DENS, NESTS AND THE LOEHR-WARRINGTON CONJECTURE

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Abstract. We prove and extend the longest-standing conjecture in 'q, t-Catalan combinatorics,' namely, the combinatorial formula for $\nabla^m s_\mu$ conjectured by Loehr and Warrington, where s_μ is a Schur function and ∇ is an eigenoperator of Macdonald polynomials.

Our approach is to establish a stronger identity of infinite series of GL_ℓ characters involving *Schur Catalanimals*; these were recently shown by the authors to represent Schur functions $s_\mu[-MX^{m,n}]$ in subalgebras $\Lambda(X^{m,n}) \subset \mathcal{E}$ isomorphic to the algebra of symmetric functions Λ over $\mathbb{Q}(q,t)$, where $\mathcal{E}$ is the elliptic Hall algebra of Burban and Schiffmann. We establish a combinatorial formula for Schur Catalanimals as weighted sums of LLT polynomials, with terms indexed by configurations of nested lattice paths called *nests*, having endpoints and bounding constraints controlled by data called a *den*.

The special case for $\Lambda(X^{m,1})$ proves the Loehr-Warrington conjecture, giving $\nabla^m s_\mu$ as a weighted sum of LLT polynomials indexed by systems of nested Dyck paths. In general, for $\Lambda(X^{m,n})$ our formula implies a new (m,n) version of the Loehr-Warrington conjecture. In the case where each nest consists of a single lattice path, the nests in a den formula reduces to our previous shuffle theorem for paths under any line. Both this and the (m,n) Loehr-Warrington formula generalize the (km, kn) shuffle theorem proven by Carlsson and Mellit (for $n = 1$) and Mellit. Our formula here unifies these two generalizations.

1. Introduction

1.1. **Background.** In this paper we prove and extend the oldest unresolved conjecture in 'q, t-Catalan combinatorics,' namely, the combinatorial formula for $\nabla^m s_\mu$ conjectured by Loehr and Warrington [15], where s_μ is a Schur function and ∇ is the operator from [2] which is important in the theory of Macdonald polynomials. Like other results and conjectures in this area, beginning with the *shuffle theorem* conjectured by Haglund et. al. [13] and proven by Carlsson and Mellit [8], the Loehr-Warrington formula is expressed as a sum over Dyck paths (in this case, systems of nested Dyck paths) of LLT polynomials weighted by monomials in q and t.

Our main result, Theorem 3.5.1, is considerably more general than the Loehr-Warrington formula. We briefly describe some of its further consequences.

The simplest case of the Loehr-Warrington formula, when $s_\mu = e_k$ is an elementary symmetric function, reduces to the original shuffle theorem. The latter is the $n = 1$ case of an extended (km, kn) *shuffle theorem* conjectured by Bergeron et. al. [3] and proven by Mellit

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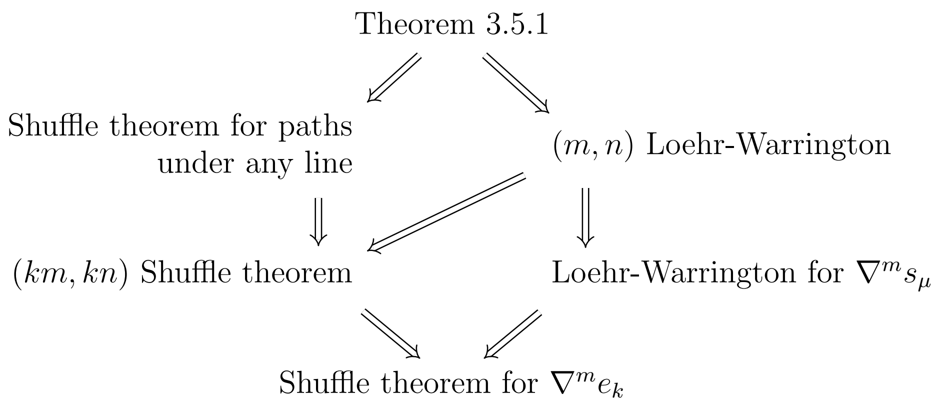
[18]. A new consequence of our main result here, not previously formulated even as a conjecture, is a corresponding (m, n) extension of the Loehr-Warrington formula, which reduces to the (km, kn) shuffle theorem when $s_\mu = e_k$.

Another generalization of the (km, kn) shuffle theorem is given by our shuffle theorem for paths under any line [6]. This too is a consequence of our main theorem here. Thus our theorem unifies a number of previous results and conjectures, as summarized in the following diagram.



1.2. **Overview.** In [4], we introduced raising operator series $H(\mathbf{z}; q, t)$ called *Catalanimals*; among them, we constructed examples for which the polynomial truncation $H(\mathbf{z}; q, t)_{\mathrm{pol}}$ is equal, up to an explicit factor of the form $\pm q^r t^s$, to $\omega \, \nabla^m s_\mu(\mathbf{z})$, where ω is the standard involution on symmetric functions.

Our main result, Theorem 3.5.1, gives a combinatorially defined expansion

$$H(\mathbf{z}; q, t)_{\mathrm{pol}} = \sum_{\pi} t^{a(\pi)} q^{\mathrm{dinv}_p(\pi)} \mathcal{G}_{\boldsymbol{\nu}(\pi)}(\mathbf{z}; q^{-1}), \tag{1}$$

in terms of LLT polynomials $\mathcal{G}_{\boldsymbol{\nu}}(\mathbf{z}; q)$, for a special class of Catalanimals $H(\mathbf{z}; q, t)$ including those for which $H(\mathbf{z}; q, t)_{\mathrm{pol}} = \pm q^r t^s \, \omega \, \nabla^m s_\mu(\mathbf{z})$, as just discussed.

The terms on the right hand side of (1) are indexed by configurations of nested lattice paths $\pi = (\pi_1, \dots, \pi_r)$, called *nests*, with endpoints and bounding constraints controlled by combinatorial data called a *den*. The statistics $a(\pi)$, $\mathrm{dinv}_p(\pi)$ generalize the 'area' and 'dinv' statistics found in the shuffle theorem and its friends. We define these combinatorial notions in §3.

