COMMUTATIVE ALGEBRA OF n POINTS IN THE PLANE

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Introduction

These lectures will address commutative algebra questions arising from the geometry of configurations of n points in the affine plane $\mathbb{C}^2$. In the first lecture, we will study the ideal of the locus where some two of the points coincide. In the process, we will be led naturally to consider the action of the symmetric group S_n, permuting the points among themselves. This provides the topic for the second lecture, in which we will study the rings of invariants and coinvariants for this action. As you can see, we have chosen to study questions that involve rather simple and naive geometric considerations. For those who have not encountered this subject before, it may come as a surprise that the theorems which give the answers are quite remarkable, and seem to be hard.

One reason for the subtlety of the theorems is that lurking in the background is the more subtle geometry of the Hilbert scheme of points in the plane. The special properties of this algebraic variety play a role in the proofs of the theorems. The involvement of the Hilbert scheme in the proofs means that at present the theorems apply only to points in the plane, even though we could equally well raise the same questions for points in $\mathbb{C}^d$, and conjecturally we expect them to have similar answers.

In the third lecture, we will change perspective slightly, by introducing the $\binom{n}{2}$ lines connecting the points in pairs, and asking for the ideal of relations among the slopes of these lines when the points are in general position (*i.e.*, no two points coincide). We present a synopsis of the beautiful and surprising results on this problem found by my former student, Jeremy Martin.

1. A subspace arrangement

Let us fix some notation. We consider ordered n-tuples of points in the plane

$$P_1, \ldots, P_n \in \mathbb{C}^2.$$

We work over $\mathbb{C}$ to keep things simple and geometrically concrete, although some of the commutative algebra results remain true over more general ground rings. Assigning the points coordinates

$$x_1, y_1, \ldots, x_n, y_n,$$

we identify the space E of all n-tuples $(P_1, \ldots, P_n)$ with $\mathbb{C}^{2n}$. The coordinate ring of E is then the polynomial ring

$$\mathbb{C}[E] = \mathbb{C}[\mathbf{x}, \mathbf{y}] = \mathbb{C}[x_1, y_1, \ldots, x_n, y_n]$$

in $2n$ variables. Let V_{ij} be the locus where $P_i = P_j$, that is, the codimension-2 subspace of E defined by the equations $x_i = x_j$, $y_i = y_j$. The locus

$$V = \bigcup_{i<j} V_{ij}$$

Date: Mar. 29, 2003.

where some two points coincide is a *subspace arrangement* of $\binom{n}{2}$ codimension-2 subspaces in E. Evidently, V is the zero locus of the radical ideal

$$I = I(V) = \bigcap_{i<j} (x_i - x_j, y_i - y_j).$$

Now we come to the central theme of today's lecture.

Question: what does the ideal I look like?

As a warm-up, let us consider the much easier case of n points on a *line*. Then we only have coordinates $x_1, \ldots, x_n$, and the analog of I is the ideal

$$J = \bigcap_{i<j} (x_i - x_j) \subseteq \mathbb{C}[\mathbf{x}].$$

This ideal has various obvious properties.

(1) J is the principal ideal $(\Delta(\mathbf{x}))$ generated by the Vandermonde determinant

$$\Delta(\mathbf{x}) = \prod_{i<j} (x_i - x_j) = \det \begin{bmatrix} 1 & x_1 & \ldots & x_1^{n-1} \\ 1 & x_2 & \ldots & x_2^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & x_n & \ldots & x_n^{n-1} \end{bmatrix}.$$

(2) J is (trivially) a free $\mathbb{C}[\mathbf{x}]$ module with generator $\Delta(\mathbf{x})$.

(3) $J^m = J^{(m)} \underset{def}{=} \bigcap_{i<j} (x_i - x_j)^m$, that is, the powers of J are equal to its symbolic powers. This is clear, since both ideals are equal to $(\Delta(\mathbf{x})^m)$.

(4) The Rees algebra $\mathbb{C}[\mathbf{x}][tJ]$ is Gorenstein. In fact, it's just a polynomial ring in $n+1$ variables.

Of course all the above properties merely follow from the fact that J is the ideal of a hyperplane arrangement. In general, one cannot say much about the ideal of an arrangement of subspaces of codimension 2 or more. However, our ideal I is rather special, so let's try to compare its properties with those listed above for J.

Beginning with property (1), we can observe that I has certain obvious elements. The symmetric group S_n acts on E, permuting the points P_i amongst themselves. In coordinates, this is the *diagonal* action:

$$\sigma x_i = x_{\sigma(i)},\ \sigma y_i = y_{\sigma(i)} \quad \text{for } \sigma \in S_n.$$

We denote the sign character of S_n by

$$\epsilon(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ is even,} \\ -1 & \text{if } \sigma \text{ is odd.} \end{cases}$$

Let

$$\mathbb{C}[\mathbf{x}, \mathbf{y}]^\epsilon = \{ f \in \mathbb{C}[\mathbf{x}, \mathbf{y}] : \sigma f = \epsilon(\sigma) f \quad \text{for all } \sigma \in S_n \}$$

be the space of *alternating polynomials*. Any alternating polynomial f satisfies

$$f(x_1, y_1, \ldots, x_i, y_i, \ldots, x_j, y_j, \ldots, x_n, y_n) = -f(x_1, y_1, \ldots, x_j, y_j, \ldots, x_i, y_i, \ldots, x_n, y_n),$$

which immediately implies that f vanishes on every V_{ij}, that is, f belongs to I.

We remark here that there is a natural vector space basis for $\mathbb{C}[\mathbf{x}, \mathbf{y}]^\epsilon$. Namely, let $\mathbf{x}^\alpha \mathbf{y}^\beta = x_1^{\alpha_1} y_1^{\beta_1} \cdots x_n^{\alpha_n} y_n^{\beta_n}$ be a monomial, and put

$$A(\mathbf{x}^\alpha \mathbf{y}^\beta) = \sum_{\sigma \in S_n} \epsilon(\sigma) \sigma(\mathbf{x}^\alpha \mathbf{y}^\beta).$$

If the exponent pairs (α_i, β_i) are not all distinct, then $A(\mathbf{x}^\alpha \mathbf{y}^\beta) = 0$. If they are all distinct, set $D = \{(\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)\} \subseteq \mathbb{N} \times \mathbb{N}$. Then $A(\mathbf{x}^\alpha \mathbf{y}^\beta)$ is given by a bivariate analog of the Vandermonde determinant

$$A(\mathbf{x}^\alpha \mathbf{y}^\beta) = \Delta_D = \det \begin{bmatrix} x_1^{\alpha_1} y_1^{\beta_1} & \ldots & x_1^{\alpha_n} y_1^{\beta_n} \\ \vdots & & \vdots \\ x_n^{\alpha_1} y_n^{\beta_1} & \ldots & x_n^{\alpha_n} y_n^{\beta_n} \end{bmatrix},$$

which only depends on D, up to sign. It is easy to see that the set of all such polynomials

$$\{\Delta_D : D \subseteq \mathbb{N} \times \mathbb{N},\ |D| = n\}$$

is a vector space basis of $\mathbb{C}[\mathbf{x}, \mathbf{y}]^\epsilon$. In particular, the ideal they generate is the same as the ideal generated by all alternating polynomials. We have just seen that this ideal is contained in I.

