mathcal{A}$ as θ varies with multiplicities color coded can be found at https://math.berkeley.edu/~zworski/multi.mp4.

The next theorem gives a useful criterion for simplicity. It is used in [BHZ22] to prove existence and *simplicity* of the first magic α and also in [BeZw23].

Theorem 2. *If $\alpha \in \mathcal{A}$ then, in the notation of (1.5),*

$$\begin{aligned} \forall\, j > 1,\ k \in \mathbb{C} \quad E_j(\alpha, k) > 0 \iff &\ \forall\, k \in \mathbb{C} \quad \dim \ker_{L^2_k(\mathbb{C}/\Lambda)} D(\alpha) = 1 \\ \iff &\ \exists\, p \in \mathbb{C} \quad \dim \ker_{L^2_p(\mathbb{C}/\Lambda)} D(\alpha) = 1. \end{aligned} \tag{1.9}$$

In other words, the simplicity of 0 as the eigenvalues of $D(\alpha)$ on $L^2_k(\mathbb{C}/\Gamma;\mathbb{C}^4)$ for all k is equivalent to the simplicity of the zero eigenvalue of $D(\alpha)$ on $L^2_p(\mathbb{C}/\Lambda;\mathbb{C}^2)$, for any one p.

The symmetries of the potential U imply that U vanishes at the *stacking point* of high symmetry: $z_S := i/\sqrt{3} = (\omega - \omega^2)/3 \in \Lambda/3$:

$$\omega z_S = z_S - 1 - \omega \equiv z_S \mod \Lambda \implies U(-z_S) = 0. \tag{1.10}$$

(To obtain this conclusion use (1.2) to see that $U(z_S + \omega\zeta) = \bar\omega U(z_S + \zeta)$.)

In the work of Tarnopolsky et al [TKV19], flat bands were characterized by vanishing of a distinguished element of the kernel of $D(\alpha)$ at the stacking points $\pm z_S$. For the potential (1.3) it was claimed that the vanishing of an eigenvector $u \in \ker_{L^2_0}(D(\alpha) - K)$ occurs precisely at z_S . This is equivalent to showing that the zero of $u \in \ker_{L^2_{-K}} D(\alpha)$ occurs precisely at z_S. We show that this is indeed true when $\alpha \in \mathcal{A}$ is simple and formulate it more generally:

Theorem 3. *Suppose the equivalent conditions in (1.9) hold. Then, non-trivial elements of (one dimensional) space* $\ker_{H^1_k(\mathbb{C}/\Lambda;\mathbb{C}^2)} D(\alpha)$ *have zeros of order one at*

$$\frac{\sqrt{3}k}{4\pi i} + \Lambda \tag{1.11}$$

and nowhere else. In particular for $k = -K$ *the zeros occur precisely at the stacking points* $z_S + \Lambda$.

Remark. We consider the zero at z_0 to be of order one if $\partial_z u(z_0) \neq 0$; the equation implies at zeros $\partial_{\bar{z}}^\ell u = 0$ for all ℓ – see Lemma 3.2. This implies that $u(z_0 + \zeta) = \zeta w(\zeta, \bar{\zeta})$, $w(0) \neq 0$ and w is holomorphic near $0 \in \mathbb{C}^2$. Theorem 3 is illustrated by Figure 5.

As a consequence we find (in §5.2)

Theorem 4. *If* $\dim \ker_{L^2_0} D(\alpha) = 1$, *then the Chern number associated with the Bloch function* $u_k \in \ker_{L^2_{\mathbf{0}}}(D(\alpha) + k)$, *is equal to one (see §5.2 for a precise formulation.)*

Remarks 1. Theorem 2 shows that the assumption of Theorem 3 is equivalent to the minimal multiplicity of the flat band, that is to $|E_j(\alpha, k)| > 1$ for $|j| > 1$.

2. Numerical results suggest that the first string of complex α's in $\mathcal{A}$ for (1.3) have higher multiplicities (see Figure 1 where double α's are indicated) and in that case the zeros of $u_K \in \ker_{H^1_0}(D(\alpha) + K)$ appear at $-z_S + \Lambda$.

3. In [BHZ22], we show that the first real angle (existence of which was first established by Watson–Luskin [WaLu21]) is in fact simple. For higher real α's for the potential (1.3) numerical experiments [Be*22] provide strong evidence of simplicity.

We also make two numerical observations presented in §5. The first one is illustrated by Figure 2 and the movie referenced there. We see that the rescaled first band is nearly constant close to magic angles and its shape is closed to that of $|U|$ after a linear changes of variables $z \mapsto k$.

The second observation concerns the behaviour of the curvature of the hermitian holomorphic line bundle (somewhat informally) defined by $k \mapsto u_k$ [Le*20] (with hermitian structured inherited from L^2). We observe that the curvature peaks at the Γ point, that is, in our notation, at $k = i$ – see Figure 3. It is also interesting to note that the curvature does not change much at different magic α's – see §5 for definition and computational details.

Comments on an earlier version of this paper. We now concentrate exclusively on the case of simple bands with an expanded discussion of multiplicities moved to

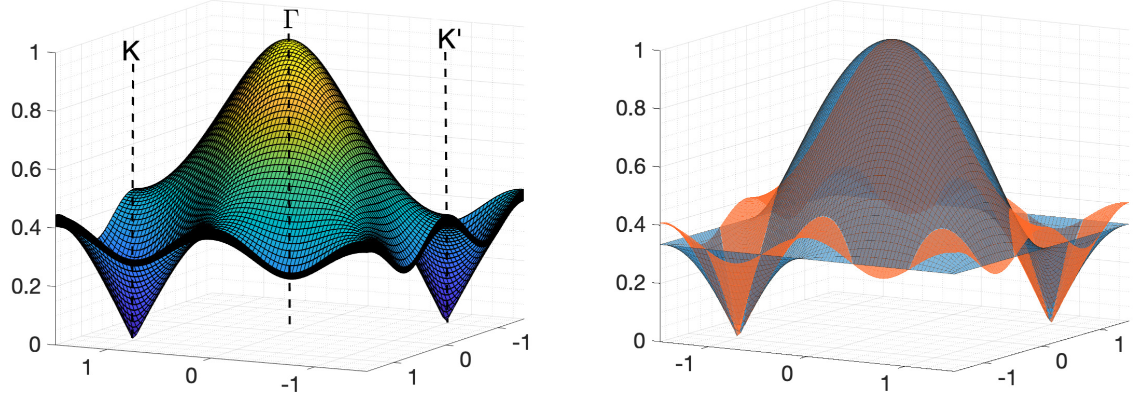

FIGURE 2. Plots of $k \mapsto E_1(\alpha,k)/(\max_k E_1(\alpha,k))$ for $0.4 < \alpha < 0.6$ (left) ($k = (\omega^2 k_1 - \omega k_2)/\sqrt{3}$, $|k_j| \leq \frac{3}{2}$ and we use the coordinates k_j). Although the band becomes flat at the first magic $\alpha \simeq 0.586$, the rescaled plots remain almost fixed and close to $k \mapsto |2\partial_z U(-4\sqrt{3}\pi i k/9)|$ (right, blue-coloured) compared with $E_1(0.58,k)$ (right, orange-coloured). For an animated version see https://math.berkeley.edu/~zworski/KKmovie.mp4.

[BHZ]. That paper will also include a modified version of generic multiplicities. Contrary to our earlier statement, certain double α's are protected (for instance the ones marked as double in Figure 1).

We conclude this introduction by discussing relation to some physics issues.

The anomalous quantum Hall effect. The analysis of the multiplicity of the flat band has immediate implications on the transport properties of twisted bilayer graphene. In the case of a simple magic angle, the two bands have Chern numbers ± 1 resulting in a net Chern number zero. While this cancellation may sound discouraging at first, it has been recently discovered that twisted bilayer graphene hosts an anomalous quantum Hall effect when it is aligned with hexagonal Boron nitride (hBN) [Se*20]. In that case, an additional sublattice potential of strength $m > 0$ is added to the Hamiltonian, that is, the Hamiltonian in (1.1) is replaced by

$$H_m(\alpha) = \begin{pmatrix} m & D(\alpha)^* \\ D(\alpha) & -m \end{pmatrix}.$$

This effective mass splits the two flat bands at zero energy to one at energy m and one at $-m$, respectively. It follows then from Theorem 4 that the anomalous Hall conductivity σ of any individual flat band at energy m has Chern number -1 which by

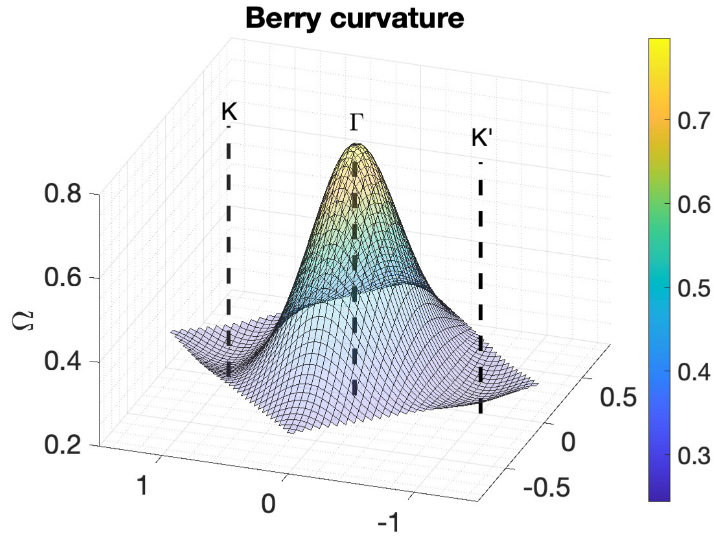

FIGURE 3. The plot of the curvature of the holomorphic line bundle corresponding to the first simple band, defined in (5.6). The extrema at K, Γ, K' follow from Prop. 5.3 and the subsequent discussion.

the Kubo formula corresponds to a Hall conductivity

$$\sigma = -\frac{e^2}{2\pi\hbar} c_1.$$

For the band at energy $-m$ the Chern number is $+1$.

Superfluid weights. Twisted bilayer graphene exhibits a form of superconductivity at the magic angles. The Bardeen-Cooper-Schrieffer (BCS) theory states [Sch64] that the critical temperature of a superconductor satisfies $T_C \propto e^{-1/(n_F U)}$ where n_F is the DOS at Fermi level and U the interaction between electrons forming a Cooper pair showing why flat bands are promising candidates for high-temperature superconductors. Although this identifies the critical temperature it does not explain whether superconductivity actually exists. Another necessary condition for superconducting states in flat bands has recently been discussed in a series of works by Peotta, Törmä, and collaborators [PT15, JPVKT16, LVPSHT16, TLP18], see also [PTB22] for an analysis in the context of moire materials, stressing the importance of the flat band geometry characterized by the *quantum geometric tensor*. The electrodynamic properties of a superconductor are captured by the London equation $j = -D_s A$ where j is the current density, A the vector potential in the London gauge and D_s the superfluid weight. In [PTB22, (22)] it is argued that, under some approximations, for a flat band with filling factor ν

$$D_s \propto \nu(1-\nu)\int_{\text{B.Z.}} g \text{ with metric } g = \partial_{\bar z}\partial_z \log(h(z))|dz|^2,$$

that is, Cooper pairs can support transport in bands with non-trivial topology. In particular, the volume of the metric is then proportional to the first Chern number

$$D_s \propto \nu(1-\nu)|c_1|,$$

which by Theorem 4 is equal to 1 for an isolated flat band emphasizing the importance of a non-zero first Chern number in such systems.

Fractional Quantum Hall effect. The multiplicity of the flat band has also implications for other many-body phenomena. Unlike the integer quantum Hall effect which can be understood in a single-particle picture, the fractional quantum Hall effect is a many-body effect conjectured to appear in twisted bilayer graphene [Le*20]. In its original formulation, Laughlin [L83] constructed under the assumption of a sufficiently large gap of the flat bands, a many-particle wavefunction using the lowest landau levels which was then generalized by Haldane and Rezayi to the torus [HR85], see also [F15]. Theorem 1 together with [BHZ22, Theorem 3] ensures the existence of such a gap at the first magic angle. Let us briefly explain the construction in [Le*20]: One defines $\Gamma_N := \frac{4\pi i N_1}{3}\mathbb{Z}\omega + \frac{4\pi i N_2}{3}\mathbb{Z}\omega^2$ with $N_s := N_1N_2$ and $\tau = N_2\omega/N_1$. If N_e is the number of electrons occupying the band, then we require $m := N_s/N_e \in 2\mathbb{N}_0+1$. The ansatz for the Laughlin state of the interacting N_e-body electron system, depends on the multiplicity and zero set of the Bloch function, identified in Theorems 3,

$$\begin{aligned}\psi(z_1,...,z_{N_e}) &= F(z_1,...,z_{N_e})\prod_{i=1}^{N_e}\frac{u(z_i)}{\vartheta_1\Big(3(z_i+z_S)/(4\pi i\omega)|\omega\Big)}\\ F(z_1,...,z_N) &= G_m(Z)\prod_{i<j} g(z_i-z_j).\end{aligned}$$

The Bloch conditions $\mathscr{L}^{(i)}_{N_1a_1+N_2a_2}\psi(z_1,...,z_N) = \psi(z_1,...,z_N)$, where $\mathscr{L}^{(i)}$ acts like $\mathscr{L}$ on the i-th coordinate z_i, is then assumed to hold for each particle and implies that g has a zero of order m. A Laughlin state is then obtained by assuming that all zeros occur at the origin which implies, by assuming g to be holomorphic, that $g(z) = \theta_1\big(3z/(4\pi i N_1\omega)|\tau\big)^m$. An easy computation shows that this leaves a m-fold degeneracy in the choice of G which is called the *topological order* of the Laughlin state.