In the case where the left hand side of (1) becomes $\pm q^r t^s \, \omega \, \nabla^m s_\mu(\mathbf{z})$, formula (1) proves the Loehr-Warrington conjecture (see Theorem 4.2.2 and §4.3).

Formula (1) also applies to more general *Schur Catalanimals* $H(\mathbf{z}; q, t) = H^{m,n}_{(\mu^\circ)^m}$, which were shown in [4] to represent (again up to a factor $\pm q^r t^s$) Schur functions $s_\mu[-MX^{m,n}]$ in subalgebras $\Lambda(X^{m,n}) \subset \mathcal{E}$ isomorphic to the algebra of symmetric functions Λ over $\mathbb{Q}(q, t)$, where $\mathcal{E}$ is the elliptic Hall algebra of Burban and Schiffmann [7]—see §2 for details. Under the action of $\mathcal{E}$ on Λ constructed by Schiffmann and Vasserot [20], the Schur Catalanimal $H = H^{m,n}_{(\mu^\circ)^m}$ satisfies $H_{\mathrm{pol}} = \pm q^r t^s \omega(s_\mu[-MX^{m,n}] \cdot 1)(\mathbf{z})$. For $n = 1$, we have $s_\mu[-MX^{m,1}] \cdot 1 = \nabla^m s_\mu$. In this case, the Schur Catalanimals are the Catalanimals referred to above.

For general n, (1) yields a combinatorial formula for $s_\mu[-MX^{m,n}]\cdot 1$, made precise in Theorem 4.2.2, which can be naturally understood as an (m,n) extension of the Loehr-Warrington conjecture. For $\mu=(1^k)$, $s_\mu=e_k$ is the k-th elementary symmetric function, and our (m,n) Loehr-Warrington theorem reduces to the (km,kn) shuffle theorem of [3, 18], just as the original Loehr-Warrington conjecture for $\nabla^m s_\mu$ reduces to the classical shuffle theorem [8, 13] for $\nabla^m e_k$.

Finally, for dens such that each nest consists of a single lattice path, (1) reduces to our shuffle theorem for paths under any line [6, Theorem 5.5.1], which also generalizes the (km,kn) shuffle theorem. Thus, we have the diagram of implications in §1.1, above, with formula (1) at the top.

As with other instances of q,t-Catalan combinatorics, the left hand side of (1) is symmetric in q and t by construction; hence the right hand side shares this symmetry. No purely combinatorial explanation of this symmetry is yet known, even in the case of the classical shuffle theorem for ∇e_k.

In addition, since LLT polynomials are q-Schur positive [12]—i.e., their coefficients in terms of Schur functions belong to $\mathbb{N}[q]$—it follows that the Catalanimals to which (1) applies are (q,t)-Schur positive. The general question of which Catalanimals are (q,t)-Schur positive seems to be a difficult one. See [6, Conjecture 7.1.1] for one conjecture in this direction.

1.3. **Method and outline.** We prove our main theorem by a method parallel to the one we used to prove the shuffle theorem for paths under any line in [6].

We obtain the combinatorial formula in Theorem 3.5.1 by taking the polynomial part of an identity between infinite series of GL_l characters. The latter identity, equation (189) in Theorem 7.1.1, expands the full Catalanimal $H(\mathbf{z};q,t)$ associated with a den as an infinite sum of *LLT series* $\mathcal{L}^\sigma_{\mathbf{r},\beta/\alpha}(\mathbf{z};q)$ weighted by powers of t.

Upon taking the polynomial part, all but a finite number of the terms $t^a\mathcal{L}^\sigma_{\mathbf{r},\beta/\alpha}(\mathbf{z};q)$ in (189) vanish. The surviving terms are indexed by nests π in the given den, and have polynomial parts $t^a\mathcal{L}^\sigma_{\mathbf{r},\beta/\alpha}(\mathbf{z};q)_{\mathrm{pol}}=t^{a(\pi)}q^{\mathrm{dinv}_p(\pi)}\mathcal{G}_{\boldsymbol{\nu}(\pi)}(\mathbf{z};q^{-1})$, yielding (1).

Given a Levi subgroup $GL_{\mathbf{r}}=GL_{r_1}\times\cdots\times GL_{r_k}$ of GL_l, the LLT series $\mathcal{L}^\sigma_{\mathbf{r},\beta/\alpha}(\mathbf{z};q)$ in l variables $\mathbf{z}=z_1,\ldots,z_l$ (Definition 5.4.1) encapsulates the matrix coefficients of multiplication by arbitrary GL_l characters with respect to chosen basis elements $E^\sigma_{\mathbf{r},\alpha}(\mathbf{z};q)$, $E^\sigma_{\mathbf{r},\beta}(\mathbf{z};q)$ of the space of virtual $GL_{\mathbf{r}}$ characters. Here $E^\sigma_{\mathbf{r},\lambda}(\mathbf{z};q)$ denotes a (twisted) *semi-symmetric Hall-Littlewood polynomial* (Definition 5.2.1).

The orthogonality of semi-symmetric Hall-Littlewood polynomials (Proposition 5.3.1) leads to a formula for LLT series in terms of these polynomials (Proposition 5.4.3). Using this formula, the desired infinite series identity (189) follows from a Cauchy identity for semi-symmetric Hall-Littlewood polynomials, Theorem 6.1.3, along with an auxiliary identity, Proposition 6.2.4, that relates semi-symmetric Hall-Littlewood polynomials with different twists.

The steps just outlined parallel those in the proof of [6, Theorem 5.5.1], although many of the details are more intricate. Readers may find the simpler argument in [6], which covers the case $\mathbf{r}=(1^l)$, a helpful guide to the argument here.