Theorem 1.1. *We have $I = (\Delta_D : D \subseteq \mathbb{N} \times \mathbb{N},\ |D| = n)$.*

As far as I know, this is not an easy theorem. We will say something about its proof later on. Before that, I would like to briefly discuss the question of finding a minimal set of generators for I, and take up the analogs of the other properties (2)–(4) that we had for J.

Note that I is a homogeneous ideal—in fact it is doubly homogeneous, with respect to the double grading given by degrees in the $\mathbf{x}$ and $\mathbf{y}$ variables separately. It follows that a set of homogeneous generators for I, for example a subset of the Δ_D's, is minimal if and only if its image is a vector space basis of

$$I/(\mathbf{x}, \mathbf{y})I.$$

It turns out that we know exactly what the size of such a minimal generating set must be, although no one has yet succeeded in finding an explicit choice of minimal generators.

Theorem 1.2. *The dimension of $I/(\mathbf{x}, \mathbf{y})I$ is equal to the* Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$.

Indeed, quite a bit more can be said. The space $M = I/(\mathbf{x}, \mathbf{y})I$ is doubly graded, say $M = \bigoplus_{r,s} M_{r,s}$. Define a "$q,t$-analog" of the Catalan number by

$$C_n(q, t) = \sum_{r,s} t^r q^s \dim M_{r,s}.$$

According to Theorem 1.2 we then have $C_n(1, 1) = C_n$. From geometric considerations involving the Hilbert scheme we have a formula for $C_n(q, t)$ [9, 11], and Theorem 1.2 is proved by specializing the formula to $q = t = 1$. The formula gives $C_n(q, t)$ as a complicated rational function of q, t that on its face is not even obviously a polynomial. However, Garsia and Haglund [5, 6] discovered a simple combinatorial interpretation of the formula, as follows. Let $\mathcal{D}$ be the set of integer sequences

$$\lambda_1 \geq \lambda_2 \geq \cdots \lambda_{n-1} \geq 0$$

satisfying

$$\lambda_i \leq n - i \quad \text{for all } i.$$

In other words, $\mathcal{D}$ is the set of partitions whose Young diagram fits inside that of the partition $(n-1, n-2, \ldots, 1)$. It is well-known that the number of these is the Catalan number C_n. For each $\lambda \in \mathcal{D}$, define

$$a(\lambda) = \sum_i (n - i - \lambda),$$

$$b(\lambda) = \left|\left\{i < j : \lambda_i - \lambda_j + i - j \in \{0, 1\}\right\}\right|.$$

Garsia and Haglund showed that

$$C_n(q, t) = \sum_{\lambda \in \mathcal{D}} q^{a(\lambda)} t^{b(\lambda)}.$$

Problem 1.1. Find a rule associating to each $\lambda \in \mathcal{D}$ an n-element subset $D(\lambda) \subseteq \mathbb{N} \times \mathbb{N}$ in such a way that $\deg_{\mathbf{y}} \Delta_{D(\lambda)} = a(\lambda)$, $\deg_{\mathbf{x}} \Delta_{D(\lambda)} = b(\lambda)$, and the set $\{\Delta_{D(\lambda)} : \lambda \in \mathcal{D}\}$ generates I.

A solution to this problem would give a new and in some sense improved proof of the Garsia-Haglund result. One can proceed similarly for the powers of I, defining

$$M^{(m)} = I^m/(\mathbf{x}, \mathbf{y})I^m$$

and

$$C_n^{(m)}(q,t) = \sum_{r,s} t^r q^s \dim M_{r,s}^{(m)}.$$

Again there is a formula for $C_n^{(m)}(q,t)$ from geometry. There is also a *conjectured* combinatorial interpretation, as follows. Let $\mathcal{D}^{(m)}$ be the set of integer sequences

$$\lambda_1 \geq \lambda_2 \geq \cdots \lambda_{n-1} \geq 0$$

satisfying

$$\lambda_i \leq m(n-i) \quad \text{for all } i.$$

In other words, we now allow partitions whose Young diagram fits inside that of $m \cdot (n-1, n-2, \ldots, 1)$. For each $\lambda \in \mathcal{D}^{(m)}$, define

$$a^{(m)}(\lambda) = \sum_i (m(n-i) - \lambda),$$
$$b^{(m)}(\lambda) = \left|\left\{i < j : \lambda_i - \lambda_j + m(i-j) \in \{0, 1, \ldots, m\}\right\}\right|.$$

Conjecture 1.1. *We have* $C_n^{(m)}(q,t) = \sum_{\lambda \in \mathcal{D}^{(m)}} q^{a^{(m)}(\lambda)} t^{b^{(m)}(\lambda)}$.

Problem 1.2. Find generators for I^m indexed by elements $\lambda \in \mathcal{D}^{(m)}$, with $\mathbf{y}$-degree equal to $a^{(m)}(\lambda)$ and $\mathbf{x}$-degree equal to $b^{(m)}(\lambda)$.

We remark that it is known that $C_n^{(m)}(q,1) = \sum_{\lambda \in \mathcal{D}^{(m)}} q^{a^{(m)}(\lambda)}$, and hence in particular that $\dim I^m/(\mathbf{x}, \mathbf{y})I^m = C_n(1,1) = |\mathcal{D}^{(m)}|$. The generating set given by a solution to Problem 1.2 would therefore be minimal, so Conjecture 1.1 would follow automatically.

Now we ask whether I has an analog of property (2) for J. It certainly cannot be that I is a free $\mathbb{C}[\mathbf{x}, \mathbf{y}]$-module, for then $\mathbb{C}[\mathbf{x}, \mathbf{y}]/I$ would have depth $2n-1$, whereas it has dimension $2n-2$. What we have instead is that I is a free module with respect to either set of variables alone.

Theorem 1.3. *The ideal I is a free $\mathbb{C}[\mathbf{y}]$-module.*

This theorem is best possible, modulo one detail. The ideal I has an extra degree of freedom: it is invariant with respect to $\mathbf{x}$-translations mapping each x_i to $x_i + a$. This invariance holds for $I/(\mathbf{y})I$ as well, and implies that $I/(\mathbf{y})I$ is a free $\mathbb{C}[x_1]$-module (say). Hence Theorem 1.3 actually implies that I is a free $\mathbb{C}[\mathbf{y}, x_1]$-module, and in particular has depth at least $n+1$. On the other hand, it is easy to see that $\Delta(\mathbf{y})$ represents a nonzero element of $I/(\mathbf{y})I$ annihilated by $(x_1 - x_2, \ldots, x_{n-1} - x_n)$. This implies that $\operatorname{depth} I/(\mathbf{y})I \leq 1$ and hence $\operatorname{depth} I = n+1$.

Next we turn to property (3), the coincidence of powers with symbolic powers.