2. Spectral theory and symmetries of the Hamiltonian

In this section we review symmetries of the Hamiltonian, present a more detailed discussion of different approaches to Floquet theory, recall the spectral characterization of magic angles and prove Theorem 1.

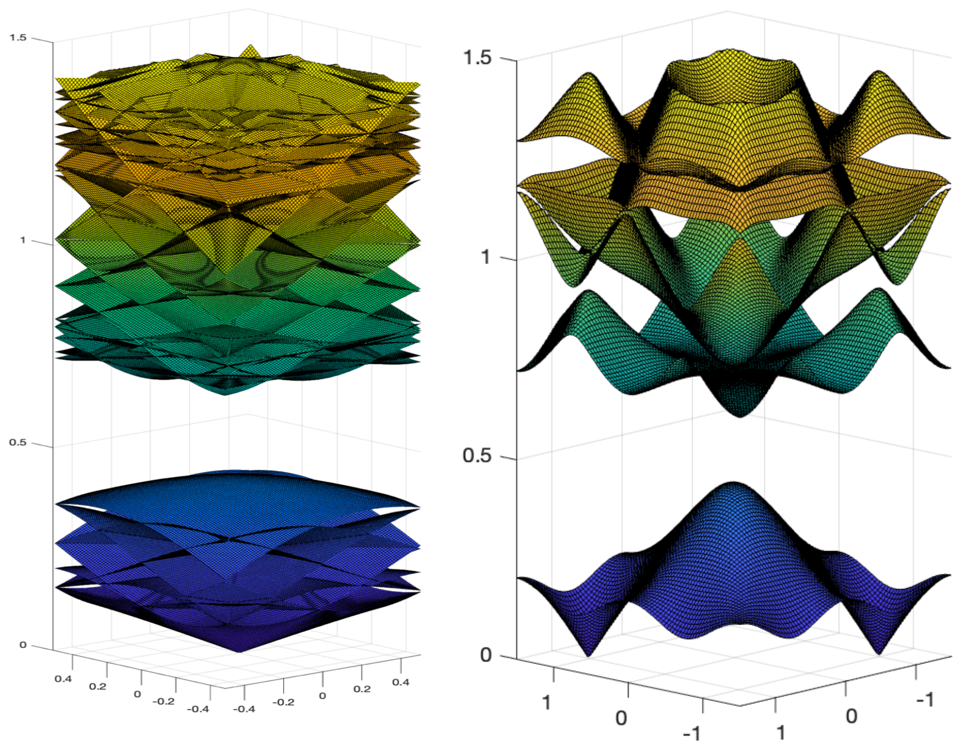

FIGURE 4. The bands are the functions $k \mapsto E_j(\alpha, k)$ where E_j are defined in (1.6). On the left the plot the first 45 bands for $\alpha = 0.3$. defined using the boundary condition $u(z+\gamma) = e^{i\langle \gamma, k\rangle} u(z)$, $\gamma \in \Gamma$, $k \in \mathbb{C}/\Gamma^*$, corresponding to the lattice of exact periodicity of $D(\alpha)$, in the convention of [Be*22] (see Appendix B). The fundamental cell of Γ^*, parametrized by $(k_1, k_2) \mapsto k = (\omega^2 k_1 - \omega k_2)/\sqrt{3}$, $|k_j| < \frac{1}{2}$. On the right the plot of $k \mapsto E_j(0.3, k)$, defined using the boundary condition (1.5) for $1 \leq j \leq 5$, where k in the fundamental cell of $3\Gamma^*$, parametrized by (k_1, k_2) $|k_j| < \frac{3}{2}$. A movie version of the picture on the right can be found at https://math.berkeley.edu/~zworski/chiral_bands.mp4. It is interesting to compare this to the case of the full Bistritzer–MacDonald model [BiMa11] https://math.berkeley.edu/~zworski/BM_bands.mp4 where, in the notation of [Be*21, (1)] we put $w_1 = \alpha$, $w_0 = 0.7\alpha$ and $\varphi = 0$. Remarkably, the low magic α's of the chiral model seem to provide a good approximation for the nearly flat bands of the Bistritzer–MacDonald model. For completeness, the bands for the anti-chiral model $w_1 = 0$, $w_0 = \alpha$, $\varphi = 0$, can be found at https://math.berkeley.edu/~zworski/antichiral_bands.mp4. As shown in [Be*21] there are no exact flat bands in that case.

2.1. **Symmetries revisited.** We already recalled that $\mathscr{L}_\gamma$ defined in (1.5) commutes with $D(\alpha)$ and (extended diagonally) with $H(\alpha)$. The rotation

$$\Omega u(z) := u(\omega z), \quad u \in \mathscr{S}'(\mathbb{C}; \mathbb{C}^2),$$

satisfies

$$\Omega D(\alpha) = \omega D(\alpha) \Omega,$$

and produces a commuting action on $H(\alpha)$ as follows

$$\mathscr{C} H(\alpha) = H(\alpha) \mathscr{C}, \quad \mathscr{C} := \begin{pmatrix} \Omega & 0 \\ 0 & \bar{\omega}\Omega \end{pmatrix} : L^2_{\rm loc}(\mathbb{C}; \mathbb{C}^4) \to L^2_{\rm loc}(\mathbb{C}; \mathbb{C}^4). \tag{2.1}$$

We then have

$$\mathscr{L}_\gamma \Omega = \Omega \mathscr{L}_{\omega\gamma}, \quad \mathscr{L}_\gamma \mathscr{C} = \mathscr{C} \mathscr{L}_{\omega\gamma}, \quad \mathscr{C} \mathscr{L}_\gamma = \mathscr{L}_{\bar{\omega}\gamma} \mathscr{C}.$$

The chiral symmetry is given by

$$\begin{gathered} H(\alpha) = -\mathscr{W} H(\alpha) \mathscr{W}, \quad \mathscr{W} := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n \times \mathbb{C}^n, \\ \mathscr{W} \mathscr{C} = \mathscr{C} \mathscr{W}, \quad \mathscr{L}_\gamma \mathscr{W} = \mathscr{W} \mathscr{L}_\gamma. \end{gathered} \tag{2.2}$$

We follow [Be*22, §2.1] combine the Λ and $\mathbb{Z}_3$ actions into a group of unitary action which commute with $H(\alpha)$:

$$\begin{gathered} G := \Lambda \rtimes \mathbb{Z}_3, \quad \mathbb{Z}_3 \ni \ell : \gamma \to \bar{\omega}^\ell \gamma, \quad (\gamma, \ell) \cdot (\gamma', \ell') = (\gamma + \bar{\omega}^\ell \gamma', \ell + \ell'), \\ (\gamma, \ell) \cdot u = \mathscr{L}_\gamma \mathscr{C}^\ell u, \quad u \in L^2_{\rm loc}(\mathbb{C}; \mathbb{C}^4). \end{gathered} \tag{2.3}$$

By taking a quotient by 3Λ we obtain a finite group acting unitarily on $L^2(\mathbb{C}/3\Lambda)$ and commuting with $H(\alpha)$:

$$G_3 := G/3\Lambda = \Lambda/3\Lambda \rtimes \mathbb{Z}_3 \simeq \mathbb{Z}_3^2 \rtimes \mathbb{Z}_3. \tag{2.4}$$

In addition to the spaces L^2_k defined in (1.5), we introduce

$$\begin{gathered} L^2_{k,p}(\mathbb{C}/\Lambda; \mathbb{C}^4) := \{ u \in L^2_{\rm loc}(\mathbb{C}; \mathbb{C}^4) : \mathscr{L}_\gamma \mathscr{C}^\ell u = e^{i\langle k, \gamma \rangle} \bar{\omega}^{\ell p} u \}, \\ H^s_{k,p} := L^2_{k,p} \cap H^s_{\rm loc}, \quad k \in (\tfrac{1}{3}\Lambda^*)/\Lambda^* \simeq \mathbb{Z}_3^2, \quad p \in \mathbb{Z}_3. \end{gathered} \tag{2.5}$$

This streamlines the notation of [Be*22] and concentrates on the most relevant representations of G_3. We use the same notation for $\mathbb{C}^2$ valued or scalar functions with $\mathscr{C}$ replaced by Ω.

We have the following orthogonal decompositions

$$\begin{gathered} L^2(\mathbb{C}/3\Lambda) = \bigoplus_{k \in \frac{1}{3}\Lambda^*/\Lambda^*} L^2_k(\mathbb{C}/\Lambda), \\ L^2_k(\mathbb{C}/\Lambda) = \bigoplus_{p \in \mathbb{Z}_3} L^2_{k,p}(\mathbb{C}/\Lambda), \quad k \in \mathcal{K}/\Lambda^*, \end{gathered} \tag{2.6}$$

where

$$\mathcal{K} := \{k \in \mathbb{C} : \omega k \equiv k \mod \Lambda^*\} = \{K, -K, 0\} + \Lambda^*. \tag{2.7}$$

Remark. Decompositions (2.6) do not provide a decomposition of $L^2(\mathbb{C}/3\Lambda)$ into representations of G_3 given by (2.4). In addition to $L^2_{k,p}$, $k \in \mathcal{K}$ and $p \in \mathbb{Z}^3$, we also have two irreducible representations of dimension three – see [Be*22, §2.2]. These representations appear in $\ker_{L^2(\mathbb{C}/3\Lambda,\mathbb{C}^2)} D(\alpha)$ when $\alpha \in \mathcal{A}$ is simple. Since this observation does not play a role in our proofs we do not provide details.

W also recall from [Be*22, §1] the additional symmetry

$$\mathscr{E} D(\alpha)\mathscr{E}^* = -D(\alpha), \quad \mathscr{E} v(z) := Jv(-z), \quad J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{2.8}$$

noting that it plays a crucial role in [Wa*21]. We have

$$\begin{gathered} L^2_{K,\ell}(\mathbb{C}/\Lambda, \mathbb{C}^2) \xrightarrow{\mathscr{E}} L^2_{-K,\ell}(\mathbb{C}/\Lambda, \mathbb{C}^2) \xrightarrow{\mathscr{E}} L^2_{K,\ell}(\mathbb{C}/\Lambda, \mathbb{C}^2), \\ L^2_{0,\ell}(\mathbb{C}/\Lambda, \mathbb{C}^2) \xrightarrow{\mathscr{E}} L^2_{0,\ell}(\mathbb{C}/\Lambda, \mathbb{C}^2). \end{gathered} \tag{2.9}$$

Finally we recall the anti-linear symmetries

$$\begin{gathered} Qv(z) = \overline{v(-z)}, \quad \mathscr{Q}u(z) := \begin{pmatrix} 0 & Q \\ Q & 0 \end{pmatrix} u(z), \\ QD(\alpha)Q = D(\alpha)^*, \quad H(\alpha)\mathscr{Q} = \mathscr{Q}H(\alpha), \end{gathered} \tag{2.10}$$

and

$$\begin{gathered} Q : L^2_{k,p}(\mathbb{C}/\Lambda; \mathbb{C}^2) \to L^2_{k,-p}(\mathbb{C}/\Lambda; \mathbb{C}^2), \\ \mathscr{Q} : L^2_{k,p}(\mathbb{C}/\Lambda; \mathbb{C}^4) \to L^2_{k,-p+1}(\mathbb{C}/\Lambda; \mathbb{C}^4), \end{gathered} \tag{2.11}$$

for $k \in \mathcal{K}$, $p \in \mathbb{Z}_3$. For another useful antilinear symmetry see [BeZw23, §3.5].

2.2. **Bloch–Floquet theory.** In [Be*22], the band theory was based on lattice of periodicity of $D(\alpha)$ and $H(\alpha)$ given by 3Λ (see Appendix A a translation of notations). That meant that Bloch eigenvalues were functions of $k \in \mathbb{C}/\frac{1}{3}\Lambda^*$, a small torus. Following the physics literature we now consider Bloch–Floquet theory based on (1.5), using the commuting operators $\mathscr{L}_\gamma$. The two approaches are equivalent but Figure 4 illustrates the advantages of the latter: the bands have a much cleaner structure and eigenvalues are functions on a larger torus, $\mathbb{C}/\Lambda^*$.