Chief among the new intricacies is that the Cauchy identity for semi-symmetric Hall-Littlewood polynomials in Theorem 6.1.3 is more subtle than the one for non-symmetric

Hall-Littlewood polynomials in [6, Theorem 5.1.1]. The new Cauchy identity involves semi-symmetric Hall-Littlewood polynomials for two separate Levi subgroups $GL_{\mathbf{r}}$ and $GL_{\mathbf{s}}$, along with the choice of minimal dominant regular weights $\rho_{\mathbf{r}}, \rho_{\mathbf{s}}$ for each of them. These choices must satisfy certain compatibilities in order for the Cauchy identity to hold. Because of this greater complexity, we are not able to give a short proof of Theorem 6.1.3 like we did for [6, Theorem 5.1.1]. Instead, we devote most of §6.1 to developing a series of properties of semi-symmetric Hall-Littlewood polynomials, which we then use to prove Theorem 6.1.3.

2. Catalanimals and LLT polynomials

2.1. **Symmetric function conventions.** The (French style) diagram of a partition λ is the set of lattice points $\{(i,j) \mid 1 \leq j \leq \ell(\lambda),\ 1 \leq i \leq \lambda_j\}$, where $\ell(\lambda)$ is the length of λ. We often identify λ and its diagram with the set of lattice squares, or *boxes*, with northeast corner at a point $(i,j) \in \lambda$. A *skew diagram* is a difference $\nu = \lambda/\mu$ of partition diagrams $\mu \subseteq \lambda$, or any translate of such a diagram. This allows for skew diagrams $\nu = \beta/\alpha$ in which the x-coordinates α_i, β_i of the left and right ends of the rows may be negative.

The *content* of a box $a = (i,j)$ in row j, column i of a (skew) diagram is $c(a) = i - j$.

Let $\Lambda = \Lambda(X)$ be the algebra of symmetric functions in infinitely many variables $X = x_1, x_2, \ldots$, with coefficients in the field $\Bbbk = \mathbb{Q}(q,t)$. We follow Macdonald's notation [17] for the graded bases of Λ, the Hall inner product $\langle -,- \rangle$ in which the Schur functions s_λ are orthonormal, and the automorphism $\omega\colon \Lambda \to \Lambda$ such that $\omega s_\lambda = s_{\lambda^*}$, where λ^* denotes the transpose of a partition λ.

Given $f \in \Lambda$ and any expression A involving indeterminates, such as a polynomial, rational function, or formal series, the plethystic evaluation $f[A]$ is defined by writing f as a polynomial in the power-sums p_k and evaluating with $p_k \mapsto p_k[A]$, where $p_k[A]$ is the result of substituting a^k for every indeterminate a occurring in A. The variables q, t from our ground field $\Bbbk$ count as indeterminates.

By convention, the name of an alphabet $X = x_1, x_2, \ldots$ stands for $x_1 + x_2 + \cdots$ inside a plethystic evaluation. Then $f[X] = f[x_1 + x_2 + \cdots] = f(x_1, x_2, \ldots) = f(X)$. A special case of this convention that will arise often is the following. We fix

$$M = (1-q)(1-t) \tag{2}$$

here and throughout. Then the evaluation $f[-MX]$ is the image of $f(X)$ under the $\Bbbk$-algebra automorphism of Λ that sends p_k to $-(1-q^k)(1-t^k)p_k$.

We also allow plethystic evaluation term by term in a symmetric formal series, provided the result makes sense formally. In particular, the series

$$\Omega = \sum_{n=0}^{\infty} h_n = \exp \sum_{k=1}^{\infty} \frac{p_k}{k} \tag{3}$$

(with $h_0 = 1$) has the property

$$\Omega[x_1 + x_2 + \cdots - y_1 - y_2 - \cdots] = \frac{\prod_i (1-y_i)}{\prod_i (1-x_i)}. \tag{4}$$

The linear operator ∇ on Λ, introduced in [2], is defined to act diagonally in the basis of modified Macdonald polynomials $\tilde{H}_\mu(X;q,t)$ [11], with

$$\nabla \tilde{H}_\mu = t^{n(\mu)} q^{n(\mu^*)} \tilde{H}_\mu, \tag{5}$$

where $n(\mu) = \sum_i (i-1)\mu_i$.

2.2. **LLT polynomials.** We recall the definition and basic properties of LLT polynomials [14], using the 'attacking inversions' formulation from [13].

Let $\boldsymbol{\nu} = (\nu_{(1)}, \ldots, \nu_{(k)})$ be a tuple of skew diagrams. We consider the set of boxes in $\boldsymbol{\nu}$ to be the disjoint union of the sets of boxes in the $\nu_{(i)}$, and define the *adjusted content* of a box $a \in \nu_{(i)}$ to be $\tilde{c}(a) = c(a) + i\,\epsilon$, where ϵ is a fixed positive number such that $k\,\epsilon < 1$.

A *diagonal* in $\boldsymbol{\nu}$ is the set of boxes of a fixed adjusted content, that is, a diagonal of fixed content in one of the $\nu_{(i)}$.

The *reading order* on $\boldsymbol{\nu}$ is the total ordering $<$ on the boxes of $\boldsymbol{\nu}$ such that $a < b \Rightarrow \tilde{c}(a) \leq \tilde{c}(b)$ and boxes on each diagonal increase from southwest to northeast. An *attacking pair* is an ordered pair of boxes (a, b) in $\boldsymbol{\nu}$ such that $a < b$ in reading order and $0 < \tilde{c}(b) - \tilde{c}(a) < 1$.

A *semistandard tableau* on the tuple $\boldsymbol{\nu}$ is a map $T\colon \boldsymbol{\nu} \to \mathbb{Z}_+$ which restricts to a semistandard Young tableau on each component $\nu_{(i)}$. The set of these is denoted $\mathrm{SSYT}(\boldsymbol{\nu})$. An *attacking inversion* in T is an attacking pair (a, b) such that $T(a) > T(b)$. The number of attacking inversions in T is denoted $\mathrm{inv}(T)$.

Definition 2.2.1. The *LLT polynomial* indexed by a tuple of skew diagrams $\boldsymbol{\nu}$ is the generating function, which is known to be symmetric [13, 14],

$$\mathcal{G}_{\boldsymbol{\nu}}(X;q) = \sum_{T \in \mathrm{SSYT}(\boldsymbol{\nu})} q^{\mathrm{inv}(T)} \mathbf{x}^T, \tag{6}$$

where $\mathbf{x}^T = \prod_{a \in \boldsymbol{\nu}} x_{T(a)}$.