Theorem 1.4. *We have* $I^m = I^{(m)} \underset{def}{=} \bigcap_{i<j} (x_i - x_j, y_i - y_j)^{(m)}$ *for all* m.

In fact, Theorems 1.1, 1.3, and 1.4 are all plainly corollaries to the following two statements.

Theorem 1.5. *For all m, the m-th power of the ideal $(\Delta_D : D \subseteq \mathbb{N} \times \mathbb{N},\ |D| = n)$ is a free $\mathbb{C}[\mathbf{y}]$-module.*

Corollary 1.1. *For all m, we have* $I^{(m)} = (\Delta_D : D \subseteq \mathbb{N} \times \mathbb{N},\ |D| = n)^m$.

On the maxim that every mathematics lecture should contain one proof, let us sketch how Theorem 1.5 implies Corollary 1.1. Abbreviating $(\Delta_D : D \subseteq \mathbb{N}\times\mathbb{N},\ |D| = n)$ to (Δ_D), we clearly have

$$(\Delta_D)^m \subseteq I^{(m)}.$$

Localizing at any point $\mathbf{P} \in E$ with not all P_i equal, one shows that both $(\Delta_D)_{\mathbf{P}}$ and $I_{\mathbf{P}}^{(m)}$ factor locally into products of the corresponding ideals in subsets of the variables. On the open set U where some $P_i \neq P_j$ we can therefore assume locally that $(\Delta_D)_{\mathbf{P}}^m = I_{\mathbf{P}}^{(m)}$, by induction on n.

Now Theorem 1.5 implies that $\mathbb{C}[\mathbf{x},\mathbf{y}]/(\Delta_D)^m$ has depth $\geq n-1$ as a $\mathbb{C}[\mathbf{y}]$-module. In particular, $(\Delta_D)^m$ cannot have an associated prime supported in $V(y_1 - y_2, \ldots, y_{n-1} - y_n)$, if $n \geq 3$. In other words, if $f \in \mathbb{C}[\mathbf{x},\mathbf{y}]$ belongs to the localization $(\Delta_D)_Q^m$ for all $Q \in (\operatorname{Spec}\mathbb{C}[\mathbf{y}]) \setminus V(y_1 - y_2, \ldots, y_{n-1} - y_n)$, then $f \in (\Delta_D)^m$. By induction this holds for all $f \in I^{(m)}$. The induction step assumes $n \geq 3$. The base cases $n = 1, 2$ are trivial. $\square$

Finally, we discuss property (4). Consider the Rees algebra $R = \mathbb{C}[\mathbf{x},\mathbf{y}][t(\Delta_D)]$, and put $X = \operatorname{Proj} R$, that is, the blowup of E at the ideal (Δ_D). Here, as above, (Δ_D) is shorthand for the ideal generated by all the alternating polynomials Δ_D. In view of Theorem 1.1, we can also identify X with the blowup of E along V, but it is preferable for geometric reasons not to take this as the definition.

The symmetric group S_n acts equivariantly on both X and E, giving a diagram

$$\begin{array}{ccc} X & \longrightarrow & E \\ \big\downarrow & & \big\downarrow \\ X/S_n & \longrightarrow & E/S_n. \end{array}$$

Now it develops that X/S_n is nothing else but the *Hilbert scheme* $\operatorname{Hilb}^n(\mathbb{C}^2)$ parametrizing 0-dimensional subschemes of length n in $\mathbb{C}^2$, or equivalently, ideals $\mathcal{J} \subseteq \mathbb{C}[x,y]$ such that $\dim_{\mathbb{C}} \mathbb{C}[x,y]/\mathcal{J} = n$. This is in fact not difficult to show, using explicit local coordinates on $\operatorname{Hilb}^n(\mathbb{C}^2)$ and the definition of X.

By a classical theorem of Fogarty [4], $\operatorname{Hilb}^n(\mathbb{C}^2)$ is non-singular and irreducible—see the Appendix for another proof using explicit local coordinates. It is also known that the locus in $\operatorname{Hilb}^n(\mathbb{C}^2)$ where the y-coordinates vanish, that is, the locus describing subschemes of $\mathbb{C}^2$ supported on the x-axis, has codimension n. From this it follows easily that

$$\dim R/(\mathbf{y}) = n + 1.$$

We come now to the most important theorem from the geometric point of view.

Theorem 1.6. *The blowup scheme X is arithmetically Gorenstein, that is, R is a Gorenstein ring.*

Let us pause to understand how this result is related to Theorem 1.5. The dimension count above shows that $(\mathbf{y})$ is a complete intersection ideal in R. Hence, if we assume Theorem 1.6 holds, then R is a free $\mathbb{C}[\mathbf{y}]$-module, which is merely a restatement of Theorem 1.5. So Theorem 1.5 is a simple corollary to Theorem 1.6.

Unfortunately for this logic, the only proof of Theorem 1.6 known at present *uses* Theorem 1.5. Specifically, although the main argument of the proof given in [10] is an induction based on elementary geometry of the Hilbert schemes, there is a key technical step that depends on Theorem 1.5. So for now we cannot elegantly deduce Theorem 1.5 from Theorem 1.6, as above, but must prove Theorem 1.5 directly.

Problem 1.3. Find an "intrinsic" proof of Theorem 1.6 that does not rely on Theorem 1.5.

In this connection we may note that there are classical theorems in commutative algebra for showing that Rees algebras are Cohen-Macaulay or Gorenstein. In particular, as W. Vasconcelos pointed out to me, since our ideal has codimension 2 it is enough to show that the Rees algebra R is Cohen-Macaulay, and it is then automatically Gorenstein (this consequence also follows from the geometry). Unfortunately, as far as I am aware, the theorems one might use to show that R is Cohen-Macaulay tend to require hypotheses on the blowup ideal, such as strong Cohen-Macaulayness, or small analytic spread, that fail drastically for our ideal

I. It is natural to inquire whether advances in singularity theory might even make it possible to show that our Rees algebra R has singularities better than Cohen-Macaulay. Could one hope to prove, for instance, that R is of F-rational type?

I'll conclude with some remarks concerning the existing proof and possible generalizations of Theorem 1.5, which for the moment remains the linchpin among the results. To prove Theorem 1.5, we first show that $(\Delta_D)^m$ is a direct summand as a graded $\mathbb{C}[\mathbf{x}, \mathbf{y}]$-module of the coordinate ring $\mathbb{C}[W]$ of an auxiliary subspace arrangement $W \subseteq E \times \mathbb{C}^{2mn}$, called a "polygraph." Then we show that $\mathbb{C}[W]$ is a free $\mathbb{C}[\mathbf{y}]$-module by explicitly constructing a basis. This requires a horribly complicated and not very illuminating induction. The basis construction is secretly modeled on a combinatorial interpretation of a formula from geometry for the Hilbert series of $\mathbb{C}[W]$. In the end, however, both the formula and the combinatorics are suppressed from the proof, as they must be, since one can only prove such formulas by assuming the theorem *a priori*.