We first recall that the eigenvalues in (1.5) are the same as the eigenvalues of

$$\begin{gathered} H_k(\alpha) : H^1_0(\mathbb{C}/\Lambda; \mathbb{C}^4) \to L^2_0(\mathbb{C}/\Lambda; \mathbb{C}^4), \\ (H_k(\alpha) - E_j(\alpha, k))e_j(\alpha, k) = 0, \quad e_j(\alpha, k) \in H^1_0(\mathbb{C}/\Lambda; \mathbb{C}^4), \\ H_k(\alpha) := e^{-i\langle z,k\rangle} H(\alpha) e^{i\langle z,k\rangle} = \begin{pmatrix} 0 & D(\alpha)^* + \bar{k} \\ D(\alpha) + k & 0 \end{pmatrix}. \end{gathered} \tag{2.12}$$

The eigenvalues of $H_k(\alpha)$ on $L^2_0(\mathbb{C}/\Lambda;\mathbb{C}^4)$ (with the domain given by $H^1_0(\mathbb{C}/\Lambda;\mathbb{C}^4)$ – see (1.5)) are given by (1.6). We note that

$$E_j(\alpha, k+p) = E_j(\alpha, k), \quad p \in \Lambda^*, \quad E_j(\alpha, \omega k) = E_j(\alpha, k), \quad k \in \mathbb{C}. \tag{2.13}$$

The last property follows from checking that $\mathscr{C} H_{\omega k}(\alpha)\mathscr{C}^* = H_k(\alpha)$, where $\mathscr{C}$ was defined in (2.1). This shows that $k \mapsto E_j(\alpha, k)$ is either singular or critical at $K, -K, 0$ ($K = 4\pi/3$ – see (1.2) and the end of Section 5.2; that is also nicely seen in the animation https://math.berkeley.edu/~zworski/KKmovie.mp4.).

The key fact used in [TKV19] and [Be*22] is the existence of protected states. We recall it in the current convention:

Proposition 2.1. *For every $\alpha \in \mathbb{C}$ there exists $u_{\pm K}(\alpha) \in H^1_0(\mathbb{C}/\Lambda;\mathbb{C}^2)$ such that $\tau(K)u_K(0) = \mathbf{e}_1$, $\tau(-K)u_{-K}(0) = \mathbf{e}_2$,*

$$(D(\alpha) \pm K)u_{\pm K}(\alpha) = 0, \quad \tau(k)v(z) := e^{i\langle z,k\rangle}v(z), \tag{2.14}$$

where we note that $\tau(k) : L^2_p \to L^2_{p+k}$, $p, k \in \mathbb{C}$. In addition,

$$\tau(\pm K)u_{\pm K}(\alpha) \in \ker_{H^1_{\pm K,0}(\mathbb{C}/\Lambda;\mathbb{C}^2)} D(\alpha), \tag{2.15}$$

and if $\tau(\pm K)u_{\pm K}(\alpha, z) = (u^\pm_1(\alpha, z), u^\pm_2(\alpha, z))^t$ then

$$u^+_2(\alpha, \pm z_S) = u^-_1(\alpha, \pm z_S) = 0, \quad z_S := i/\sqrt{3}, \quad \omega z_S = z_S - (1+\omega). \tag{2.16}$$

Proof. We decompose $\ker_{H^1(\mathbb{C}/3\Lambda;\mathbb{C}^4)} H(\alpha)$ into representation of G_3 (see (2.4) and [Be*22, §2.2] for a review of representations of G_3 – we only use representation appearing in (2.6) so that is all that is needed here). From (2.2) we see that the spectrum of $H(\alpha)$ restricted to representations of G_3 is symmetric with respect to the origin. The kernel of $H(0)$ on $H^1(\mathbb{C}/3\Lambda;\mathbb{C}^4)$ is given by by the standard basis vectors in $\mathbb{C}^4$, $\mathbf{e}_j$. They satisfy

$$\mathbf{e}_1 \in H^1_{K,0}, \quad \mathbf{e}_2 \in H^1_{-K,0}, \quad \mathbf{e}_3 \in H^1_{K,1}, \quad \mathbf{e}_4 \in H^1_{-K,1},$$

and all these spaces are mutually orthogonal. Since the spectrum of $H(\alpha)|_{L^2_{k,p}}$ is even, continuity of eigenvalues shows that $\dim\ker_{L^2_{\pm K,p}} H(\alpha) \geq 1$, $\alpha \in \mathbb{C}$, $p = 0, 1$. Since $\tau(\mp K) : \ker_{H^1_{\pm K}} H(\alpha) \to \ker_{H^1_0}(H(\alpha) \pm K)$ this gives (2.14) and (2.15).

For (2.16) we give an argument in the case of u_K: $u_2(\pm z_S) = u_2(\pm\omega z_S) = u_2(\pm z_S \mp (1+\omega))$ and in view of $(u_1, u_2)^t \in L^2_K$, the right hand side is equal to $e^{\mp 2i\langle(1+\omega),K\rangle}u_2(\pm z_S)$. Since $e^{i\langle 1+\omega,K\rangle} = e^{\frac{4}{3}i\pi \operatorname{Re}(1+\omega)} = \omega$, we see that $u_2(\pm z_S) = 0$. □

As a consequence of Proposition 2.1 we have

$$\forall\, \alpha \in \mathbb{C} \quad \operatorname{Spec}_{L^2_0(\mathbb{C}/\Lambda;\mathbb{C}^2)} D(\alpha) \supset \mathcal{K}_0 := \{K, -K\} + \Lambda^*. \tag{2.17}$$

2.3. **Spectral characterization of magic angles.** In [TKV19] magic angles were computed by analyzing $u_{\pm K}$ (see Proposition 2.1) and identifying $\mathcal{A}$ with the zeros of the Wronskian,

$$v(\alpha) = W(\tau(K)u_K(\alpha), \tau(-K)u_{-K}(\alpha)), \quad W(v,w) := \det(u,v), \quad u,v \in \mathbb{C}^2. \tag{2.18}$$

The function $\alpha \mapsto v(\alpha) \in \mathbb{C}$ was also identified with the physical quantity called the *Fermi velocity* [TKV19, (7),(8)]. That led to a rough (three digits) computation of the first five α's [TKV19] and then a computer assisted rigorous proof of the existence of the first magic α [WaLu21]. Proposition 2.3 below shows that we can choose $u_K(\alpha)$ so that $v(\alpha)$ in an entire function.

The approach taken in [Be*21, Be*22] was different and was based on identifying magic α's with reciprocals of eigenvalues of a family of compact operators. Crucially, the eigenvalues are independent of the elements of the family and that lies behind Theorem 1. We recall this in a form generalizing (1.1):

$$\begin{gathered} D_V(\alpha) := 2D_{\bar z} + \alpha V(z), \quad V(z) = \begin{pmatrix} 0 & U_+(z) \\ U_-(z) & 0 \end{pmatrix}, \quad U_\pm(\omega z) = \omega U_\pm(z), \\ U_\pm(z+\gamma) = e^{\pm i\langle \gamma, K\rangle} U_\pm(z), \quad \gamma \in \Lambda. \end{gathered} \tag{2.19}$$

We also define $H(\alpha)$ and note that the results of the previous sections apply without modification. We define the set $\mathcal{A} := \mathcal{A}(V)$ by (1.7). Since

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} D_V(\alpha) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = D_V(-\alpha),$$

we still have the symmetry $\mathcal{A}(V) = -\mathcal{A}(V)$.

If in addition to (2.19) we also have

$$\overline{U_\pm(\bar z)} = -U_\pm(-z) \iff V(z) = -\overline{V(-\bar z)}, \tag{2.20}$$

then we get $\widetilde\Gamma D_V(\alpha)\widetilde\Gamma = -D_V(-\bar\alpha)$, $\widetilde\Gamma v(z) := \overline{v(\bar z)}$, and hence $\mathcal{A}(V) = \overline{\mathcal{A}(V)}$.

The following result is a generalized formulation of [Be*22, Theorem 2]. To state it we define

$$\begin{gathered} R(k) := (2D_{\bar z} - k)^{-1} : L^2_p(\mathbb{C}/\Lambda, \mathbb{C}^2) :\longrightarrow L^2_p(\mathbb{C}/\Lambda; \mathbb{C}^2), \quad p \in \mathbb{C}, \\ k \notin \mathcal{K}_0 + p, \quad \mathcal{K}_0 := \{K, -K\} + \Lambda^*. \end{gathered} \tag{2.21}$$

This follows from the fact that $(2D_{\bar z} - k)^{-1} : L^2_0 \to L^2_0$ for $k \notin \mathcal{K}_0$. We then have $\tau(p) : L^2_0 \to L^2_p$ and $\tau(-p)R(k)\tau(p) = R(k-p)$.

Proposition 2.2. *In the notation of* (2.19) *and* (2.21) *the following compact operators are well defined*

$$T_k := R(k)V : L^2_p(\mathbb{C}/\Lambda, \mathbb{C}^2) \to L^2_p(\mathbb{C}/\Lambda, \mathbb{C}^2), \quad k \notin \mathcal{K}_0 + p. \tag{2.22}$$

Moreover,

$$\operatorname{Spec}_{L^2_p}(T_k) = \operatorname{Spec}_{L^2_q}(T_{k'}), \quad k \notin \mathcal{K}_0 + p, \quad k' \notin \mathcal{K}_0 + q, \tag{2.23}$$

$$\mathcal{A}(V) = \{\alpha \in \mathbb{C} : \alpha^{-1} \in \operatorname{Spec}_{L^2_p}(T_k)\}, \quad k \notin p + \mathcal{K}_0, \quad p \in \mathbb{C}, \tag{2.24}$$

and

$$\operatorname{Spec} D_V(\alpha) = \begin{cases} \mathcal{K}_0, & \alpha \notin \mathcal{A}(V) \\ \mathbb{C}, & \alpha \in \mathcal{A}(V), \end{cases} \tag{2.25}$$

with simple eigenvalues when $\alpha \notin \mathcal{A}(V)$.

Proof. We first note that the definition of $\mathscr{L}_\gamma$ in (1.5) and (2.19) show that $\mathscr{L}_\gamma V = V \mathscr{L}_\gamma$ and hence $V : L^2_p \to L^2_p$. This and (2.21) give the mapping property (2.22).

We first consider (2.23) and (2.24) for $q = p = 0$ and $k \notin \mathcal{K}_0$ (this spectral characterization was proved in [Be*22] but we include a streamlined proof using the current convention). For a fixed $k \notin \mathcal{K}_0$, we define a discrete set $\mathcal{A}_k := \{\alpha \in \mathbb{C} : -\alpha^{-1} \in \operatorname{Spec}_{L^2_0} T_k\}$. For $\alpha \notin \mathcal{A}_k$ the spectrum of $D(\alpha)$ is then discrete since

$$D(\alpha) - z = (D(0) - k)(I + K(z)), \quad K(z) := \alpha T_k + R(k)(k - z), \tag{2.26}$$

and $z \mapsto K(z)$ is a holomorphic family of compact operators with $I + K(k)$ invertible (since $-\alpha^{-1} \notin \operatorname{Spec}_{L^2_0}(T_k)$). But that implies (see for instance [DyZw19, Theorem C.8]) that $(D(\alpha) - z)^{-1} = (I + K(z))^{-1} R(k)$ is a meromorphic family of operators, and in particular, the spectrum of $D(\alpha)$ is discrete.

We now put

$$\Omega := \{\alpha \in \mathbb{C} \setminus \mathcal{A}_k : \operatorname{Spec}_{L^2_0}(D(\alpha)) = \mathcal{K}_0 \text{ with simple eigenvalues}\},$$

noting that $0 \in \Omega$. We claim that Ω is open and closed in the relative topology of the connected topological space $\mathbb{C} \setminus \mathcal{A}_k$. That will imply that $\Omega = \mathbb{C} \setminus \mathcal{A}_k$.

To prove the claim, we note that for $\alpha_0 \in \Omega$ there exists a neighbourhood of α_0, U, such that for $\alpha \in U$, the spectrum of $D(\alpha)$ is discrete and changes continuously with α. From Proposition 2.1 we also know that $\operatorname{Spec}_{L^2_0}(D(\alpha)) \supset \mathcal{K}_0$. But as it is equal to $\mathcal{K}_0$ at $\alpha = \alpha_0$ it has be equal to $\mathcal{K}_0$ in U. To see that Ω is closed, assume that $\{\alpha_j\}_{j=1}^\infty \in \Omega$, $\alpha_j \to \alpha_0 \in \mathbb{C} \setminus \mathcal{A}_k$. But this means that there exists an open neighbourhood of α_0, U, such that for $\alpha \in U$ the spectrum is discrete and hence depends continuously on α. Since $\operatorname{Spec}_{L^2_0}(D(\alpha_j)) = \mathcal{K}_0$, we conclude that $\operatorname{Spec}_{L^2_0} D(\alpha_0) = \mathcal{K}_0$ (all with agreement of simple multiplicities), that is, $\alpha_0 \in \Omega$.

It remains to show that $\mathcal{A}_k$ is independent of k. For that we note that $-\alpha^{-1} \in \operatorname{Spec}_{L^2_0} T_k$ is equivalent to $k \in \operatorname{Spec}_{L^2_0}(D(\alpha))$ (see $K(k)$ in (2.26)). Since $k \notin \mathcal{K}_0$ the spectrum cannot be discrete, as then it would be equal to $\mathcal{K}_0$. Hence, it has to be equal to $\mathbb{C}$ (if there were any points at which $D(\alpha) - z$ were invertible then the compactness of the inverse and an argument similar to that after (2.26) would show the spectrum is discrete). But that means that any $k' \in \operatorname{Spec}_{L^2_0}(D(\alpha))$ and the equivalence above shows that $-\alpha^{-1} \in \operatorname{Spec}_{L^2_0}(T_{k'})$.