A similar formula expresses $\omega\, \mathcal{G}_{\boldsymbol{\nu}}(X;q)$ as a generating function for tableaux, as follows. Fix an ordered alphabet $\mathcal{A}_-$ of 'negative' letters $\overline{1} < \overline{2} < \cdots$ (since $\mathcal{G}_{\boldsymbol{\nu}}(X;q)$ is symmetric, the choice of ordering is arbitrary).

A *negative tableau* on $\boldsymbol{\nu}$ is a map $T\colon \boldsymbol{\nu} \to \mathcal{A}_-$ that is strictly increasing on rows and weakly increasing on columns. Let $\mathrm{SSYT}_-(\boldsymbol{\nu})$ be the set of these. An attacking inversion in a negative tableau is an attacking pair (a, b) such that $T(a) \geq T(b)$ (like for ordinary tableaux except that equal negative entries also count as inversions). The number of attacking inversions is again denoted $\mathrm{inv}(T)$.

Proposition 2.2.2 ([6, Corollary 4.1.3])**.** *Setting $x_{\overline{i}} = x_i$ for indices $\overline{i} \in \mathcal{A}_-$, we have*

$$\omega\, \mathcal{G}_{\boldsymbol{\nu}}(X;q) = \sum_{T \in \mathrm{SSYT}_-(\boldsymbol{\nu})} q^{\mathrm{inv}(T)} \mathbf{x}^T. \tag{7}$$

As in [6], formula (7) leads to the following corollary.

Corollary 2.2.3 ([6, Lemma 4.1.6])**.** *The LLT polynomial $\mathcal{G}_{\boldsymbol{\nu}}(X;q)$ is a linear combination of Schur functions s_λ with $\ell(\lambda) \leq l$, where $l = \sum_i \ell(\nu_{(i)})$ is the total number of rows in the skew diagrams $\nu_{(i)}$.*

We also need the invariance of $\mathcal{G}_{\boldsymbol{\nu}}(X;q)$ under shifted rotations of $\boldsymbol{\nu}$.

Proposition 2.2.4. *Let $\boldsymbol{\nu} = (\nu_{(1)}, \dots, \nu_{(k)})$ and set $\boldsymbol{\nu}' = (\nu^+_{(j+1)}, \dots, \nu^+_{(k)}, \nu_{(1)}, \dots, \nu_{(j)})$ for any $1 \le j < k$, where $\nu^+_{(i)}$ is a translation of $\nu_{(i)}$ such that the content of every box is increased by 1. Then $\mathcal{G}_{\boldsymbol{\nu}}(X;q) = \mathcal{G}_{\boldsymbol{\nu}'}(X;q)$.*

Proof. From the construction of $\boldsymbol{\nu}'$, there is a natural bijection between boxes of $\boldsymbol{\nu}$ and boxes of $\boldsymbol{\nu}'$ that preserves the reading order and the set of attacking pairs. This induces a bijection $\mathrm{SSYT}(\boldsymbol{\nu}) \cong \mathrm{SSYT}(\boldsymbol{\nu}')$ that preserves $\mathbf{x}^T$ and $\mathrm{inv}(T)$. □

2.3. **Catalanimals.** Let l be a positive integer and let $R_+ = R_+(GL_l) = \{\alpha_{ij} = \varepsilon_i - \varepsilon_j \mid i < j\}$ be the set of positive roots for GL_l, where ε_i denotes the i-th unit coordinate vector in $\mathbb{Z}^l$. Given subsets R_q, R_t, $R_{qt} \subseteq R_+$ and a weight $\lambda \in \mathbb{Z}^l$, we define the *Catalanimal* $H(R_q, R_t, R_{qt}, \lambda)$ of length l as in [4] to be the symmetric rational function in l variables $\mathbf{z} = z_1, \dots, z_l$ given by

$$H(R_q, R_t, R_{qt}, \lambda) \overset{\mathrm{def}}{=} \sum_{w \in S_l} w\left(\frac{\mathbf{z}^\lambda \prod_{\alpha \in R_{qt}} (1 - q\,t\,\mathbf{z}^\alpha)}{\prod_{\alpha \in R_+} (1 - \mathbf{z}^{-\alpha}) \prod_{\alpha \in R_q} (1 - q\,\mathbf{z}^\alpha) \prod_{\alpha \in R_t} (1 - t\,\mathbf{z}^\alpha)} \right), \tag{8}$$

where $\mathbf{z}^\lambda$ stands for $z_1^{\lambda_1} \cdots z_l^{\lambda_l}$. The defining formula can also be written

$$H(R_q, R_t, R_{qt}, \lambda) = \boldsymbol{\sigma}\left(\frac{\mathbf{z}^\lambda \prod_{\alpha \in R_{qt}} (1 - q\,t\,\mathbf{z}^\alpha)}{\prod_{\alpha \in R_q} (1 - q\,\mathbf{z}^\alpha) \prod_{\alpha \in R_t} (1 - t\,\mathbf{z}^\alpha)} \right), \tag{9}$$

where

$$\boldsymbol{\sigma}(f) = \sum_{w \in S_l} w\left(\frac{f}{\prod_{\alpha \in R_+} (1 - \mathbf{z}^{-\alpha})} \right) \tag{10}$$

is the Weyl symmetrization operator for GL_l. Recall that $\boldsymbol{\sigma}(\mathbf{z}^\lambda) = \chi_\lambda$ is an irreducible GL_l character if λ is a dominant weight. For an arbitrary weight $\mu \in \mathbb{Z}^l$, either $\boldsymbol{\sigma}(\mathbf{z}^\mu) = \pm\chi_\lambda$ for a suitable dominant weight λ, or $\boldsymbol{\sigma}(\mathbf{z}^\mu) = 0$.