I think that some of the complexity of the existing proof may eventually be removed. I also think that most of the phenomena concerning the ideal I should persist if we take points in $\mathbb{C}^d$ for general d, instead of $\mathbb{C}^2$. If so, we will need proofs that do not refer to the Hilbert scheme, secretly or otherwise. Here are some specific problems motivated by my thoughts along these lines.

Problem 1.4. Is it possible to dispense with the polygraph and construct a free $\mathbb{C}[\mathbf{y}]$-module basis of $(\Delta_D)^m$ directly? It would already be interesting to accomplish this for $d = 2$. In this case, the geometry does provide a formula for the Hilbert series, but an obstacle to using it is that we don't have a combinatorial interpretation, and therefore no clue how to index the basis elements.

Problem 1.5. Our subspace arrangement V can be written as $\mathbb{C}^2 \otimes V'$, where V' is the hyperplane arrangement $V' = \bigcup_{i<j} V(x_i - x_j)$ in $\mathbb{C}^n$. Here, for any subspace arrangement $A = \bigcup_k A_k \subseteq \mathbb{C}^n$, we denote by $\mathbb{C}^d \otimes A$ the arrangement of subspaces $\mathbb{C}^d \otimes A_k \subseteq \mathbb{C}^d \otimes \mathbb{C}^n = \mathbb{C}^{dn}$.

(a) Is it true more generally that for all d, the ideal of $\mathbb{C}^d \otimes V'$ is a free $\mathbb{C}[\mathbf{x}]$-module, where $\mathbf{x}$ is one of the d sets of n coordinates on $\mathbb{C}^{dn}$?

(b) The hyperplane arrangement V' is the Coxeter arrangement of type A_{n-1}. What if we consider instead the Coxeter arrangements of other types?

(c) Are there general criteria for a hyperplane arrangement $A \subseteq \mathbb{C}^n$ to have the property that the ideal I_d of $\mathbb{C}^d \otimes A$ is a free module over the coordinate ring of $\mathbb{C}^n$, for all d?

(d) Exercise: show that a hyperplane arrangement with the property in (c) must be *free* in the sense used in the theory of hyperplane arrangements [16]. Freeness as a hyperplane arrangement is not sufficient for (c), however.

2. A ring of invariants

As in Lecture 1, let $E = \mathbb{C}^{2n}$ be the space of n-tuples $(P_1, \ldots, P_n)$ of points in the plane. The action of the symmetric group S_n on E has already made an appearance in our study of the ideal of the locus where points coincide. In this lecture we will discuss some other features of this action. We will begin with a review of some general theory of invariants and coinvariants of linear representations of finite groups, then turn to particulars of the representation of S_n on E.

For the moment, we consider an arbitrary finite group G, acting linearly on a finite-dimensional vector space $V = k^n$. Our only assumption will be that $\operatorname{char} k$ does not divide $|G|$. Then all finite-dimensional representations of G are completely reducible, *i.e.*, they are direct sums of irreducible representations. In particular, each homogeneous component of the ring $k[V]$ of polynomial functions on V is completely reducible. Of special interest is the subring of *invariants* $k[V]^G$. It follows from complete reducibility that $k[V]^G$ is a direct summand of $k[V]$ as a G-module, and also as a $k[V]^G$-module. The projection of $k[V]$ on its summand $k[V]^G$ is given explicitly by the *Reynolds operator*

$$\mathbf{R}f = \frac{1}{|G|} \sum_{g \in G} g \cdot f,$$

which will be important in what follows.

A second ring associated with the action of G on V is the ring of *coinvariants* , defined as

$$R_G = k[V]/I_G,$$

where $I_G = k[V] \cdot (k[V]^G_+)$ is the ideal generated by all homogeneous invariants of positive degree. Geometrically, these rings have the following interpretation (at least when k is algebraically closed). The space of G-orbits V/G has a natural structure of algebraic variety, with regular functions given by the G-invariant functions on V. Thus its coordinate ring is the ring of invariants:

$$k[V]^G = k[V/G].$$

The homogeneous maximal ideal $k[V]^G_+$ in $k[V]^G$ is the ideal of the origin $0 \in V/G$ (the G-orbit consisting only of the origin in V). Then the scheme-theoretic fiber $\pi^{-1}(0)$ of the natural projection

$$\pi\colon V \to V/G$$

has coordinate ring equal to the ring of coinvariants,

$$R_G = k[\pi^{-1}(0)].$$

The two constructions are related by a famous lemma of Hilbert.

Lemma 2.1 (Hilbert)**.** *Homogeneous invariants $f_1, \ldots, f_r$ of positive degree generate $k[V]^G$ as a k-algebra if and only if they generate I_G as an ideal.*

Proof. If $k[V]^G = k[f_1, \ldots, f_r]$, then every homogeneous invariant of positive degree is a polynomial without constant term in the f_i's. This shows that $I_G \subseteq (f_1, \ldots, f_r)$, and the reverse inclusion is trivial.

For the converse, suppose to the contrary that $I_G = (f_1, \ldots, f_r)$ but $k[V]^G \neq k[f_1, \ldots, f_r]$. Let h be a homogeneous invariant of minimal degree, say d, not contained in $k[f_1, \ldots, f_r]$. Certainly $d > 0$, so $h \in I_G$, and we can write

$$h = \sum_i a_i f_i,$$

where we can assume without loss of generality that a_i is homogeneous of degree $d - \deg f_i$. Applying the Reynolds operator to both sides gives

$$h = \sum_i (\mathbf{R}a_i) f_i.$$

But each $\mathbf{R}a_i$ is a homogeneous invariant of degree $< d$, hence belongs to $k[f_1, \ldots, f_r]$. This contradicts the assumption $h \notin k[f_1, \ldots, f_r]$. $\square$

It is natural to ask for a bound on the degrees of a minimal set of homogeneous generators for $k[V]^G$, or equivalently for I_G. To give precise bounds for particular G and V is in general a difficult problem. One has the following global bound, which was proved by Noether in characteristic 0.

Theorem 2.1. *The ring of invariants $k[V]^G$ is generated by homogeneous elements of degree at most $|G|$.*

Let us pause to discuss a more modern proof of this theorem, based on a beautiful lemma of Harm Derksen. To state the lemma we need some additional notation. Let $x_1, \ldots, x_n$ be a basis of coordinates on V, so $k[V] = k[\mathbf{x}]$. We introduce a second copy of V, with coordinates $y_1, \ldots, y_n$. Then the coordinate ring $k[V \times V]$ is identified with the polynomial ring $k[\mathbf{x}, \mathbf{y}]$. For each $g \in G$, let

$$J_g = (x_i - gy_i : 1 \leq i \leq n) \subseteq k[\mathbf{x}, \mathbf{y}] \tag{1}$$

be the ideal of the subspace $W_g = \{(v, gv) : v \in V\} \subseteq V \times V$.