In particular, $\operatorname{Spec}_{L^2_0}(T_{k_1}) = \operatorname{Spec}_{L^2_0}(T_{k_2})$ for any $k_j \notin \mathcal{K}_0$. To establish (2.23) we can take $q = 0$ and note that that $\tau(-p) : L^2_p \to L^2_0$ (see (2.14)) $\tau(p) T_{k_1} \tau(p)^{-1} = T_{k_1+p} : L^2_p \to L^2_p$, $k_1 \notin \mathcal{K}_0$ (and hence $k_1 + p \notin \mathcal{K}_0 + p$). Hence to see (2.23) with $q = 0$ we take $k_2 = k'$ and $k_1 = k - p$. □

Proof of Theorem 1. This is immediate from from (2.25): if $E_1(\alpha, k) = 0$ for $k \notin \mathcal{K}_0$, $0 \in \operatorname{Spec}_{L^2_0} H_k(\alpha)$, then $\ker_{H^1_0}(D(\alpha) + k)$ or $\ker_{H^1_0}((D(\alpha) + k)^*)$ are non zero. Since, $D(\alpha) + k$ is a Fredholm operator of index zero (see [Be*22, Proposition 2.3]) the two statements are equivalent. But $\ker(D(\alpha) + k) \neq \{0\}$, $k \notin \mathcal{K}_0$, implies in view of (2.25) that $E_1(\alpha, k) \equiv 0$, $k \in \mathbb{C}$. □

Combining Propositions 2.1 and 2.2 we obtain a stronger statement about protected states:

Proposition 2.3. *Suppose that $D(\alpha)$ is given by (1.1), with U satisfying (1.2). Then, for $\alpha \notin \mathcal{A}$, $u_{\pm K}(\alpha)$ are unique up to multiplicative constants, and we can choose*

$$u_{-K}(\alpha) = \tau(K) \mathscr{E} \tau(K) u_K(\alpha). \tag{2.27}$$

Moreover, $\alpha \mapsto u_{\pm K}(\alpha)$ can be chosen to be holomorphic as a function of $\alpha \in \mathbb{C}$ with values in $H^1_0(\mathbb{C}/\Lambda; \mathbb{C}^2)$.

Proof. Since for $\alpha \notin \mathcal{A}$, the eigenvalues of $D(\alpha)$ are simple and the right hand side of (2.27) has all the properties of $u_{-K}(\alpha)$ in Proposition 2.1, we can choose it to be $u_{-K}(\alpha)$.

To find a holomorphic family $\alpha \mapsto u_K(\alpha)$ we proceed as follows. We first note that for $\alpha_0 \notin \mathcal{A}$, $(\tau(K) u_K(\alpha_0 + \zeta), 0)^t$ spans $\ker \widetilde{H}(\alpha_0, \zeta)|_{H^1_{K,0}}$,

$$\widetilde{H}(\alpha_0, \zeta) := \begin{pmatrix} 0 & D(\alpha_0 + \bar{\zeta})^* \\ D(\alpha_0 + \zeta) & 0 \end{pmatrix}.$$

Since $\zeta \mapsto H(\alpha_0, \zeta)$ is a holomorphic family of operators it follows that we can choose $\zeta \mapsto \tau(K) u_K(\alpha_0 + \zeta)$ holomorphic in ζ for $|\zeta| < \delta$ (note that $(\tau(K) u_K(\alpha_0 + \zeta), 0)^t \in \ker H(\alpha_0 + \zeta)|_{H^1_{K,0}}$). When $\alpha_0 \in \mathcal{A}$, $\zeta \mapsto H(\alpha_0, \zeta)$ is a holomorphic family of operators, which is self-adjoint for $\zeta \in \mathbb{R}$. Rellich's theorem [Ka80, Chapter VII, Theorem 3.9], then shows that an element of the kernel of $H(\alpha_0, \zeta)|_{H^1_{K,0}}$ can be chosen to be holomorphic near $\zeta = 0$. In view of simplicity for $0 < |\zeta| < \delta$, it has to coincide with a choice of $\tau(K) u_K(\alpha_0 + \zeta)$.

The local constructions above and a partition of unity on $\mathbb{C}$ show that we can choose $\tau(K) \widetilde{u}_K \in C^\infty(\mathbb{C}; H^1_{K,0})$ and it remains to modify it so that it becomes holomorphic.

We have

$$0 = \partial_{\bar\alpha}(D(\alpha)\tau(K)u_K(\alpha)) = D(\alpha)(\tau(K)\partial_{\bar\alpha}u_K(\alpha)).$$

For $\alpha \notin \mathcal{A}$ the kernel on H^1_K is one dimensional and hence

$$\partial_{\bar\alpha}\widetilde{u}_K(\alpha) = f(\alpha)\widetilde{u}_K(\alpha), \quad \alpha \notin \mathcal{A}, \quad f(\alpha) = \frac{\langle \partial_{\bar\alpha}\widetilde{u}_K(\alpha), \widetilde{u}_K(\alpha)\rangle}{\|u_K(\alpha)\|^2}. \tag{2.28}$$

In the formula for $f(\alpha)$, the right hand side is smooth in α and that shows that the first formula in (2.28) holds for all $\alpha \in \mathbb{C}$. The equation $\partial_{\bar\alpha}F(\alpha) = f(\alpha)$ (see for instance [HöI, Theorem 4.4.6] applied with $P = \partial_{\bar\alpha}$ and $X = \mathbb{C}$) can be solved with $F \in C^\infty(\mathbb{C})$. This shows that $u_K(\alpha) = \exp(-F(\alpha))\widetilde{u}_K(\alpha)$ is indeed holomorphic. ☐

3. Theta function argument revisited

In [TKV19] a theta function argument was used to explain the formation of flat bands and in [Be*22] that approach was shown to be equivalent to the spectral characterisation in Proposition 2.2. We review it here from the point of view of §2 and [Le*20], where the holomorphic dependence of eigenvectors on the Floquet parameter (Bloch pseudo-momentum) k was stressed.

3.1. **Theta functions.** To simplify notation we put $\theta(z) := \theta_1(z|\omega) := -\theta_{\frac12,\frac12}(z|\omega)$, and recall that

$$\begin{gathered}\theta(z) = -\sum_{n\in\mathbb{Z}} \exp(\pi i(n+\tfrac12)^2\omega + 2\pi i(n+\tfrac12)(z+\tfrac12)), \quad \theta(-z) = -\theta(z)\\ \theta(z+m) = (-1)^m\theta(z), \quad \theta(z+n\omega) = (-1)^n e^{-\pi i n^2\omega - 2\pi i z n}\theta(z),\end{gathered} \tag{3.1}$$

and that θ vanishing simply on Λ and nowhere else see [Mu83].

We now define

$$F_k(z) = e^{\frac{i}{2}(z-\bar z)k}\frac{\theta(z - z(k))}{\theta(z)}, \quad z(k) := \frac{\sqrt{3}k}{4\pi i}, \quad z : \Lambda^* \to \Lambda. \tag{3.2}$$

Then, using (3.1) and differentiating in the sense of distributions,

$$\begin{gathered}F_k(z+m+n\omega) = e^{-nk\operatorname{Im}\omega}e^{2\pi i n z(k)}F_k(z) = F_k(z),\\ (2D_{\bar z} + k)F_k(z) = c(k)\delta_0(z), \quad c(k) := 2\pi i\theta(z(k))/\theta'(0).\end{gathered} \tag{3.3}$$

This follows from the fact that $1/(\pi z)$ is a fundamental solution of $\partial_{\bar z}$ – see for instance [HöI, (3.1.12)]. In other, words, for $k \notin \Lambda^*$, F_k gives the Green kernel of $2D_{\bar z} + k$ on the torus $\mathbb{C}/\Lambda$:

$$(D_{\bar z} + k)^{-1}f(z) = c(k)^{-1}\int_{\mathbb{C}/\Lambda} F_k(z - z')f(z')dm(z),$$

$dm(z) = dxdy$, $z = x + iy$. For future use we record some properties of F_k:

Lemma 3.1. *For $u \in C^\infty(\mathbb{C})$, we have, in the sense of distributions, and in the notation of (3.3),*

$$(2D_{\bar z} + q - \ell)\left(\frac{F_q(z-z_0)}{F_\ell(z-z_0)}u(z)\right) = \frac{F_q(z-z_0)}{F_\ell(z-z_0)}2D_{\bar z}u(z) + c(q,\ell)u(z_1)\delta(z-z_1), \quad (3.4)$$

where $z_1 = z(p) + z_0$ and $c(k,p) = 2\pi i\theta(z(q-\ell))/\theta'(0)$. In particular, by taking $\ell = 0$ and $q = k$,

$$(2D_{\bar z} + k)\left(F_k(z-z_0)u(z)\right) = F_k(z-z_0)2D_{\bar z}u(z) + c(k)u(z_0)\delta(z-z_0). \quad (3.5)$$

The following simple lemma is implicit in [TKV19]:

Lemma 3.2. *Suppose that that $w \in C^\infty(\mathbb{C};\mathbb{C}^2)$ and that $(D(\alpha)+k)w = 0$ for some k and that $w(z_0) = 0$. Then $w(z) = (z-z_0)w_0(z)$, where $w_0 \in C^\infty(\mathbb{C};\mathbb{C}^2)$.*

Proof. The conclusion of the lemma is equivalent to $(2D_{\bar z})^\ell w(z_0) = 0$ for all ℓ. Since $(2D_{\bar z})^\ell w(z) = (2D_{\bar z})^{\ell-1}\left[(U-k)\,w\right](z)$ that follows by induction on ℓ. ☐

These two lemmas are the basis of the *theta function argument* in [TKV19] (see also [DuNo80] for an earlier version of a similar method). Suppose $D(\alpha)u = 0$, $u \in H^1_0$. and $u(z_0) = 0$. Lemma 3.2 shows that, near z_0, $u(z) = (z-z_0)w(z)$, $w \in C^\infty$. But then (3.5) shows that

$$(D(\alpha)+k)(F_k(z-z_0)u(z)) = 0, \quad F_k(z-z_0)u(z) \in H^1_0,$$

and from an element of the kernel of u on H^1_0 we obtained eigenfunction in H^1_0 for all k. (Strictly speaking we do not even need Lemma 3.2 since elliptic regularity guarantees smoothness of $z \mapsto F_k(z-z_0)u(z)$.)

We will also need the properties of F_k when k is translated, this will allow us to define a natural hermitian line bundle over $\mathbb{C}/\Lambda$ studied in subsection 5.2:

Lemma 3.3. *For $p \in \Lambda^*$,*

$$\begin{gathered} F_{k+p}(z) = e_p(k)^{-1}\tau(p)F_k(p), \\ e_p(k) := \frac{\theta(z(k))}{\theta(z(k+p))} = (-1)^n(-1)^m e^{i\pi n^2\omega + 2\pi z(k)}, \end{gathered} \quad (3.6)$$

where $z(p) = m + n\omega$, $n, m \in \mathbb{Z}$.

Proof. Since, for $k \notin \Lambda^*$, $(2D_{\bar z}+k+p)\tau(p)c(k)^{-1}F_k = \delta_0$ and $(2D_{\bar z}+k+p)c(k+p)^{-1}F_{k+p} = \delta_0$, the uniqueness of the kernel of the resolvent of $2D_{\bar z}$ shows that

$$F_{k+p}(z) = \frac{c(k+p)}{c(k)}[\tau(p)F_k](z) = \frac{\theta(z(k+p))}{\theta(z(k))}[\tau(p)F_k](z),$$

and (3.6) follows. ☐

3.2. **Flat bands and theta functions.** We now reformulate the characterization of magic angles using the vanishing of $u_K(\alpha)$ (in [Be*22] this was established only for $\alpha \in \mathbb{R}$):

Proposition 3.4. *Let $\alpha \mapsto u_K(\alpha) \in \ker_{H_0^1(\mathbb{C}/\Lambda,\mathbb{C}^2)}(D(\alpha)+K)$ be a smooth family given in Proposition 2.3. Then*

$$\begin{aligned} \alpha \in \mathcal{A} &\iff \exists\, \varepsilon \in \{\pm 1\} \quad u_K(\alpha, \varepsilon z_S) = 0, \quad z_S := i/\sqrt{3} \\ &\iff \exists\, z_0 \quad u_K(\alpha, z_0) = 0. \end{aligned} \tag{3.7}$$

Proof. Suppose first that there exists z_0 at which $u_K(\alpha)$ vanishes. Since $(D(\alpha) + K)u_K = 0$ we then see that for every $k' \in \mathbb{C}$

$$(D(\alpha) + K + k')(F_{k'}(z - z_0)u_K(z)) = 0, \tag{3.8}$$

and the solution of this elliptic equation is automatically in H_0^1 (since $u_K \in H_0^1$ and the scalar valued function $F_{k'}$ is periodic by (3.3)).

Hence $\operatorname{Spec}_{L_0^2} D(\alpha) = \mathbb{C}$. Using (2.27) and putting $u_K = (u_1, u_2)^t$, the Wronskian (2.18), which is constant (apply $\partial_{\bar z}$ to both sides and use periodicity), is given by

$$v(\alpha) = u_1(z)u_1(-z) + u_2(-z)u_2(z) = \begin{cases} u_1(z_0)u_1(-z_0) + u_2(-z_0)u_2(z_0) = 0, \\ \quad u_1(z_S)u_1(-z_S), \end{cases}$$

where we used (2.16). Hence u_K has to vanish at either z_S or $-z_S$.

