Math 256B. Homework 10

Due Wednesday 10 April

1. Let X be an integral scheme, let $\mathscr{L}$ be a line sheaf on X, let s be a nonzero rational section of $\mathscr{L}$, and let $D = (s)$. Show that $\mathscr{O}_X(D) \cong \mathscr{L}$.

2. Hartshorne II Ex. 6.6. You may ignore the last sentence of (d).

   However, note that the definition of the group law on X comes from the group operation on $\operatorname{Cl}^0 X$ and the bijection (and not from the line-and-chord operation). Also, make it clear whether you are adding divisors or using the group operation on X.

3(NC). Let X be a variety over an algebraically closed field k, and let $U = \operatorname{Spec} A$ be a nonempty open affine subset of X.

   (a). Let $P \in X$ be a point not in U. Show that there is a function $f \in A$ that does not extend to a regular function at P (i.e., $f \notin \mathscr{O}_{X,P}$). Moreover, f can be chosen from any given generating set for A over k.

   [**Hint:** Hartshorne II Ex. 4.2 or Vakil 2023 11.4.A (=Vakil 2017 10.2.A) may be useful (and may be used without proof). But if you use one of them, say which one you are using and how you are using it.]

   (b). Assume now that X is a nonsingular curve (not necessarily projective). Choose a closed embedding $i\colon U \to \mathbb{A}^n_k$ over k for some n, let V be the corresponding closed subscheme of $\mathbb{A}^n_k$, and let Y be the closure of V in $\mathbb{P}^n_k$ (with reduced induced subscheme structure). (Here we regard $\mathbb{A}^n_k$ as the open subscheme $D_+(x_0)$ of $\mathbb{P}^n_k$ in the usual way.) As was noted in class, the map $i\colon U \to V$ extends uniquely to a morphism $j\colon X \to Y$. Show that $j^{-1}(V) = U$ (or equivalently, $j^{-1}(Y \setminus V) = X \setminus U$).

4(NC). Hartshorne III Ex. 6.3.