Lemma 2.2 (Derksen [3])**.** *Let $J = \bigcap_{g \in G} J_g$, with J_g as above. Then $k[\mathbf{x}] \cap (J + (\mathbf{y})) = I_G$.*

Proof. If $f(\mathbf{x})$ is a homogeneous invariant of positive degree, then $f(\mathbf{y}) \in (\mathbf{y})$, and $f(\mathbf{x}) - f(\mathbf{y}) \in J$, since $f(\mathbf{x}) - f(\mathbf{y})$ vanishes on setting $\mathbf{y} = g\mathbf{x}$ for any $g \in G$. This shows $I_G \subseteq k[\mathbf{x}] \cap (J + (\mathbf{y}))$.

For the reverse inclusion, suppose $f(\mathbf{x}) \in J + (\mathbf{y})$, so

$$f(\mathbf{x}) = \sum_i a_i(\mathbf{x}) b_i(\mathbf{y}) + p(\mathbf{x}, \mathbf{y}), \tag{2}$$

where $p(\mathbf{x}, \mathbf{y}) \in J$ and we can assume $b_i(\mathbf{y})$ homogeneous of positive degree. Let $\mathbf{R_y}$ be the Reynolds operator for the action of G on the $\mathbf{y}$ variables only. The ideal J is invariant for this action, so $\mathbf{R_y} J \subseteq J$. Hence, applying $\mathbf{R_y}$ to both sides in (2) yields

$$f(\mathbf{x}) = \sum_i a_i(\mathbf{x}) \mathbf{R_y} b_i(\mathbf{y}) + q(\mathbf{x}, \mathbf{y})$$

with $q(\mathbf{x}, \mathbf{y}) \in J$. In particular, $q(\mathbf{x}, \mathbf{x}) = 0$. Substituting $\mathbf{y} \mapsto \mathbf{x}$ on both sides now exhibits f as an element of I_G. $\square$

We remark that J is the ideal of the subspace arrangement $W = \bigcup_g W_g$, which we will call *Derksen's arrangement.* It is the arrangement in $V \times V$ whose projection on the first factor V has finite fiber over each point v, identified set-theoretically with the orbit Gv (by projecting on the second factor). Derksen's Lemma says that the scheme-theoretic 0-fiber of the projection $W \to V$ is isomorphic to the scheme-theoretic 0-fiber of $\pi \colon V \to V/G$, that is, to $\operatorname{Spec} R_G$.

Derksen's lemma has the following easy analog for the product ideal.

Lemma 2.3. *Let $d = |G|$ and let $J' = \prod_g J_g$, with J_g as in* (1)*. Then $k[\mathbf{x}] \cap (J' + (\mathbf{y})) = (\mathbf{x})^d$.*

Proof. Note that $k[\mathbf{x}] \cap (J' + (\mathbf{y}))$ is the set of polynomials $\{f(\mathbf{x}, 0) : f(\mathbf{x}, \mathbf{y}) \in J'\}$ (this holds with any ideal in the role of J'). Since J' is generated by products of d linear forms, this shows $k[\mathbf{x}] \cap (J' + (\mathbf{y})) \subseteq (\mathbf{x})^d$. For the reverse inclusion, fix any monomial $\mathbf{x}^\alpha$ of degree d, and write it as a product of individual variables

$$\mathbf{x}^\alpha = x_{i_1} x_{i_2} \cdots x_{i_d}.$$

Let $g_1, \ldots, g_d$ be an enumeration of all the elements of G, and consider the polynomial

$$f(\mathbf{x}, \mathbf{y}) = \prod_j (x_{i_j} - g_j y_{i_j}).$$

The j-th factor belongs to J_{g_j}, so $f(\mathbf{x}, \mathbf{y}) \in J'$, and clearly $f(\mathbf{x}, 0) = \mathbf{x}^\alpha$. $\square$

Now $J' \subseteq J$, so Lemmas 2.2 and 2.3 imply $(\mathbf{x})^d \subseteq I_G$. Hence I_G is generated by its homogeneous elements of degree at most d, proving Theorem 2.1. In fact, we have proved something stronger.

Corollary 2.1. *The ring of coinvariants R_G is zero in degrees $\geq |G|$.*

The degree bound in Theorem 2.1 is tight only when G is a cyclic group. For arbitrary G and V, rather little is known about how to describe $k[V]^G$ and R_G more fully. Of the two, the ring of invariants is better understood. In particular, we have the Eagon-Hochster theorem:

Theorem 2.2 (Eagon-Hochster [12])**.** *The ring of invariants $k[V]^G$ is Cohen-Macaulay.*

My hope in this lecture is to persuade you that $k[V]^G$ and R_G can have surprisingly rich structure for naturally occurring group representations, and that the problem of describing them is deserving of further study. We now turn to the particular case $G = S_n$, and fix $k = \mathbb{C}$. As we did in Lecture 1, let's warm up in the easier situation of n points on a line. This means we consider the representation of S_n on $V = \mathbb{C}^n$, permuting the coordinates $x_1, \ldots, x_n$. We make several observations.

(I) The ring of invariants $\mathbb{C}[\mathbf{x}]^{S_n}$ is the polynomial ring $\mathbb{C}[e_1, \ldots, e_n]$ freely generated by the elementary symmetric functions $e_j = e_j(\mathbf{x})$. This is the *fundamental theorem of symmetric functions.* Its Hilbert series is

$$\frac{1}{(1-q)(1-q^2)\cdots(1-q^n)},$$

which can also be written as

$$h_n(1, q, q^2, \ldots), \tag{3}$$

where $h_n(z_1, z_2, \ldots)$ denotes the complete homogeneous symmetric function of degree n in infinitely many variables.

(II) By Lemma 2.1, $I_{S_n}(\mathbf{x}) = (e_1, \ldots, e_n)$. In particular it is a complete intersection ideal. Hence $R_{S_n}(\mathbf{x})$ is an Artinian local complete intersection ring. It can be described quite precisely. For example, since $\deg e_j = j$, the Hilbert series of $R_{S_n}(\mathbf{x})$ is given by the q-analog of $n!$, namely,

$$[n]_q! = \frac{(1-q)(1-q^2)\cdots(1-q^n)}{(1-q)^n} = [n]_q[n-1]_q \cdots [1]_q,$$

where $[k]_q = 1 + q + \cdots + q^{k-1}$. Hence

$$\dim_{\mathbb{C}} R_{S_n}(\mathbf{x}) = n!.$$

(III) Since $\mathbb{C}[\mathbf{x}]$ is a graded Cohen-Macaulay ring, and $e_1, \ldots, e_n$ is a homogeneous system of parameters, it follows that $\mathbb{C}[\mathbf{x}]$ is a free $\mathbb{C}[\mathbf{x}]^{S_n}$-module, with basis given by any $n!$ homogeneous elements forming a vector space basis of $R_{S_n}(\mathbf{x})$. It is easy using standard techniques to determine the character of the polynomial ring $\mathbb{C}[\mathbf{x}]$ as a graded S_n representation, and from this to determine the corresponding graded character of $R_{S_n}(\mathbf{x})$. The answer can be expressed as follows. The irreducible representations V_λ of S_n are indexed by partitions λ of the integer n. For each λ, define