It remains to show that if $u_K(z_S)u_K(-z_S) \neq 0$ then $\alpha \notin \mathcal{A}$. That is equivalent to the Wronskian, $v(\alpha) \neq 0$ in which case we can express $(D(\alpha) - k)^{-1}$, $k \notin \mathcal{K}$ using u_K and u_{-K} – [Be*22, Proposition 3.3]. ☐

4. Proofs of Theorems 2 and 3

The theta function argument of [TKV19] which we reviewed in the previous section relies on vanishing of both component of $u_K \in L_0^2(\mathbb{C}/\Lambda;\mathbb{C}^2)$, $(D(\alpha) + K)u_K = 0$, or equivalently of vanishing of $u_{-K} = \tau(K)\mathscr{E}\tau(K)u_K$ – see Proposition 3.4, Figure 5 and the movie referenced there.

Proof of Theorem 2. We first note that if $E_j(\alpha, p) > 0$ for $j > 1$ and $\alpha \in A$ then $E_1(\alpha, p) = 0$ is a double eigenvalue of $H_p(\alpha)$. But that means that $D(\alpha) + p$ on L_0^2 is one dimensional (see the Proof of Theorem 1 in §2).

Hence we need to show that if $\alpha \in \mathcal{A}$ and $\dim \ker_{L_p^2(\mathbb{C}/\Gamma;\mathbb{C}^2)} D(\alpha) = 1$, for a *fixed* $p \in \mathbb{C}$ then $E_j(\alpha, k) > 0$ for all k and $j > 1$. To do that we proceed by contradiction and suppose that there exists k such that $E_1(\alpha, k) = E_2(\alpha, k)$.

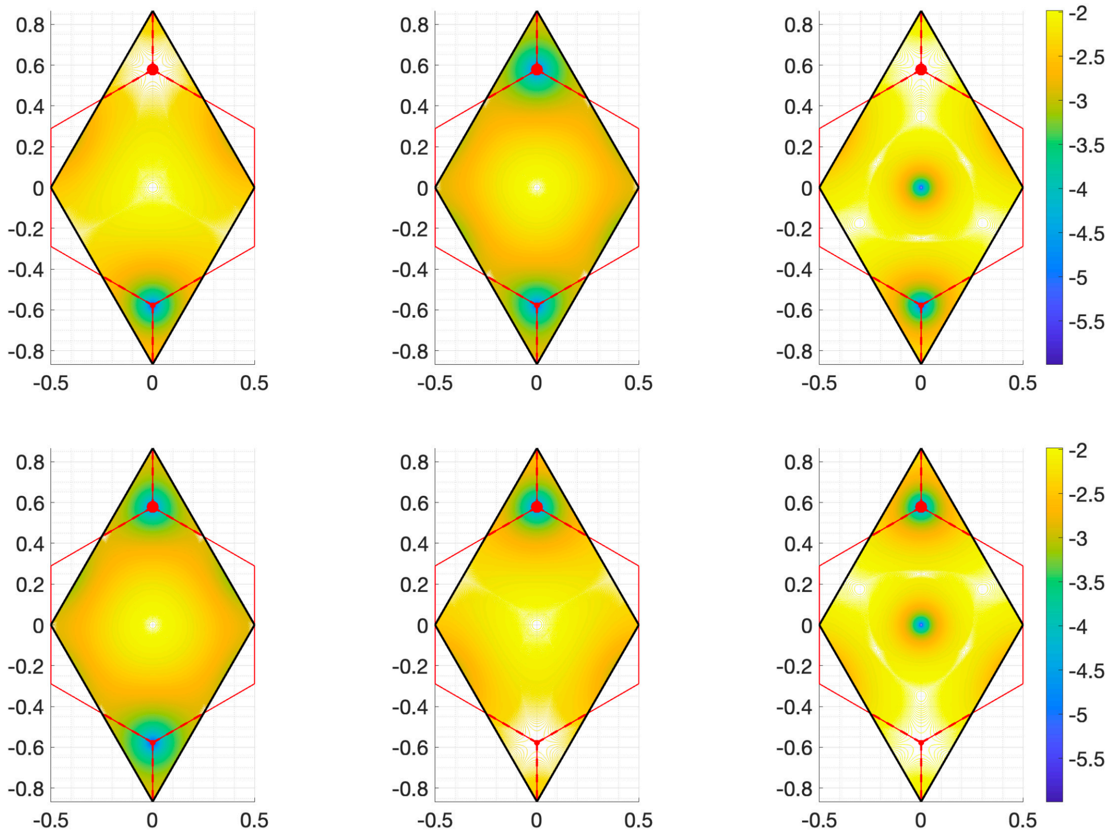

FIGURE 5. On top/bottom, the first/second components of $\log|u_\bullet|$ for $\bullet = K, -K, 0$, respectively, where $u_\bullet$ spans the kernel of $D(\alpha) - \bullet$ on $L^2_0(\mathbb{C}/\Lambda;\mathbb{C}^2)$, and α is the first real magic angle for (1.3); u_K, u_{-K}, u_0 vanish at $-z_S, z_S$ (marked by $\bullet$), and 0, respectively. We also indicate (–) the hexagon spanned by $\pm z_S + \Lambda$. The states $u_{\pm K}$ exist for all α's (Proposition 2.3) and, in the case of a simple $\alpha \in \mathcal{A}$ have zeros at $\pm z_S$ (Theorem 3); see https://math.berkeley.edu/~zworski/magic.mp4 for the plot of $\log|u_{-K}|$ as α changes.

First consider the easy case of $k = p$ and we have two independent v_j, $j = 1, 2$ in $\ker_{L^2_0}(D(\alpha) + p)$. Then $\widetilde{v}_j(z) = \tau(p)v_j(z)$ satisfy $D(\alpha)\widetilde{v}_j = 0$ and $v_j \in L^2_k = L^2_p$, which gives the desired contradiction.

Now assume that $k \neq p$. Propositions 2.3 and 3.4 give a nontrivial $u_K \in L^2_0$ such that $u_K(\varepsilon z_S) = 0$ where $\varepsilon \in \{\pm 1\}$. Put $z_0 := \varepsilon z_S$, $u_K(z_0) = 0$. Using (3.5) we define

$$v(z) := F_{k-K}(z - z_0)u_K, \quad v \in L^2_0(\mathbb{C}/\Gamma;\mathbb{C}^2), \quad (D(\alpha) + k)v = 0. \tag{4.1}$$

Since $E_2(\alpha, k) = 0$, there exists $w \in L^2_0$, independent of v and such that $(D(\alpha)+k)w = 0$. If $v = (\varphi_1, \varphi_2)$ and $w = (\psi_1, \psi_2)$, we form the Wronskian $W := \varphi_1\psi_2 - \varphi_2\psi_1$ which satisfies

$$(2D_{\bar z} + 2k)W = 0, \quad W(z+\gamma) = W(z), \quad \gamma \in \Lambda. \tag{4.2}$$

(Since $\mathscr{L}_\gamma u = u$, $\varphi_1(z+\gamma) = e^{-i\langle \gamma, K\rangle}\varphi_1(z)$ and $\varphi_2(z+\gamma) = e^{i\langle \gamma, K\rangle}\varphi_2(z)$, and similarly for ψ_1 and ψ_2. That shows periodicity of W.) The definition of F_{k-K} in (3.2) shows that

$$F_{k-K}(z_1 - z_0) = 0, \quad z_1 := z_0 + z(k-K), \tag{4.3}$$

so that (4.1) gives $v(z_1) = 0$. This implies that $W(z_1) = 0$. If $2k \notin \Lambda^*$, $W \equiv 0$ since $2D_{\bar z} + 2k$ is invertible. Otherwise we note that $W(z) = e^{-i\langle 2k, z - z_1\rangle} W(z_1) = 0$.

Since $W = 0$,

$$w(z) = g(z)v(z), \quad g \in C^\infty(\Omega), \quad \partial_{\bar z} g|_\Omega = 0, \quad g(z+\gamma) = g(z), \quad z \in \Lambda, \tag{4.4}$$

where $\Omega := \complement\{z : v(z) = 0\}$. Also $g \not\equiv 1$ as v and w are independent. We claim that g is a meromorphic function on $\mathbb{C}/\Lambda$. For that fix any $\underline{z} \in \mathbb{C}$ and write $w = (w_1, w_2)^t$, $v = (v_1, v_2)^t$. Then $g = w_1/v_1 = w_2/v_2$, and $w_1(\underline{z}+\zeta) = G_1(\zeta, \bar\zeta)$, $v_1(-\underline{z}-\zeta) = G_2(\zeta, \bar\zeta)$, where $G_j : B_{\mathbb{C}^2}(0,\delta) \to \mathbb{C}$ are holomorphic functions (this follows from real analyticity of w and v, which is a consequence of the ellipticity of the equation and analyticity of U – see [HöI, Theorem 8.6.1]). The definition of g and the fact that $\partial_{\bar z} g = 0$ away from zeros of v shows that $G_1(\zeta, \xi) = g(\underline{z} + \zeta)G_2(\zeta, \xi)$. We can then choose ξ_0 such that $G_2(\zeta, \xi_0)$ is not identically zero (if no such ξ_0 existed, $v_1 \equiv 0$, and hence, from the equation, $v \equiv 0$). But then $\zeta \mapsto g(\underline{z} + \zeta) = G_1(\zeta, \xi_0)/G_2(\zeta, \xi_0)$ is meromorphic near $\zeta = 0$ and, as $\underline{z}$ was arbitrary, everywhere.

Suppose first that $w(z_1) = 0$, where z_1 was defined in (4.3). Then, using (3.4) (with $q = p + K$ and $\ell = k - K$) and the fact that $v(z_1) = w(z_1) = 0$,

$$\widetilde{v}(z) := \frac{\tau(p)F_{p-K}(z - z_0)}{F_{k-K}(z - z_0)} v(z), \quad \widetilde{w}(z) := \frac{\tau(p)F_{p-K}(z - z_0)}{F_{k-K}(z - z_0)} w(z),$$

satisfy

$$D(\alpha)\widetilde{v} = 0, \quad D(\alpha)\widetilde{w} = 0, \quad \widetilde{v},\ \widetilde{w} \in L^2_p(\mathbb{C}/\Gamma; \mathbb{C}^2). \tag{4.5}$$

Since $\widetilde{v} \not\parallel \widetilde{w}$, this contradicts $\dim \ker_{L^2_p(\mathbb{C}/\Gamma;\mathbb{C}^2)} D(\alpha) = 1$.

Now, suppose that $w(z_1) \neq 0$. Since $v(z_1) = 0$, $g(z)$ (defined in (4.4) and shown to be meromorphic) has to have a pole at z_1 and consequently $g(z_2) = 0$ for some $z_2 \not\equiv z_1$ mod Λ. That shows that $w(z_2) = 0$ and we define z_3 (unique mod Λ and not congruent to z_0) so that $F_{k-K}(z_2 - z_3) = 0$. Hence, the following functions are smooth,

$$\widetilde{v}(z) := \frac{\tau(p)F_{p-K}(z - z_0)}{F_{k-K}(z - z_0)} v(z), \quad \widetilde{w}(z) := \frac{\tau(p)F_{p-K}(z - z_3)}{F_{k-K}(z - z_3)} w(z),$$

and (4.5) holds. If $\widetilde{v} \not\parallel \widetilde{w}$ we have a contradiction.

Hence, suppose that $\widetilde{v} \parallel \widetilde{w}$. Then, using (4.4)

$$g(z) = c_0 \frac{F_{p-K}(z - z_0)F_{k-K}(z - z_3)}{F_{p-K}(z - z_3)F_{k-K}(z - z_0)} = c_1 \frac{\theta(z - z_0 - z(p-K))\theta(z - z_3 - z(k-K))}{\theta(z - z_3 + z(p-K))\theta(z - z_0 + z(k-K))}.$$

Defining z_4, z_5 by

$$F_{p-K}(z_4 - z_0) = 0, \quad F_{p-K}(z_5 - z_3) = 0$$

we obtain $w(z_4) = v(z_5) = 0$. If $z_4 \neq z_2$ or $z_5 \neq z_1$ we put

$$\widetilde{w}_1(z) := \frac{\tau(p)F_{p-K}(z - z_6)}{F_{k-K}(z - z_6)} w(z), \quad \widetilde{v}_1(z) = \frac{\tau(p)F_{p-K}(z - z_7)}{F_{k-K}(z - z_7)} v(z),$$

and

$$F_{k-K}(z_j - z_{j+2}) = 0, \ j = 4, 5.$$

So, when $z_4 \neq z_2$, $\widetilde{w}$ and $\widetilde{w}_1$ are independent and satisfy (4.5). Similarly, for $\widetilde{v}$ and $\widetilde{v}_1$ when $z_5 \neq z_1$. This again provides a contradiction.