Expanding the factors $(1 - q\,\mathbf{z}^\alpha)^{-1} = 1 + q\,\mathbf{z}^\alpha + \cdots$ and $(1 - t\,\mathbf{z}^\alpha)^{-1} = 1 + t\,\mathbf{z}^\alpha + \cdots$ as geometric series before applying $\boldsymbol{\sigma}$, we can regard (9) as a *raising operator series*, expressing $H(R_q, R_t, R_{qt}, \lambda)$ as an infinite formal linear combination $\sum_\mu a_\mu \chi_\mu$ of irreducible GL_l characters with coefficients $a_\mu \in \mathbb{Z}[q,t]$.

The *polynomial characters* of GL_l are the irreducible characters χ_μ with $\mu \in \mathbb{N}^l$; thus μ is an integer partition with at most l parts and possible trailing zeroes, and χ_μ is equal to the Schur function $s_\mu(z_1, \dots, z_l)$. The *polynomial part*

$$H(R_q, R_t, R_{qt}, \lambda)_{\mathrm{pol}} \tag{11}$$

of a Catalanimal is the truncation of its raising operator series expansion to terms $a_\mu \chi_\mu$ for polynomial characters χ_μ. Then $H(R_q, R_t, R_{qt}, \lambda)_{\mathrm{pol}}$ is a symmetric polynomial, homogeneous of degree $|\lambda| = \sum_i \lambda_i$, in the variables $z_1, \dots, z_l$.

We will need several results from [4] concerning Catalanimals and their connection with the elliptic Hall algebra $\mathcal{E}$ of Burban and Schiffmann [7] (or Schiffmann algebra). Before stating them, we fix notation and recall some facts about $\mathcal{E}$.

For each pair of coprime integers (m,n), the Schiffmann algebra $\mathcal{E}$ contains a distinguished subalgebra $\Lambda(X^{m,n})$ isomorphic to the algebra of symmetric functions. By a theorem of Schiffmann and Vasserot [20], the 'right half-plane' subalgebra $\mathcal{E}^+ \subseteq \mathcal{E}$ generated by the $\Lambda(X^{m,n})$ for $m > 0$ is isomorphic to the shuffle algebra of Feigin et al. [9], Feigin and Tsymbauliak [10], and Negut [19]. This shuffle algebra has many different realizations. Here we use the realization $\mathcal{S}_{\widehat{\Gamma}}$ in [4, §3.2]; it is a graded algebra whose degree l component is a certain subspace $\mathcal{S}^l_{\widehat{\Gamma}} \subseteq \Bbbk(z_1,\ldots,z_l)^{S_l}$ of the space of symmetric rational functions in l variables. The isomorphism that we use is the one denoted

$$\psi_{\widehat{\Gamma}}\colon \mathcal{S}_{\widehat{\Gamma}} \xrightarrow{\cong} \mathcal{E}^+ \tag{12}$$

in [4, equations (26, 28)]. In [20], Schiffmann and Vasserot also constructed an action $\mathcal{E}$ on $\Lambda(X)$. We use the version of this action given by [6, Proposition 3.3.1].

The results stated below summarize everything we need to know for the purposes of this paper about the above algebras, isomorphism and action. For more details, the reader can consult [7, 19, 20]; the translation between the notation in these papers and ours can be found in [6, §§3.2–3.3], the defining relations of $\mathcal{E}$ written in our notation are in [5, §3.2], and the relationship between $\mathcal{S}_{\widehat{\Gamma}}$ and the shuffle algebra studied by Negut in [19] is explained in [4, §3.6].

Proposition 2.3.1 ([4, Proposition 4.1.3])**.** *Let $H = H(R_q, R_t, R_{qt}, \lambda)$ be a* tame *Catalanimal as in* [4, Definition 4.1.2]*—that is, the root sets satisfy $[R_q, R_t] \subseteq R_{qt}$, where $[R_q, R_t] = (R_q + R_t) \cap R_+$. Then H, considered as a symmetric rational function, is an element of $\mathcal{S}_{\widehat{\Gamma}}$, and as such represents an element $\psi_{\widehat{\Gamma}}(H) \in \mathcal{E}^+$ of the Schiffmann algebra.*

Proposition 2.3.2. *If a Catalanimal $H = H(R_q, R_t, R_{qt}, \lambda)$ of length l belongs to $\mathcal{S}_{\widehat{\Gamma}}$, and $\zeta = \psi_{\widehat{\Gamma}}(H)$ is the corresponding element of $\mathcal{E}^+$, then ζ acting on $1 \in \Lambda(X)$ satisfies*

$$\omega(\zeta \cdot 1)(z_1, \ldots, z_l) = H_{\mathrm{pol}}. \tag{13}$$

In addition, $\omega(\zeta \cdot 1)$ is a linear combination of Schur functions s_μ indexed by partitions μ with at most l parts, so it is determined by (13)*.*

Proof. This follows from [6, Proposition 3.5.2] in the same way that [4, Proposition 3.5.2] does. ☐

Proposition 2.3.3 ([4, Lemma 3.5.1])**.** *For any symmetric function f, the element $f[-MX^{m,1}] \in \mathcal{E}$ acting on $1 \in \Lambda(X)$ is given by*

$$f[-MX^{m,1}] \cdot 1 = \nabla^m f(X). \tag{14}$$

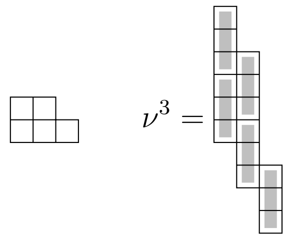
2.4. **Schur Catalanimals.** Given any LLT polynomial $\mathcal{G}_{\boldsymbol{\nu}}(X;q)$, we constructed Catalanimals $H^{m,n}_{\boldsymbol{\nu}^m}$ in [4] such that $\psi_{\widehat{\Gamma}}(H^{m,n}_{\boldsymbol{\nu}^m})$ is equal, up to a sign and a monomial factor in q, t, to $\mathcal{G}_{\boldsymbol{\nu}}[-MX^{m,n}]$. For the proof of the Loehr-Warrington conjecture we need the special case of this result when $\boldsymbol{\nu}$ is a single diagram and the LLT polynomial $\mathcal{G}_{\boldsymbol{\nu}}(X;q)$ is a Schur function. To describe this case, we recall some combinatorial notions from [4, §§8.1–8.2].