$$f_\lambda(q) = (1-q)(1-q^2)\cdots(1-q^n)s_\lambda(1, q, q^2, \ldots),$$

where $s_\lambda(z_1, z_2, \ldots)$ is the *Schur symmetric function* indexed by λ in infinitely many variables. Then $f_\lambda(q)$ is a polynomial with positive integer coefficients, and $f_\lambda(1)$ is the number of standard Young tableau of shape λ, which is also equal to $\dim V_\lambda$. Let $m(V_\lambda, R_{S_n}(\mathbf{x})_d)$ denote the multiplicity of V_λ in a decomposition of the degree d homogeneous component $R_{S_n}(\mathbf{x})_d$ as a direct sum of irreducible representations of S_n. Then these multiplicities are given by

$$\sum_d m(V_\lambda, R_{S_n}(\mathbf{x})_d)q^d = f_\lambda(q).$$

This is a very precise answer, as $f_\lambda(q)$ has an explicit combinatorial description, and it is possible to produce a correspondingly explicit decomposition of $R_{S_n}(\mathbf{x})$ into irreducibles with generators indexed by suitable combinatorial data. It would take us too far afield to go into this here, but see *e.g.* [1] for more details. We only note that ignoring the grading gives

$$m(V_\lambda, R_{S_n}(\mathbf{x})) = f_\lambda(1) = \dim V_\lambda,$$

so $R_{S_n}(\mathbf{x})$ is a graded version of the regular representation of S_n (the representation of S_n by left multiplication on its group algebra $\mathbb{C}S_n$).

(IV) Derksen's arrangement W is a complete intersection in $\mathbb{C}^n \times \mathbb{C}^n$, defined by the ideal $(e_i(\mathbf{x}) - e_i(\mathbf{y}) : 1 \le i \le n)$. In particular, its coordinate ring $\mathbb{C}[W]$ is Cohen-Macaulay, and since $(\mathbf{y})$ is obviously a system of parameters, $\mathbb{C}[W]$ is a free $\mathbb{C}[\mathbf{y}]$-module.

The above special properties of the invariants and coinvariants of S_n on $\mathbb{C}^n$ are consequences of the fact that S_n acts on $\mathbb{C}^n$ as a group generated by *complex reflections*: linear transformations that fix a hyperplane pointwise. In the case of S_n, the reflections are the transpositions (i, j), which fix every vector on the hyperplane $x_i = x_j$. By general results of Steinberg, Chevalley, Shepard and Todd, every complex reflection group G has $k[V]^G$ a polynomial ring, I_G a complete intersection ideal, and R_G isomorphic to a graded version of the regular representation of G. Moreover, each of these properties holds *only* for complex reflection groups, and there is a complete classification of such groups [2, 17, 19, 20].

Finally we come to the situation that we set out to study in the first place, namely, the action of S_n on $E = \mathbb{C}^{2n}$. Note that this is *not* an action generated by complex reflections. In fact, every element of S_n acts on E with determinant 1, while a nontrivial complex reflection has determinant $\neq 1$. The determinant 1 property does have a useful consequence, however, owing to the following refinement of the Eagon-Hochster theorem.

Theorem 2.3 (Watanabe [21]). *The canonical module of $k[V]^G$ is the module of covariants $k[V]^\epsilon$, where ϵ denotes the determinant character $\epsilon(g) = \det_V(g)$. In particular if G acts on V by endomorphisms with determinant 1, then $k[V]^G$ is Gorenstein.*

There is an old theorem of Weyl giving a (minimal) generating set for the ring of invariants $\mathbb{C}[E]^{S_n}$.

Theorem 2.4 (Weyl [22]). *The ring of invariants $\mathbb{C}[\mathbf{x}, \mathbf{y}]^{S_n}$ is generated by the polarized power-sums*

$$p_{r,s} = \sum_{i=1}^n x_i^r y_i^s, \quad 1 \leq r + s \leq n.$$

The analogous theorem holds in d sets of variables. Note that the actual degree bound on the generators in this case, namely n, is very much smaller than the order of the group!

It turns out to be almost as easy to determine the Hilbert series of $\mathbb{C}[E]^{S_n} = \mathbb{C}[\mathbf{x}, \mathbf{y}]^{S_n}$ as it is for $\mathbb{C}[\mathbf{x}]^{S_n}$. In fact, we can compute its Hilbert series as a *doubly* graded ring, by degree in the $\mathbf{x}$ and $\mathbf{y}$ variables separately. It is given by the following analog of (3).

$$\sum_{r,s} \dim(\mathbb{C}[\mathbf{x}, \mathbf{y}]^{S_n})_{r,s} q^r t^s = h_n(1, q, q^2, \ldots, t, qt, q^2t, \ldots, t^2, qt^2, q^2t^2, \ldots).$$

There is a similar formula for the Hilbert series of the ring of invariants $\mathbb{C}[\mathbf{x}, \mathbf{y}, \ldots, \mathbf{z}]^{S_n}$ in d sets of variables, as an $\mathbb{N}^d$-graded ring. So we have good analogs of observation (I) for the invariants of n points in the plane or more generally in $\mathbb{C}^d$.

The interesting surprises appear when we turn to analogs of observations (II) and (III), on the ring of coinvariants. We now drop the modifier $\mathbf{x}$ from the notation and write simply R_{S_n} for the ring of coinvariants $\mathbb{C}[E]/I_{S_n}$.

Around 1991, Garsia and I were led to investigate R_{S_n} because of its connection with a problem on Macdonald polynomials. For small values of n, we used a computer to determine its dimension and S_n character in each (double) degree. Immediately we noticed some amazing coincidences between our data and well-known combinatorial numbers. We publicized our early findings informally, leading various other people, especially Ira Gessel and Richard Stanley, to discover still more such coincidences. Eventually I published a compilation of these discoveries, all of which were then just conjectures, in [8].

Later, Procesi pointed out to us the fact that the Hilbert scheme $\mathrm{Hilb}^n(\mathbb{C}^2)$ provides a nice resolution of singularities of E/S_n, as discussed in Lecture 1, and observed how this should be useful in attacking the conjectures. Assuming the validity of some geometric hypotheses that would make Procesi's method work, I was soon able to find a formula for the doubly graded character of R_{S_n} in terms of Macdonald polynomials. Garsia and I then proved that the earlier combinatorial conjectures would all follow from the master formula. Recently [10, 11] I succeeded in proving the needed geometric hypotheses, which by this time were the only missing pieces remaining.

There is not room here to discuss in full the geometry of the Hilbert scheme and the combinatorial theory of Macdonald polynomials. I will only summarize some of the facts about R_{S_n} that have been established using these methods.