It remains to show that $z_4 = z_2$ and $z_5 = z_1$ is impossible when $k \neq p$. We know that, by (3.2), $F_{k-K}(z(k-K)) = 0$ and $F_{p-K}(z(p-K)) = 0$. Then from (4.3), $z_1 = z_0 + z(k-K)$, $z_3 = z_2 - z(k-K)$, $z_4 = z_0 + z(p-K)$, and $z_5 = z_3 + z(p-K) = z_2 - z(k-K) + z(p-K) = z_1 = z_0 + z(k-K)$. Now, we use $z_0 + z(p-K) = z_4 = z_2$ to finally deduce that $z(k-K) = z(p-K) \Rightarrow k = p$, concluding the proof. □

Proof of Theorem 3. We will rely on Lemma 3.2 in several places. We write $u(z) := \tau(-K)u_{-K} = (\psi_1(z), \psi_2(z)) \in \ker_{L^2_{-K,0}(\mathbb{C}/\Lambda;\mathbb{C}^2)} D(\alpha)$ and assume that $u(z_0) = 0$. We recall from Proposition 2.3 that u has to vanish at z_S or at $-z_S$.

We first show that $z_0 = \pm z_S$. Suppose otherwise and that, in addition, $z_0 \neq 0$. In that case, $\omega^j z_0$ are three distinct points on $\mathbb{C}/\Lambda$ adding up to 0. Hence, there exists a Λ-periodic meromorphic function g_{z_0} with simple poles at $\omega^j z_0 + \Lambda$ which satisfies $g_{z_0}(\omega z) = g_{z_0}(z)$. This is a general fact (see [Mu83, §I.6]) and we can take

$$g_{z_0}(z) = c \prod_{j=0}^{2} \frac{\theta(z\bar{\omega}^j + z_0)}{\theta(z\bar{\omega}^j - z_0)},$$

But this means that $\widetilde{u}(z) := g_{z_0}(z)u(z)$ satisfies $D(\alpha)\widetilde{u} = 0$ (see Lemma 3.2) and $\widetilde{u} \in L^2_{-K,0}(\mathbb{C}/\Gamma)$, $\widetilde{u} \not\parallel u$. Since we assumed simplicity, this is impossible.

We now need to eliminate the possibility that $z_0 = 0$. From the vanishing of the Wronskian (2.18) ($\alpha \in \mathcal{A}$) and (2.27) and we see that $\mathscr{E}u(z) = f(z)u(z)$ where

$$f(z) := \frac{\psi_2(-z)}{\psi_1(z)}. \tag{4.6}$$

This function is satisfies

$$f(z + \gamma) = e^{-i\langle \gamma, K \rangle} f(z), \quad \gamma \in \Lambda, \quad f(\omega z) = f(z), \quad f(z)f(-z) = -1. \tag{4.7}$$

In fact, the holomorphy away from the zeros of ψ_1 follows from calculating $D_{\bar{z}} f$ the equations for ψ_j and the vanishing of the Wronskian (2.18). The latter also shows the functional equation for $z \mapsto -z$ and from the fact that $u \in L^2_{-K,0}(\mathbb{C}/\Lambda)$ we deduce quasi-periodicity and invariance under $z \mapsto \omega z$. We also see that f is meromorphic using the same argument as in the proof of Theorem 2 (see (4.4)). In particular, the functional equation shows that f is regular at 0.

With this in place, we now show that a zero at $z_0 = 0$ is impossible. We claim that

$$\psi_1(0) = 0 \implies \partial_z^k \partial_{\bar z}^\ell \psi_1(0) = 0, \ k \leq 2, \ \ell \geq 0. \tag{4.8}$$

This implies that if $\psi_1(0) = 0$ then $\psi_j(z) = z^3 \widetilde{\psi}_j(z)$. But this means that

$$\widetilde{u}(z) := \wp'(z;\omega,1) u \in L^2_{-K,0}(\mathbb{C}/\Gamma_3), \tag{4.9}$$

and satisfies $D(\alpha)\widetilde{u} = 0$. Projective uniqueness of u (uniqueness up to a multiplicative constant) shows that this is impossible. (Here $\wp(z;\omega_1,\omega_2)$ is the Weierstrass $\wp$-function – see [Mu83, §I.6]. It is periodic with respect to $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ and its derivative has a pole of order 3 at $z = 0$. For $\omega = e^{2\pi i/3}$ we also have $\wp'(\omega z;\omega,1) = \wp'(z;\omega,1)$.)

To prove (4.8) we consider expansions at $(z,\bar z) = (0,0)$: denoting by $\equiv$ congruency modulo 3 and using properties of U and $\psi_1(\omega z) = \psi_1(z)$, we obtain

$$\psi_1(z) = \sum_{k \equiv \ell} a_{k\ell} z^k \bar z^\ell, \quad U(z) = \sum_{p \equiv q+1} b_{pq} z^p \bar z^q, \quad f(-z) = \sum_{k \equiv 0} f_k z^k. \tag{4.10}$$

The equation $2D_{\bar z}\psi_1(z) + \alpha U(z) f(-z) \psi_1(-z) = 0$ then becomes

$$\sum_{k \equiv \ell} \left[(2\ell/i) a_{k\ell} z^k \bar z^{\ell-1} + \alpha(-1)^k \sum_{p \geq 1} \sum_{q \geq 0} \sum_{r \geq 0} b_{pq} f_r a_{k\ell} z^{k+r+p} \bar z^{\ell+q} \right] = 0$$

The vanishing of the coefficients of $z^k \bar z^\ell$ then gives (with the convention that $a_{k\ell} = 0$ for $k < 0$ or $\ell < 0$)

$$a_{k,\ell+1} = \sum_{r \leq k-1, s \leq \ell} g^{k\ell}_{rs} a_{rs}, \tag{4.11}$$

where $g^{k\ell}_{rs}$ are some constant depending on k, l, r and s. By assumption $a_{00} = 0$ and from (4.10), $a_{10} = a_{20} = 0$. Hence, (4.11) shows that $a_{k\ell} = 0$ for $k \leq 2$ and all ℓ, proving (4.8).

Hence, $u(z_0) = 0$ implies that $z_0 = \pm z_S$. We now see that the zero can occur at only one of the two points. Indeed, if u vanishes at both $-z_S$ and z_S then (note that $z(K) = -z_S$)

$$\begin{aligned} \widetilde{u}(z) &:= F_K(z - z_S) F_{-K}(z + z_S) u(z) \\ &= e^{-i(z_S - \bar z_S)K} \frac{\theta(z)^2}{\theta(z - z_S)\theta(z + z_S)} u(z) \in \ker_{L^2_{-K}(\mathbb{C}/\Lambda;\mathbb{C}^2)} D(\alpha), \end{aligned} \tag{4.12}$$

and $\widetilde{u} \not\parallel u$. But this contradicts simplicity.

To show that u has to vanish at z_S we analyse f (defined in (4.6)) near $\pm z_S$. From (4.7) and the fact that $\omega z_S = z_S - (1 + \omega)$, we obtain (see (4.7)),

$$f(z_S + \zeta) = f(\omega z_S + \omega\zeta) = f(z_S - 1 - \omega - \omega\zeta) = \omega f(z_S + \omega\zeta),$$

that is $f(z_S+\omega\zeta)=\bar\omega f(z_S+\zeta)$, and, in view of the functional equation in (4.7), $f(-z_S+\omega\zeta)=\omega f(-z_S+\zeta)$. Hence, there exists $k_0\in\mathbb{Z}$ such that

$$f(-z_S+\zeta)=\sum_{k\geqslant -k_0}\zeta^{-2+3k}f_k,\quad f(z_S-\zeta)=\sum_{\ell\geqslant k_0}\zeta^{2+3\ell}g_\ell,\quad f_{-k_0}g_{k_0}=-1. \tag{4.13}$$

We also note that, (2.16) and the definition of $u=\tau(-K)u_{-K}=(\psi_1,\psi_2)^t$ gives

$$\psi_1(z)=f(z)^{-1}\psi_2(-z)=-f(-z)\psi_2(-z),\quad \psi_2(\mp z_S)=0. \tag{4.14}$$

Suppose that $u(z_S)\neq 0$ (which is equivalent to $u(-z_S)=0$). Then (4.14) shows that $f(z)$ has a pole at $-z_S$. The expansion (4.13) implies that the pole is of order at least 2. But using (4.14) again,shows that $-z_S$ is a zero of order at least 2 of the function ψ_2, in the sense that $\psi_2(-z_S+\zeta)=\zeta^2\widetilde\Psi_2(\zeta)$, $\Psi\in C^\omega$, near $\zeta=0$. Moreover, (4.7) shows that z_S is a zero of order at least 2 of f and from $\psi_1(-z)=-f(z)\psi_2(z)$, we deduce that $\psi_1(-z_S+\zeta)=\zeta^2\Psi_1(\zeta)$, $\Psi_1\in C^\omega$.

We have therefore proved that $\zeta^{-2}u(-z_S+\zeta)$ is smooth near 0. But this implies that

$$\widetilde u(z)=\wp(z+z_S;\omega,1)u(z)\in L^2_{-K}(\mathbb{C}/\Gamma)$$

which solves $D(\alpha)\widetilde u=0$, and $\widetilde u\not\parallel u$, a contradiction. This implies that u vanishes only at the point $z=z_S$.

We want to show that $\partial_z u(z_S)\neq 0$. Since $u\in L^2_{-K,0}$, we check that

$$\psi_1(z_S+\omega\zeta)=\psi_1(z_S+\zeta),\quad \psi_2(\omega z_S+\zeta)=\bar\omega\psi_2(z_S+\zeta).$$

Since $u(-z_S)\neq 0$ and $\psi_2(-z_S)=0$, we see that $\psi_1(-z_S)\neq 0$. We conclude from (4.14) that f vanishes at $-z_S$, so that using (4.13), $f(-z_S-\zeta)=\sum_{k\geq 0}F_k\zeta^{1+3k}$. We then have

$$\psi_2(z_S+\zeta)=f(-z_S-\zeta)\psi_1(-z_S-\zeta)=\left(\sum_{k\geq 0}F_k\zeta^{1+3k}\right)(\gamma+\mathcal{O}(|\zeta|)),\ \ \gamma:=\psi_1(-z_S)\neq 0.$$

We conclude that if $\partial_z u(z_S)=\partial_\zeta u(z_S+\zeta)|_{\zeta=0}=0$ then $F_0=0$. Since we also have

$$\psi_1(z_S+\zeta)=-f(-z_S-\zeta)\psi_2(-z_S-\zeta)=-\left(\sum_{k\geq 0}F_k\zeta^{1+3k}\right)\psi_2(-z_S-\zeta),$$

we conclude that $\zeta^{-3}u(z_S+\zeta)$ is smooth near 0. But this gives a contradiction as in (4.9).

The final conclusion (1.11) follows from (3.8) applied with $k'-K=k$, $z_0=z_S$, and the fact that $F_{k'}(z-z_S)$ vanishes simply and uniquely at $z_S+z(k')=z(k)=\sqrt{3}k/4\pi i+\Lambda$ (see (3.2)). □

Remark. Rather than considering the vanishing of $\tau(-K)u_{-K} \in L^2_{-K,0}$ we could look at u_0, $\ker_{L^2_0}(D(\alpha)) = \mathbb{C}u_0$, $\alpha \in \mathcal{A}$. One can easily show (see [BeZw23, Proposition 3.6]) that $u_0 \in L^2_{0,2}$ and that $\mathscr{E}u_0 = \pm iu_0$ (see (2.9) and note that $\mathrm{Spec}_{L^2_0}(\mathscr{E}) = \{i, -i\}$). That implies that u_0 vanishes at zero and that other zeros are symmetric with respect to the origin. But 0 is the only zero as the same argument as in (4.12) would contradict simplicity. Since, again by simplicity, $\tau(-K)u_{-K} = c_0\tau(-K)F_{-K}(z)u_0$, that gives a different (and perhaps simpler) proof that $\tau(-K)u_{-K}$ vanishes only at z_S.

As suggested by Mengxuan Yang, we can then see directly that the u_0 vanishes simply at 0 (which then implies that u_{-K} vanishes simply at z_S). For that consider $u_1 \in C^\infty(\mathbb{C})$ such that $u_0(z) = zu_1(z)$ (this follows from Lemma 3.2). But then $D(\alpha)(zu_1(z)) = zD(\alpha)u_1(z) = 0$, and as u_1 is smooth, $D(\alpha)u_1(z) = 0$ for $z \in \mathbb{C}$. Hence, if $u_1(0) = 0$ then Lemma 3.2 shows that $u(z) = zu_1(z) = z^2u_2(z)$ and the $\wp'$-function argument (see (4.9)) contradicts simplicity.

We opted for a direct discussion of $\tau(K)u_K$ (and the proof of simplicity of the zero) as that protected state which exists for all α and its zero were central in the original physics presentation [TKV19].

5. Theorem 4 and two numerical observations

Here we present two numerical observations about the structure of flat bands and compute the Chern number of the flat band.

5.1. **Fixed shape of the rescaled flat band.** We define rescaled bands as follows:

$$\widehat{E}_j(\alpha, k) := \frac{E_j(\alpha, k)}{\max_k E_1(\alpha, k)}, \quad \alpha \in \mathbb{R}. \tag{5.1}$$

and notice that for α near α's near elements of $\mathcal{A}_\mathbb{R}$,

$$E_1(\alpha, k) \simeq |U(z(k))|, \quad z(k) := \frac{\sqrt{3}k}{4\pi i}k : \Lambda^* \to \Lambda, \tag{5.2}$$

see Figure 2 and, for an animated version `https://math.berkeley.edu/~zworski/KKmovie.mp4`. We note that the $|U(z(k))|$ is the simplest function with symmetries of $E_1(k)$ and conic singularities at $\pm K$.