Given $(m,n) \in \mathbb{Z}_+ \times \mathbb{Z}$, we define the sequence of m integers as in [4, (104)]

$$\mathbf{b}(m,n)_i = \lceil in/m \rceil - \lceil (i-1)n/m \rceil \qquad (i = 1, \ldots, m). \tag{15}$$

The *m-stretching* of a (skew) diagram ν is the skew diagram ν^m constructed by dilating ν vertically by a factor of m in the following way: for each box x of content c in ν, the m-stretching ν^m has m boxes of contents $mc - m + 1, \ldots, mc - 1, mc$ in the same column as x. For example, the 3-stretching of the partition diagram $\nu = (3, 2)$ is shown here, with shaded bars showing the three boxes in ν^3 that correspond to each box in ν.

(16) $\nu =$, $\nu^3 =$



We define

$$\gamma(\nu) = (\gamma_1, \ldots, \gamma_h) \tag{17}$$

to be the sequence of lengths of diagonals $\{x \in \nu \mid c(x) = c\}$ in increasing order of the content c, and set

$$n'(\gamma(\nu)) = \sum_{i=1}^{h} \binom{\gamma_i}{2}. \tag{18}$$

The *magic number* $p(\nu)$ is the sum of the lengths of the diagonals that do not contain the first box in a row of ν. Note that the diagonals of ν^m correspond to diagonals of ν, each repeated m times vertically. Using this one sees that $n'(\gamma(\nu^m)) = m\, n'(\gamma(\nu))$ and $p(\nu^m) = p(\nu)$.

A more subtle property of the magic number, which follows from [4, Lemma 7.2.2], is that if ν° is the 180° rotation of ν, then $p(\nu^\circ) = p(\nu)$.

Definition 2.4.1. Given a (skew) diagram ν and coprime integers m, n with $m > 0$, the *(skew) Schur Catalanimal* $H^{m,n}_{\nu^m} = H(R_q, R_t, R_{qt}, \lambda)$ is the tame Catalanimal of length $l = m\,|\nu| = |\nu^m|$ constructed as follows, where ν^m is the m-stretching of ν.

The root sets and weight are defined with reference to the partition of $[l] = \{1, \ldots, l\}$ into intervals of lengths $\gamma(\nu^m)$. For the root sets, we take $\alpha_{ij} \in R_q = R_t$ if $i < j$ are in distinct blocks of this partition, and $\alpha_{ij} \in R_{qt}$ if $i < j$ are in distinct, non-adjacent blocks; equivalently, $R_{qt} = [R_q, R_t]$.

The weight λ is defined to be constant on blocks, as follows: for every $i \in [l]$ belonging to the k-th block of the partition, we set

$$\begin{aligned} \lambda_i = \chi(&D_k \text{ contains the first box in a row of } \nu^m) \\ &- \chi(D_k \text{ contains the last box in a row of } \nu^m) + \mathbf{b}(m,n)_{\mathrm{mod}_m(c)}, \end{aligned} \tag{19}$$

where D_k is the k-th diagonal of ν^m, c is the content of boxes on that diagonal, and $\mathrm{mod}_m(c)$ is the integer $j \in [m]$ such that $j \equiv c \pmod{m}$.

When μ is a (non-skew) partition diagram, we call $H^{m,n}_{\mu^m}$ a *Schur Catalanimal* and $H^{m,n}_{(\mu^\circ)^m}$ the *opposite Schur Catalanimal*, where μ° is the 180° rotation of μ.

Theorem 2.4.2 ([4, Theorem 8.3.1])**.** *The Schur Catalanimal $H^{m,n}_{\mu^m}$ satisfies the identity*

$$s_\mu[-MX^{m,n}] = (-1)^{p(\mu)}(q\,t)^{p(\mu)+m\,n'(\gamma(\mu))}\psi_{\widehat{\Gamma}}(H^{m,n}_{\mu^m}). \tag{20}$$

This identity also holds with the opposite Schur Catalanimal $H^{m,n}_{(\mu^\circ)^m}$ in place of $H^{m,n}_{\mu^m}$.

3. Nests in a den formula

Our main combinatorial result, Theorem 3.5.1 below, is an identity expanding the polynomial parts of certain tame Catalanimals as weighted sums of LLT polynomials indexed by configurations of nested lattice paths. In this section, we define the required combinatorial notions and then state the theorem. The proof will be given in §7.

3.1. **Dens and nests.** We begin by defining the data that will serve as input to Theorem 3.5.1.

Definition 3.1.1. A *den* is a tuple $(h, p, \mathbf{d}, \mathbf{e})$, where h is a positive integer, p is an irrational real number, and $\mathbf{d} = (d_0, \dots, d_h)$ and $\mathbf{e} = (e_0, \dots, e_h)$ are sequences of integers, subject to the following conditions:

$$(d_i - d_j + 1)/(j - i) > p \quad \text{for} \quad 0 \le i < j \le h - 1; \tag{21}$$

$$(e_i - e_j - 1)/(j - i) < p \quad \text{for} \quad 1 \le i < j \le h; \tag{22}$$

$$d_0 > e_0, \quad d_h < e_h, \quad \text{and} \quad \sum_{i=0}^{h} d_i = \sum_{i=0}^{h} e_i. \tag{23}$$

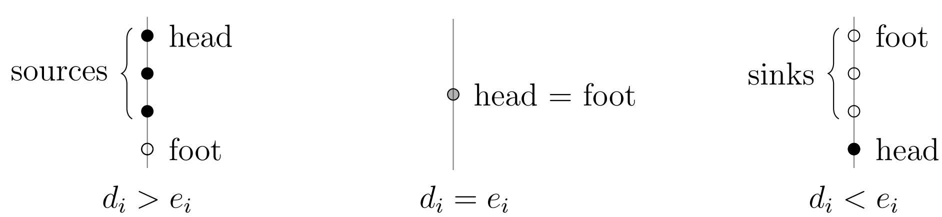
The reason for assuming p irrational is to avoid having to disambiguate equalities that might otherwise occur in comparisons such as those in (21, 22).