Theorem 2.5. *The coinvariant ring R_{S_n} for S_n acting on $\mathbb{C}^{2n}$ has length*

$$\dim_{\mathbb{C}} R_{S_n} = (n+1)^{n-1}.$$

Ignoring the grading, the representation of S_n on R_{S_n} is isomorphic to the sign representation tensored by the obvious permutation representation of S_n on the finite Abelian group $Q/(n+1)Q$, where $Q = \mathbb{Z}^n/\mathbb{Z}\cdot(1,1,\dots,1)$. Retaining the grading by $\mathbf{x}$ degree only, one has the Hilbert series

$$\sum_d \dim(R_{S_n})_{(d,-)} q^d = F_n(q),$$

where $F_n(q)$ is the generating function enumerating rooted forests on the vertex set $\{1,\dots,n\}$ by number of inversions, or equivalently, enumerating parking functions on n cars by weight (see [8] *for definitions and details).*

Here we should mention the connection between R_{S_n} and the ideal I studied in the previous lecture, given by the following proposition, which is easy to prove.

Proposition 2.1. *Homogeneous S_n-alternating polynomials $f_1,\dots,f_r \in \mathbb{C}[E]$ minimally generate the ideal I in Theorem 1.1 if and only if their images modulo I_{S_n} form a basis of the space of S_n-alternating elements of R_{S_n}.*

In particular, Theorem 1.2 is really a statement about the character of R_{S_n}. Like Theorem 2.5, it follows from the master formula for the character of R_{S_n} given by the geometry of the Hilbert scheme.

I think it should be possible to obtain at least some of the above results on R_{S_n}, and maybe some new ones, or analogous ones for other groups, without invoking Hilbert scheme and Macdonald polynomial machinery. In particular, it seems to me that there is room for purely algebraic approaches. One encouraging sign is a recent preprint by Iain Gordon [7], where he obtains an extension of the $(n+1)^{n-1}$ theorem, in a slightly weakened form, to any Weyl group. This is especially notable in that for the Weyl groups of type G_2, F_4, and D_n, it is known that there is no suitable geometric analog of the Hilbert scheme.

To close, let me suggest some open problems that might repay further study.

Problem 2.1. Can one determine the dimension and Hilbert series of R_{S_n} inductively by fitting it into an exact complex with other terms built out of the coinvariant rings R_{S_k} for $k < n$? A specific conjecture along these lines in [8] remains open.

Problem 2.2. Describe the minimal free resolution of $\mathbb{C}[\mathbf{x},\mathbf{y}]^{S_n}$ with respect to the minimal generators given by Theorem 2.4. One could also consider this problem in d sets of variables, although $d = 2$ may be nicer, since the ring of invariants is Gorenstein. I don't think a good description is known even for the first syzygies.

Problem 2.3. Let W be the Derksen arrangement for S_n acting on E, say with coordinates $\mathbf{x},\mathbf{y},\mathbf{x}',\mathbf{y}'$ on $E\times E$. Is $\mathbb{C}[W]$ a free $\mathbb{C}[\mathbf{y}]$-module? What about the same problem for fiber powers $W\times_E W\times_E\cdots\times_E W$? An affirmative answer would be equivalent to sheaf cohomology vanishing properties for certain vector bundles on the Hilbert scheme. Are there similar results in d sets of variables, with E replaced by $\mathbb{C}^{dn}$? Are there similar results for other Weyl groups G, with E the direct sum of two (or more) copies of the defining representation?

3. A remarkable Gröbner basis

This lecture will be an overview of some results by Jeremy Martin. I'll give less detailed notes here than for the previous two lectures, referring you to [14, 15] for the full story. Martin's results concern the situation where we introduce not only the points $P_1,\dots,P_n \in \mathbb{C}^2$ but also lines L_{ij} connecting them in pairs. That is, L_{ij} is a line passing through P_i and P_j. When P_i and P_j are distinct, of course, L_{ij} is determined. When they coincide, the line L_{ij} can pass through them with any slope, introducing an extra degree of freedom.

The locus of all configurations of points and lines as above is the *picture space* $\mathcal{X}(K_n)$. One thinks of these configurations as plane "pictures" of the complete graph K_n on n vertices, with edges represented by lines. To specify a picture, we need to give the coordinates $x_1, y_1, \ldots, x_n, y_n$ of the n points, together with the slopes m_{ij} of the $\binom{n}{2}$ lines. In principle, the slopes m_{ij} lie on a projective line $\mathbb{P}^1$. However, we will be interested only in local questions, so we will consider the affine open set in $\mathcal{X}(K_n)$ where $m_{ij} \neq \infty$. It is the locus cut out (set-theoretically, at least) by the equations

$$y_j - y_i = m_{ij}(x_j - x_i) \quad \text{for all } i,\ j.$$

Now $\mathcal{X}(K_n)$ is in general not irreducible. For example, $\mathcal{X}(K_4)$ has two irreducible components, each of dimension 8: the generic component—the closure of the locus where all the points are distinct, and the lines are determined—and another component where all four points coincide, and the six lines have arbitrary slopes. Martin has given a complete combinatorial description of the component structure of $\mathcal{X}(G)$ for any graph G, which we won't discuss in this lecture. Instead we will concentrate on his results describing the generic component $\mathcal{V}(K_n)$ of $\mathcal{X}(K_n)$, which we call the *graph variety*. Note that $\mathcal{V}(K_n)$ is, essentially by definition, the simultaneous blowup of $\mathbb{C}^{2n}$ along the coincidence subspaces $V_{ij} = V(x_i - x_j, y_i - y_j)$ discussed in Lecture 1. This is, however, quite a different thing from the blowup along the union of these subspaces, which is the variety X from Lecture 1.

Proposition 3.1. *The graph variety $\mathcal{V}(K_n)$ is cut out set-theoretically in $\mathcal{X}(K_n)$ by the equations in the variables m_{ij} giving the algebraic relations among the slopes that hold when the points P_i are in general position (i.e., no two coincide).*

In view of this proposition, the key issue is to understand the ideal of relations among the slope variables m_{ij}. Although the problem of describing all relations among the slopes of the $\binom{n}{2}$ lines connecting n points in general position in the plane is very classical in nature, there seems to have been almost no earlier work on it. In more geometric terms, the projection of the graph variety $\mathcal{V}(K_n)$ on the slope coordinates is a variety $\mathcal{S}(K_n)$, called the *slope variety*, whose ideal $I(\mathcal{S}_n)$ is the ideal of all algebraic dependencies among the rational functions $(y_j - y_i)/(x_j - x_i)$. We want to describe this ideal.

The first result tells us which subsets of the variables m_{ij} are minimally algebraically dependent—*i.e.*, are circuits of the algebraic dependence matroid of the quantities $(y_j - y_i)/(x_j - x_i)$.

Theorem 3.1. *The variables m_{ij} corresponding to a set of edges $E \subseteq E(K_n)$ are minimally algebraically dependent if and only if*

(1) $|E| = 2|V(E)| - 2$, *and*
(2) $|F| \leq 2|V(F)| - 3$ *for all* $\emptyset \subset F \subset E$,

where $V(E)$ denotes the set of all endpoints of the edges in E.

This result is particularly interesting because there is another well-known algebraic dependence matroid whose characterization (due to Laman) is exactly the same: that is the *rigidity* matroid of algebraic dependencies among the squared-lengths $(x_i - x_j)^2 + (y_i - y_j)^2$ of the line segments connecting the points (for points with real coordinates).