The following heuristic explanation was suggested by Ledwith et al [Le*22]. Assuming that the flat band is simple consider perturbation theory of $H_k(\alpha)$ near $\widetilde{\alpha} \in \mathcal{A}_\mathbb{R}$:

$$\begin{gathered} \partial_\alpha E_1(\widetilde{\alpha}, k) = \frac{|\langle Vu_k, v_{\bar{k}}\rangle|}{\|u_k\|\|v_{\bar{k}}\|}, \quad u_k, v_k \in L^2_0, \quad V(z) := \begin{pmatrix} 0 & U(z) \\ U(-z) & 0 \end{pmatrix}, \\ (D(\widetilde{\alpha}) - k)u_k = 0, \quad (D(\widetilde{\alpha})^* - \bar{k})v_k = 0, \end{gathered} \tag{5.3}$$

where using (2.10) we can take $v_k := Qu_k$. A numerically evaluated graph of $k \mapsto \partial_\alpha E_1(\widetilde{\alpha}, k)$ is shown in Figure 6. Since $u_k = F_k u_0$, $\ker_{L^2_0} D(\alpha) = u_0$, the theta function

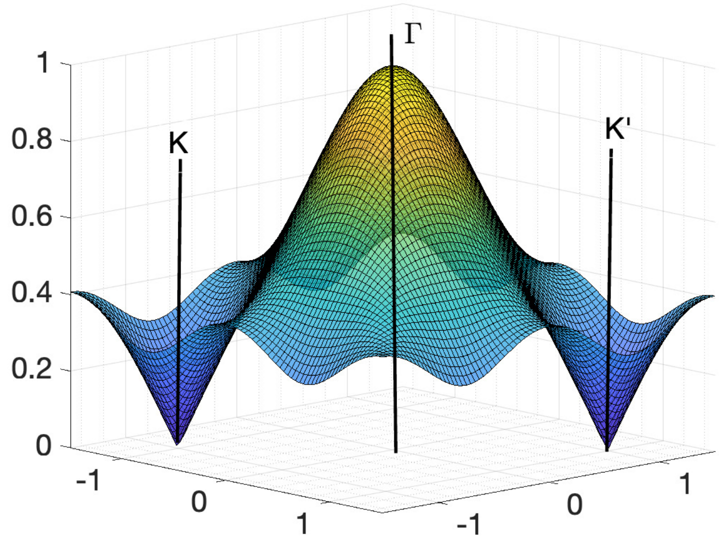

FIGURE 6. Normalized $\partial_\alpha E_1(\widetilde{\alpha}, k)$ at first magic angle. The protected zero energy states at the K and K' points are preserved, the maximum is attained at the Γ-point.

factors act in some sense as an FBI/Bargmann transform (see [Zw12, Chapter 13]). A (very formal) application of stationary phase method could then reproduce the potential U.

5.2. **The Chern connection and curvature.** The second numerical observation concerns the behaviour of the curvature of a connection on the natural hermitian bundle associated to the flat band. Since the bundle is holomorphic we use the Chern connection but, as is always the case, the resulting curvature is the same as the Berry curvature – see (5.11) for a direct verification in our case.

The numerical observation is shown in Figure 3 (a three dimensional plot of the curvature for one magic angle) and Figure 7 (the two dimensional plots for the first magic angles). We note that the absolute maximum appears at the Γ point, that is the center of the k-space hexagon spanned by translates of K and K' (equal to $-K$ in our coordinates), and the minima at K and K' – the vertices of the hexagon (the Dirac points). This is supposed to correspond to the fact that the bands are closest at Γ and farthest apart at the Dirac points (see the movie linked to Figure 2). So far, we only show that Γ, K and K' (that is $\mathcal{K}$ – see (2.7)) are critical points for the curvature which follows from Proposition 5.3 below.

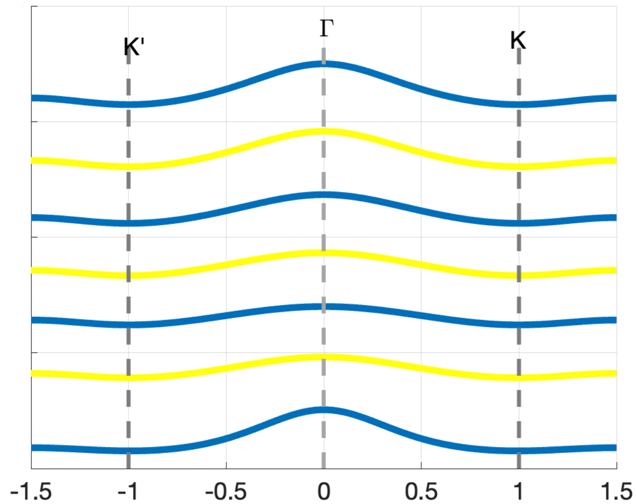

FIGURE 7. Cross-section of curvature for $k_x = 0$ for the first seven magic angles in increasing order. The extrema at K, Γ, K' follow from Prop. 5.3 and the subsequent discussion.

To describe the objects involved we need to define the hermitian holomorphic line bundle associated to the flat band. For general definitions and basic facts we refer to the self-contained appendix.

We assume that $\alpha_0 \in \mathcal{A}$ is simple in the sense of Theorem 3: $\dim \ker_{L^2_0}(D(\alpha_0)+k) = 1$ for all $k \in \mathbb{C}$. We remark that in [BHZ22, Theorem 3] we established simplicity of the first real magic α for the potential used in [TKV19]. Numerical calculations suggest that all real magic α's for that potential are indeed simple.

We recall from §3.1 (see (3.8)) that

$$\ker_{L^2_0}(D(\alpha_0) + k) = \mathbb{C}F_{k-K}(z - z_S)u_K(z).$$

We then put

$$[u(k)](z) = u(k, z) := F_{k-K}(z - z_S)u(z), \quad \mathscr{L}_\gamma u(k) = u(k), \quad \gamma \in \Lambda. \tag{5.4}$$

We also note that (3.6) implies

$$u(k + p) = e_p(k)^{-1}\tau(p)u(k). \tag{5.5}$$

Following the standard construction (see for instance [Pa07, §2.1]) we define

$$\begin{gathered} L := \left\{ [k, v]_\tau \in (\mathbb{C} \times L^2_0(\mathbb{C}/\Lambda; \mathbb{C}^2))/ \sim_\tau : v \in \ker_{L^2_0(\mathbb{C}/\Lambda;\mathbb{C}^2)}(D(\alpha_0) + k) \right\}, \\ [k, v]_\tau = [k', v']_\tau \iff (k, v) \sim_\tau (k', v') \iff \exists\, p \in \Lambda \; k' = k + p, \quad v' = \tau(p)v. \end{gathered} \tag{5.6}$$

We have

Lemma 5.1. *Definition* (5.6) *gives a holomorphic line bundle over* $\mathbb{C}/\Lambda$*,*

$$f : L \to \mathbb{C}/\Lambda, \quad f : [k, v]_\tau \to [k] \in \mathbb{C}/\Lambda.$$

The corresponding family of multipliers in (B.1) *is given by* $k \mapsto e_p(k)$*.*

Proof. The action of the discrete group Λ, $\lambda : (k, v) \mapsto (k + p, \tau(p)v)$ on the (trivial) complex line bundle

$$\widetilde{L} := \{(k, \kappa u(k)) : k \in \mathbb{C}, \quad \kappa \in \mathbb{C}\} \simeq \mathbb{C}_k \times \mathbb{C}_\kappa, \tag{5.7}$$

(where $u(k)$ is defined in (5.4)) is free and proper, and the quotient map is given by $\pi_\tau(k, \kappa u(k)) = [k, \kappa u(k)]_\tau$. Hence its quotient by that action, L, is a smooth complex manifold of dimension 2.

The map $(p, k) \mapsto e_p(k)$ satisfies conditions in (B.1),

$$e_{p+p'}(k) = \frac{\theta(z(k))}{\theta(z(k + p + p'))} = \frac{\theta(z(k + p))}{\theta(z(k + p + p'))} \frac{\theta(z(k))}{\theta(z(k + p))} = e_{p'}(k + p)e_p(k),$$

and for $p \in \Lambda$ we define $\varphi_p : \widetilde{L} \to \widetilde{L}$ as in (B.1): $\varphi_p(k, \kappa u(k)) = (k + p, e_p(k)\kappa u(k))$, $\kappa \in \mathbb{C}$. We then have $\pi_\tau(\varphi_p(k, \kappa u(k))) = \pi_\tau(k, e_p(k)\kappa u(k))$ and this gives L the structure of a complex line bundle over $\mathbb{C}/\Lambda$ □

Remark. As is implicit in the above proof, the multiplier $e_p(k)$ is the multiplier of the antiholomorphic theta line bundle over $\mathbb{C}/\Lambda^*$.

The hermitian structure is inherited from $L^2(\mathbb{C}/\Lambda)$ and the resulting hermitian structure on $\widetilde{L}$ of (5.7). In coordinates (k, κ) on $\widetilde{L}$, we get

$$h(k) = \|u(k)\|^2_{L^2(\mathbb{C}/\Lambda)},$$

where we note that (5.4) shows that $u(k)$ is well defined on $L^2_0(\mathbb{C}/\Lambda)$. This gives us also a hermitian structure on L: from (5.5) we see that

$$h(k) = |e_p(k)|^2 h(k + p), \quad p \in \mathbb{Z} \oplus \omega\mathbb{Z}. \tag{5.8}$$

To h we associate the Chern connection (B.5) and the curvature Ω, (B.6). The general formula (B.8) then reproduces the calculation from [Le*20] (where (5.8) was used directly):

$$c_1(L) = \frac{i}{2\pi} \int_{\mathbb{C}/\mathbb{Z}\omega\oplus\mathbb{Z}} \Omega = -1. \tag{5.9}$$

This proves Theorem 4.

Remark. We should stress that $k \mapsto u(k)$ is *not* a holomorphic section of L. In fact, as indicated by the Chern number, the line bundle L does not have any holomorphic sections. The dual line bundle corresponding to the kernel $D(\alpha)^* + \bar{k}$ has the Chern number equal to 1, and hence has holomorphic sections which can be expressed using theta functions.

We also have an explicit formula for Ω in terms of $u(k)$:

$$\Omega := H(k)d\bar{k} \wedge dk, \quad d\bar{k} \wedge dk = 2id \operatorname{Re} k \wedge d \operatorname{Im} k,$$
$$H(k) = \partial_k \partial_{\bar{k}} \log h(k) = \|u(k)\|^{-4} \left(\|u(k)\|^2 \|\partial_k u(k)\|^2 - |\langle \partial_k u(k), u(k) \rangle|^2 \right) \geq 0, \tag{5.10}$$

and (unlike $u(k)$) $H \in C^\infty(\mathbb{C}/\Lambda; \mathbb{R})$. We note that this is equivalent to the standard formula for the Berry curvature [Si83] (valid also in non-holomorphic situations):

$$H(k) = -\operatorname{Im}\langle \partial_{k_1} \varphi(k), \partial_{k_2} \varphi(k) \rangle_{L^2(\mathbb{C}, \mathbb{C}/3\Lambda)}, \quad \varphi(k) := u(k)/\|u(k)\|. \tag{5.11}$$

(This is a special case of a general fact.) We also recall the well known independence of $H(k)$ of the phase of φ:

Lemma 5.2. *Suppose that $\gamma(k) \in C^\infty(\mathbb{C}; \mathbb{R})$ and that $\varphi(k) \in C^\infty(\mathbb{C}; L^2_0(\mathbb{C}/\Lambda; \mathbb{C}^2))$, $\|\varphi(k)\|_{L^2(\mathbb{C}/\Lambda; \mathbb{C}^2)} \equiv 1$. Then, putting $k_1 = \operatorname{Re} k$, $k_2 = \operatorname{Im} k$,*

$$\operatorname{Im}\langle \partial_{k_1} \varphi(k), \partial_{k_2} \varphi(k) \rangle = \operatorname{Im}\langle \partial_{k_1} (e^{i\gamma(k)} \varphi(k)), \partial_{k_2} (e^{i\gamma(k)} \varphi(k)) \rangle. \tag{5.12}$$

Proof. The difference the two sides in (5.12) is given by

$$\operatorname{Im} \left(-\gamma_{k_1} \gamma_{k_2} \langle \varphi, \varphi \rangle + i\gamma_{k_1} \langle \varphi, \varphi_{k_2} \rangle - i\gamma_{k_2} \langle \varphi_{k_1}, \varphi \rangle \right) = \tfrac{1}{2} \left(\gamma_{k_1} \partial_{k_2} \|\varphi\|^2 - \gamma_{k_2} \partial_{k_1} \|\varphi\|^2 \right)$$

and this vanishes as $\varphi(k)$ is L^2-normalized. $\square$

Simplicity of α_0 has the following consequence:

Proposition 5.3. *Suppose that H is given by (5.10) with $u(k)$ defined in (5.4). Then*

$$H(\omega k) = H(k). \tag{5.13}$$

Proof. Since $[D(\alpha)u](\omega z) = \omega D(\alpha)[u(\omega \bullet)](z)$,

$$0 = [(D(\alpha) - k)u(k)](\omega z) = \omega(D(\alpha) - \bar{\omega} k)[u(k)(\omega \bullet)](z),$$

and simplicity shows that $u(\omega k, z) = \rho(k) u(k, \bar{\omega} z)$, and $\|u(\omega k)\| = |\rho(k)| \|u(k)\|$, $h(\omega k) = |\rho(k)|^2 h(k)$. In particular, $|\rho(k)| > 0$ and, as a function on $\mathbb{C}$, $\rho(k)/|\rho(k)| = e^{i\gamma(k)}$ for some $\gamma \in C^\infty(\mathbb{C}; \mathbb{R})$. The conclusion then follows from Lemma 5.2 and (5.11). $\square$

This proposition shows that elements of $\mathcal{K}$ (that is, Γ, K and K' – see (2.7)) are are critical points of H: suppose that $p \in \mathcal{K}$; then (since $\omega p \equiv p \mod \Lambda^*$ and H is Λ^*-periodic),

$$H(p + \kappa) = H(\omega p + \omega \kappa) = H(p + \omega \kappa),$$

which implies that $\partial_k H(p) = \omega \partial_k H(p)$, $\partial \bar{k} H(p) = \bar{\omega} \partial_{\bar{k}} H(p)$, that is that $d_k H(p) = 0$. This provides a partial explanation of Figures 3 and 7.