With any den we also define the following auxiliary notions. The lattice points (i, d_i) are *heads*, and (i, e_i) are *feet*. Points $\{(i, j) \mid e_i < j \le d_i\}$ weakly below a head and strictly above a foot on the same vertical line $x = i$ are *sources*. Points $\{(i, j) \mid d_i < j \le e_i\}$ weakly below a foot and strictly above a head are *sinks*. We also set

$$\mathbf{g} = (g_1, \dots, g_h), \quad \text{where} \quad g_k = \sum_{i=0}^{k-1}(d_i - e_i). \tag{24}$$

A den can be pictured by plotting the heads, feet, sources and sinks, as shown for example in Figure 1. On each line $x = i$ for $0 \le i \le h$, the head, foot, and any sources or sinks are arranged in one of the ways shown here.

(25)



Condition (21) means that for heads P left of Q, excluding the last head (h, d_h), some line of slope $-p$ passes above Q and below $P + (0, 1)$. Similarly, condition (22) means that for

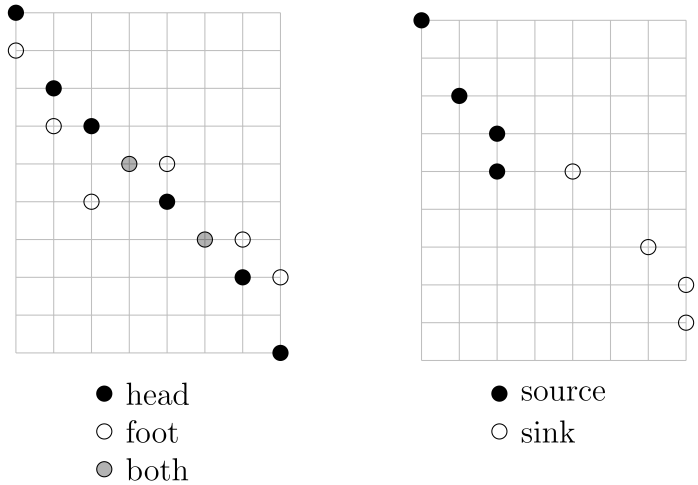
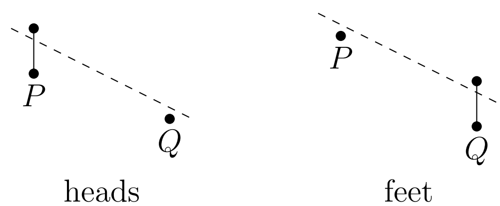


FIGURE 1. Heads, feet, sources and sinks in the den with $h = 7$, $\mathbf{d} = (9, 7, 6, 5, 4, 3, 2, 0)$, $\mathbf{e} = (8, 6, 4, 5, 5, 3, 3, 2)$. These data define a valid den for any $p \in (1, \frac{6}{5})$.

feet P left of Q, excluding the first foot $(0, e_0)$, some line of slope $-p$ passes above P and below $Q + (0, 1)$, as pictured here.

(26)



Condition (23) says that there is at least one source on the line $x = 0$ and at least one sink on the line $x = h$, and that the total number of sources is equal to the total number of sinks. Conditions (21) and (22) imply

$$d_j - e_j \leq d_i - e_i + 1 \tag{27}$$

for all $0 < i < j < h$. If there is a sink with x coordinate i and a source with x coordinate j, then $d_i - e_i < 0$ and $d_j - e_j > 0$. In particular, no source and sink can be on the same vertical line $x = i$, and (27) implies that all sources are strictly left of all sinks.

Next we define the systems of nested lattice paths that will be attached to a den.

Definition 3.1.2. An *east end path* is a lattice path with south $(0, -1)$ and east $(1, 0)$ steps that ends with an east step. East end paths π, π' are *nested with π below π'* if

(i) the interval $[a', b']$ of x-coordinates of points of π' is contained in the interval $[a, b]$ of x-coordinates of points of π, and
(ii) for every integer $i \in [a', b']$ the respective intervals $[v_i, w_i]$ and $[v'_i, w'_i]$ of y-coordinates of points on $\pi \cap (x = i)$ and $\pi' \cap (x = i)$ satisfy $v_i < v'_i$ and $w_i < w'_i$.

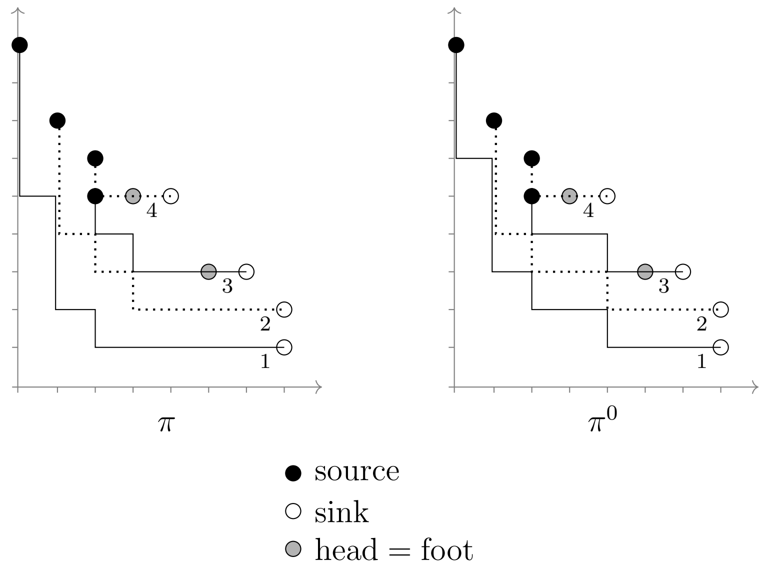
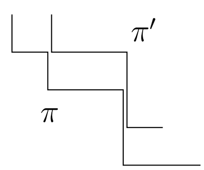


FIGURE 2. A typical nest π in the den in Figure 1, and the unique nest π^0 such that $a(\pi^0) = 0$. The sequence $\mathbf{g} = (1, 2, 4, 4, 3, 3, 2)$ associated with the den gives the number of east steps at each x-coordinate.