The next result, which is a key one, is an explicit description of the polynomial giving the algebraic dependence among the slopes in a rigidity-circuit. First one shows that every rigidity circuit is the edge-disjoint union of two spanning trees on a common set of vertices. Conversely, every minimal such union is a rigidity circuit.

Now consider any two disjoint spanning trees S and T on the same vertex set, and fix an arbitrary orientation of the edges of each tree. For each edge $f \in S$, there are unique coefficients $c_{ef} \in \{0, \pm 1\}$ such that

$$f - \sum_{e \in T} c_{ef} e \tag{4}$$

is a directed cycle. Let us abbreviate $x_e = x_j - x_i$, $y_e = y_j - y_i$ for a directed edge $e = (i, j)$. Then for a cycle as in (4), we have

$$y_f = \sum_{e \in T} c_{ef} y_e$$
$$x_f = \sum_{e \in T} c_{ef} x_e.$$

Now since $y_f = m_f x_f$ and $y_e = m_e x_e$, we have an identity between two expressions for y_f

$$\sum_{e \in T} c_{ef} m_e x_e = m_f \sum_{e \in T} c_{ef} x_e,$$

or

$$\sum_{e \in T} c_{ef} (m_e - m_f) x_e = 0.$$

This of course is not yet an equation among the variables m_{ij}. However, if S and T are trees on $d+1$ vertices, then we have d such equations, one for each f, which we can regard as linear equations in the d "unknowns" x_e. When the points are in general position, they obviously have a non-zero solution, since the x_e's do not vanish. Hence the $d \times d$ matrix

$$M_{ST} = [c_{ef}(m_e - m_f)]_{f \in S, e \in T}$$

must be singular. Its determinant

$$D_{S \cup T}(m)$$

is a polynomial of degree d in the slope variables m_e for e in our rigidity circuit $S \cup T$, and this polynomial belongs to $I(\mathcal{S}_n)$.

Theorem 3.2. *The determinants $D_{S \cup T}$ enjoy the following properties:*

1. *Up to sign, $D_{S \cup T}$ depends only on the union $S \cup T$, and not on the decomposition into trees S, T.*
2. *Every term of $D_{S \cup T}$ is a square-free monomial $\pm \prod_{e \in S'} m_e$, where S' is a spanning tree in $S \cup T$ whose complement is also a spanning tree.*
3. *$D_{S \cup T}$ is irreducible if and only if $S \cup T$ is a rigidity circuit, and in that case it generates the principal ideal of algebraic dependencies among the slope variables m_e for $e \in S \cup T$.*

One particularly simple class of rigidity circuits consists of the *wheels*. A wheel is a graph consisting of a cycle (the *rim*) and one additional vertex (the *hub*) with edges to all the rim vertices (the *spokes*). With this terminology established, we can state Martin's main theorem.

Theorem 3.3. *The polynomials D_W for W a wheel generate $I(\mathcal{S}_n)$. In fact, they form a Gröbner basis for this ideal, with respect to the graded lexicographic term order on the obvious lexicographic ordering of the variables m_{ij}. Moreover, the initial ideal* $\mathrm{in}(I(\mathcal{S}_n))$*, and hence also $I(\mathcal{S}_n)$ itself, is Cohen-Macaulay, of dimension $2n-3$ and degree*

$$M_{2n-4} = (2n-5)(2n-7) \cdots 3 \cdot 1.$$

Let us say just a few words about the proof of this theorem, which involves a beautiful interplay of commutative algebra and combinatorics. By Theorem 3.2, the initial term of D_W is a square-free "tree monomial" $m_T = \prod_{e \in T} m_e$, for some tree. Martin proves first that for wheels, the initial terms belong, not to arbitrary trees, but to trees which are *paths*, of the following special form.

Definition 3.1. A *Martin path* in the graph K_n on vertices $\{1, \ldots, n\}$ is a path $Q = (x, v, \ldots, w, y)$ such that (1) x and y are the two largest vertices of Q, and (2) assuming without loss of generality that $x < y$, then $v < w$.

Now the initial ideal $\operatorname{in}(D_W)$ of the ideal generated by wheel polynomials D_W is the square-free monomial ideal generated by monomials M_Q for Q a Martin path. Hence

$$R_\Delta = \mathbb{C}[\mathbf{m}]/\operatorname{in}(D_W)$$

is the *Stanley-Reisner ring* of the simplicial complex Δ on the edge set of K_n, whose faces are those subgraphs $H \subseteq K_n$ that contain no Martin path. Martin proves next that this simplicial complex has the most optimal properties one could desire.

Proposition 3.2. *Every maximal subgraph of K_n containing no Martin path—that is, every facet of the simplicial complex Δ—has $2n-3$ edges. The number of these facets is M_{2n-4}. Moreover the complex Δ is shellable.*

Shellability is a combinatorial property of a simplicial complex which implies in particular that it is *Cohen-Macaulay, i.e.*, the link of each face has only one non-zero reduced homology group. By a theorem of Hochster (see [18]), the latter property is equivalent to the Stanley-Reisner ring being Cohen-Macaulay. So Proposition 3.2 shows that the ideal

$$J = \operatorname{in}(D_W : \text{all wheels } W)$$

is Cohen-Macaulay, of dimension $2n-3$ and degree M_{2n-4}.

Finally, Martin uses a geometric argument to give a lower bound on the degree of the slope variety $\mathcal{S}_n$.

Proposition 3.3. *The slope variety $\mathcal{S}_n$ has dimension $2n-3$ and degree at least M_{2n-4}.*

Let us see where the above results leave us. We have two ideals, $J = \operatorname{in}(D_W)$, and $I = \operatorname{in} I(\mathcal{S}_n)$, and from the facts established so far we have:

(i) $J \subseteq I$,
(ii) J is unmixed (since it is Cohen-Macaulay),
(iii) $\dim J = \dim I$,
(iv) $\deg J \leq \deg I$.

Together, these imply $J = I$, and Theorem 3.3 follows.

To close, I'll mention a striking combinatorial fact, which Martin left as a conjecture at the end of his thesis, but has since proved. The number M_{2n-4} is the number of *matchings* on $2n-4$ vertices, that is, graphs in which every vertex is the endpoint of exactly one edge. The Hilbert series of the slope variety may be written

$$\frac{h_n(q)}{(1-q)^{2n-3}},$$

where $h_n(q)$ is a polynomial with positive integer coefficients (because the ring is Cohen-Macaulay) and $h_n(1) = M_{2n-4}$. Hence

$$h_n(q) = a_0 + a_1 q + a_2 q_2 + \cdots$$

is a *q-analog* of the number of matchings M_{2n-4}. It turns out that it coincides with a combinatorial q-analog studied long ago by Kreweras and Poupard [13].

Theorem 3.4. *The coefficient a_l in the polynomial $h_n(q)$ is the number of matchings on the integers $\{1, \ldots, 2n-4\}$ with l* long edges*, where an edge i, j is* long *if $|i-j| \neq 1$.*

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