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Appendix A: translation between different conventions

We compare the coordinates use (1.2) to those in [Be*22], and implicitly in the physics literature – [TKV19]. One of the advantages of using the lattice Λ is the more straightforward connection with θ functions.

In [Be*22] we considered the following operator built from the potential U_0:

$$\begin{gathered} \widetilde{D}(\alpha) := \begin{pmatrix} 2D_{\bar\zeta} & \alpha U_0(\zeta) \\ \alpha U_0(-\zeta) & 2D_{\bar\zeta} \end{pmatrix}, \quad \overline{U_0(\bar\zeta)} = U_0(\zeta), \\ U_0\left(\zeta + \tfrac{4\pi i}{3}(a_1\omega + a_2\omega^2)\right) = \bar\omega^{a_1+a_2} U_0(\zeta), \quad U_0(\omega\zeta) = \omega U_0(\zeta). \end{gathered} \tag{A.1}$$

We then have periodicity with respect to

$$\Gamma := 4\pi i(\omega\mathbb{Z} + \omega^2\mathbb{Z}) = 4\pi i\Lambda$$

and twisted periodicity with respect to $\Gamma/3$. The dual lattices are given by

$$\Gamma^* := \tfrac{1}{\sqrt3}(\omega\mathbb{Z} \oplus \omega^2\mathbb{Z}) = \tfrac{1}{\sqrt3}\Lambda, \quad (\tfrac13\Gamma)^* = 3\Gamma^* = \sqrt3\Lambda.$$

This means that to switch to (twisted) periodicity with respect to Λ we need a change of variables:

$$\zeta = \tfrac43\pi i z, \quad \tfrac13\Gamma = \tfrac43\pi i\Lambda, \quad 3\Gamma^* = (\tfrac13\Gamma)^* = \sqrt3\Lambda = \frac{3}{4\pi i}\Lambda^*. \tag{A.2}$$

Then

$$\widetilde{D}(\alpha) = -\frac{3}{4\pi i}\begin{pmatrix} 2D_{\bar z} & \alpha U(z) \\ \alpha U(-z) & 2D_{\bar z} \end{pmatrix}, \quad U(z) := -\tfrac43\pi i U_0\left(\tfrac43\pi i z\right). \tag{A.3}$$

The twisted periodicity condition in (A.1) corresponds to the condition in (1.2) since

$$\bar\omega^{a_1+a_2} = e^{i\langle a_1\omega + a_2\omega^2, K\rangle}, \quad K = \tfrac{4}{\sqrt3}\pi i(-\tfrac13 - \tfrac23\omega) = \tfrac43\pi.$$

Floquet theory using $\mathscr{L}_\gamma$ defined in (1.5) is equivalent to the Floquet theory based on $\widetilde{\mathcal{L}}_{\mathbf{a}}$ used in [TKV19]: for $u \in L^2_{\rm loc}(\mathbb{C};\mathbb{C}^2)$,

$$\widetilde{\mathscr{L}}_{\mathbf{a}} u := \begin{pmatrix} \omega^{a_1+a_2} & 0 \\ 0 & 1 \end{pmatrix} u(\zeta + \mathbf{a}), \quad \mathbf{a} = \tfrac43\pi i(\omega a_1 + \omega^2 a_2) \in \tfrac13\Gamma, \ a_j \in \mathbb{Z}. \tag{A.4}$$

The Floquet theory based on L^2_0 defined using $\widetilde{\mathcal{L}}_{\mathbf{a}}$ gives different values of $k \in 3\Gamma^*$ for protected states. That is easily seen by considering the spectrum of $2D_{\bar z}$ on that L^2_0 which is given (modulo $3\Gamma^*$) by

$$\widetilde{K} = \frac{3}{4\pi i} K = -i, \quad \widetilde{K}' = 0,$$

with the $\widetilde{\Gamma}$ point corresponding to i (see also [BeZw23, Proposition 3.2]).

Appendix B: holomorphic line bundles over tori

Suppose Λ is a lattice $\mathbb{Z} \oplus \omega\mathbb{Z}$, $\operatorname{Im}\omega > 0$ (for us it will be $\omega = e^{2\pi i/3}$). A holomorphic line bundle L, $f : L \to \mathbb{C}/\Lambda$ can be described using a pullback by the canonical projection $\pi : \mathbb{C} \to \mathbb{C}/\Lambda$, that is a (trivial) line bundle $\pi^* L$ over $\mathbb{C}$ for which the following diagram commutes:

$$\begin{array}{ccc} \pi^* L & \longrightarrow & L \\ \big\downarrow & & \big\downarrow f \\ \mathbb{C} & \xrightarrow{\ \pi\ } & \mathbb{C}/\Lambda \end{array}$$

We can identify $\pi^* L$ with $\mathbb{C} \times \mathbb{C}$ and write its elements as (z, ζ). Every line bundle over $\mathbb{C}/\Lambda$ is associated to an entire non vanishing function $z \mapsto e_\lambda(z)$ such that

$$\varphi_\lambda(z,\zeta) = (z+\lambda, e_\lambda(z)\zeta), \quad e_{\lambda+\lambda'}(z) = e_{\lambda'}(z+\lambda) e_\lambda(z), \quad \lambda, \lambda' \in \Lambda,$$

$$\begin{array}{ccc} \pi^* L & \xrightarrow{\ \varphi_\lambda\ } & \pi^* L \\ \big\downarrow & & \big\downarrow \\ L & \xrightarrow{\ \mathrm{id}\ } & L \end{array} \tag{B.1}$$

In other words L is the set of equivalence classes, $[(z,\zeta)]_\Lambda$ where

$$(z,\zeta) \sim (z',\zeta') \iff \exists\, \lambda \in \Lambda \ \ (z,\zeta) = \varphi_\lambda(z',\zeta').$$

We then have

$$C^\infty(\mathbb{C}/\Lambda; L) \simeq \{ u \in C^\infty(\mathbb{C}) : \forall\, \lambda \in \Lambda,\ u(z+\lambda) = e_\lambda(z) u(z) \}. \tag{B.2}$$

Holomorphic sections are defined by replacing $C^\infty(\mathbb{C})$ with $\mathscr{O}(\mathbb{C})$, the space of entire functions on $\mathbb{C}$.

The functions $e_\lambda(z)$ are not unique: if $g \in \mathscr{O}(\mathbb{C})$ then $\widetilde{e}_\lambda(z) := e^{g(z+\lambda)} e_\lambda(z) e^{-g(z)}$ gives the same line bundle.

Remark. The Appell–Humbert theorem completely characterizes the allowed functions $e_\lambda(z)$. Here we will concentrate on the specific $e_\lambda(z)$ arising from the eigenfunctions.

B.1. **Hermitian structure and the Chern connection.** Hermitian structure provides a notion of length on the fibers of L, $p^{-1}(z)$ locally described by (with $|\zeta|^2 = \bar{\zeta}\zeta$, $\zeta \in \mathbb{C}$),

$$\begin{gathered} \|[(z,\zeta)]\|^2 = h(z)|\zeta|^2, \\ \|[(z,\zeta)]\|^2 = \|[\varphi_\lambda(z,\zeta)]\|^2 \iff h(z) = h(z+\lambda)|e_\lambda(z)|^2. \end{gathered} \tag{B.3}$$

Conversely any positive smooth function $h(z)$ satisfying the condition in (B.3) defines a hermitian metric on L.

Connections on L are identified with connections on $\pi^*L \simeq \mathbb{C} \times \mathbb{C}$. The latter are given by $\eta \in C^\infty(\mathbb{C}, T^*\mathbb{C})$ so that we can define the actual connection:

$$D_\eta s = ds + s\eta \in C^\infty(\mathbb{C}, T^*\mathbb{C}), \quad s \in C^\infty(\mathbb{C}, \mathbb{C}).$$

This gives a connection on L provided that

$$d(s(z+\lambda)) + \eta(z+\lambda)s(z+\lambda) = e_\lambda(z)(ds(z) + \eta(z)s(z)),$$

that is when

$$\eta(z+\lambda) = \eta(z) - e_\lambda(z)^{-1}e'_\lambda(z)dz. \tag{B.4}$$

The *Chern connection* is defined by

$$\eta(z) = \partial(\log h(z)) = h(z)^{-1}\partial_z h(z)dz, \tag{B.5}$$

(here we denote by $\partial f = \partial_z f(z)dz$, the $(1,0)$-differential) and we easily check (B.4) using (B.3) and the holomorphy of $z \mapsto e_\lambda(z)$:

$$\begin{aligned} \eta(z+\lambda) &= \partial_z \log h(z+\lambda)dz = \partial_z(-\log(e_\lambda(z)\overline{e_\lambda(z)}) + \log h(z))dz \\ &= \eta(z) - e_\lambda(z)^{-1}e'_\lambda(z)dz. \end{aligned}$$

B.2. **Curvature and Chern numbers.** In this simplest case the curvature is just the differential of η and it is a well defined (unlike η) $(1,1)$-differential form on $\mathbb{C}/\Lambda$:

$$\Omega := \bar{\partial}\eta = \bar{\partial}\partial(\log h(z)) = \partial_{\bar{z}}\partial_z(\log h(z))d\bar{z} \wedge dz. \tag{B.6}$$

Indeed, the holomorphy of $z \mapsto e_\lambda(z)$ gives

$$\partial_{\bar{z}}\partial_z \log h(z+\lambda) = \partial_{\bar{z}}\left(\partial_z \log h(z) - e_\lambda(z)^{-1}e'_\lambda(z)\right) = \partial_{\bar{z}}\partial_z \log h(z).$$

The Chern number (since we are in complex dimension one) is defined as

$$c_1(L) := \frac{i}{2\pi}\int_{\mathbb{C}/\Lambda} \Omega \in \mathbb{Z}. \tag{B.7}$$

To see that $c_1(L)$ is an integer, we choose a fundamental domain of Λ, F, and apply Stokes's theorem: we can take $F = [0,1) + \omega[0,1)$,

$$\begin{aligned} c_1(L) &= \frac{i}{2\pi}\int_F \partial_{\bar z}\partial_z \log h \, d\bar z \wedge dz = \frac{i}{2\pi}\oint_{\partial F} \partial_z \log h(z) dz \\ &= \frac{i}{2\pi}\int_0^1 \left(\partial_z \log h(t) + \omega\partial_z \log h(1+t\omega) - \partial_z \log h(\omega + t) - \omega\partial_z \log h(t\omega)\right) dt \end{aligned} .$$

Now,

$$\begin{aligned} \omega\partial_z \log h(1+t\omega) &= \partial_t \left(\log h(t\omega) - \log e_1(t\omega)\right), \\ \partial_z \log h(\omega+t) &= \partial_t \left(\log h(t) - \log e_\omega(t)\right), \end{aligned}$$

and

$$\begin{aligned} c_1(L) &= \frac{i}{2\pi}\int_0^1 \partial_t \left(\log e_\omega(t) - \log e_1(\omega t)\right) dt \\ &= \frac{i}{2\pi}\left(\log e_\omega(1) - \log e_\omega(0) + \log e_1(0) - \log e_1(\omega)\right), \end{aligned} \tag{B.8}$$

where we choose entire functions $\log e_\lambda(z)$, which are determined up to an integral multiple of $2\pi i$. From (B.1) we see that $e_\omega(1)e_1(0) = e_{\omega+1}(0) = e_1(\omega)e_\omega(0)$, and that implies (by taking logarithms) that the right hand side of (B.8) is an integer. (We note that 0 can be replaced by any $z \in \mathbb{C}$.)

The hermitian metric on L is called *strictly positive* if $\Delta \log h < 0$, that is, the locally defined function $-\log h$ is strictly subharmonic. In this case, Ω also defines a Kähler structure (of course in the very special one dimensional case):

$$g = -\partial_{\bar z}\partial_z \log h(z)|dz|^2.$$

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