An example of a pair of nested east end paths is shown below. Note that nested paths can share south steps, but not east steps.

(28)



It is not hard to see that nesting is transitive, i.e., if π is nested below π' and π' is nested below π'', then π is nested below π''.

Definition 3.1.3. A *nest* in a den $(h, p, \mathbf{d}, \mathbf{e})$ is a system of nested east end paths $\pi = (\pi_1, \ldots, \pi_r)$ from the sources to the sinks, numbered with π_k nested below π_l for $k < l$, which satisfies the condition $j \leq d_i$ for every lattice point (i, j) other than the sink on each of the paths π_k. In other words, all non-sink lattice points in π lie weakly below the heads.

Figure 2 shows two nests belonging to the den in Figure 1. We have marked the head (= foot) on each line $x = i$ that has no source or sink in order to make visible the condition that paths in the nest must lie weakly below the heads.

Remark 3.1.4. (i) The numbering of the paths π_i from nested below to nested above implies that π_i starts at the i-th source from left to right, with sources on the same vertical line numbered bottom to top. Similarly, π_i ends at the i-th sink from right to left, again from bottom to top on vertical lines.

(ii) Because paths in a nest are nested, any non-sink lattice point (i, j) on a path automatically lies weakly below the head (i, d_i) unless the head equals the foot (i, e_i). Only when

$d_i = e_i$, so there is no source or sink on the line $x = i$, does the requirement $j \le d_i$ impose an extra condition.

For example, the head and foot at $(3,5)$ in the den in Figure 1, shown as the upper left gray dot in Figure 2, prohibits the highest path π_4 from starting with an east step to $(3,6)$, although nesting alone would allow this. Similarly, the head and foot at $(5,3)$ prohibits π_3 from passing through $(5,4)$.

(iii) The number g_k in (24) counts sources minus sinks with x-coordinate less than k. Hence, for every nest π in the den, g_k is the number of paths π_j in π that have an east step from $x = k-1$ to $x = k$, or equivalently, have a non-sink lattice point on the line $x = k-1$. These are the first g_k paths $\pi_1, \dots, \pi_{g_k}$, since if π_j has an east step from $x = k-1$ to $x = k$, then so does every path π_i nested below π_j. Since all sources are left of all sinks, and there is assumed to be at least one source at $x = 0$ and at least one sink at $x = h$, the sequence $\mathbf{g}$ is positive and unimodal, with maximum equal to the number of sources (or sinks). These properties can be seen in Figure 2.

(iv) It is possible to have an 'abandoned' den with no nests. The most obvious way this can happen is if some source is lower than the matching sink.

3.2. **Parameterizing nests.** Nests in a den can be parameterized by tuples of partitions satisfying certain inequalities, as follows.

Lemma 3.2.1. *Let $(h, p, \mathbf{d}, \mathbf{e})$ be a den, with $\mathbf{g}$ as in (24).*
(i) If $\pi = (\pi_1, \dots, \pi_r)$ is a nest in the den, then

(a) for $1 \le k \le h$, π_i has a non-sink lattice point on the line $x = k-1$ if and only if $1 \le i \le g_k$;

(b) there are unique partitions $\lambda_{(1)}, \dots, \lambda_{(h-1)}$ of length $\ell(\lambda_{(k)}) \le \min(g_k, g_{k+1})$ such that for $1 \le k \le h$ and $1 \le i \le g_k$, the y-coordinates of all non-sink lattice points of π_i on the line $x = k-1$ form the interval

$$I_{k,i} = [e_k - g_k + i - (\lambda_{(k)})_i,\ e_{k-1} - g_{k-1} + i - (\lambda_{(k-1)})_i] \tag{29}$$

where we set $\lambda_{(0)} = \lambda_{(h)} = \varnothing$, extend partitions with trailing zeroes if needed, and set $g_0 = 0$.

(ii) Set $\lambda_{(0)} = \lambda_{(h)} = \varnothing$ and $g_0 = 0$. Let $\lambda_{(1)}, \dots, \lambda_{(h-1)}$ be partitions of length $\ell(\lambda_{(k)}) \le \min(g_k, g_{k+1})$ such that for $1 \le k \le h$ and $1 \le i \le g_k$,

$$e_k - g_k - (\lambda_{(k)})_i \le e_{k-1} - g_{k-1} - (\lambda_{(k-1)})_i \tag{30}$$

(so the intervals $I_{k,i}$ are non-empty). Then there is a unique nest $\pi = (\pi_1, \dots, \pi_r)$ in the den such that the y-coordinates of all non-sink lattice points of π_i on the line $x = k-1$ form the interval $I_{k,i}$ for all $1 \le k \le h$ and $1 \le i \le g_k$.

Proof. Given a nest π, part (i)(a) holds by Remark 3.1.4 (iii), and the east steps from $x = k-1$ to $x = k$ in π are on paths π_1 through π_{g_k}. Let $y_{k,1} < \dots < y_{k,g_k}$ be the y-coordinates of these east steps. The right endpoint of any east step is weakly below the foot on the same vertical line, so $y_{k,g_k} \le e_k$. Hence,

$$y_{k,i} \le e_k - g_k + i \tag{31}$$

for all k and $i \le g_k$.

Every east step from $x = h - 1$ to $x = h$ ends at a sink, so $y_{h,i} = e_h - g_h + i$ is fixed for all $i \leq g_h$, independent of the nest. If $k < h$ and $g_k > g_{k+1}$, there are $g_k - g_{k+1}$ sinks on the line $x = k$. In this case the top $g_k - g_{k+1}$ east steps from $x = k - 1$ to $x = k$ are fixed, with $y_{k,i} = e_k - g_k + i$ for $g_{k+1} < i \leq g_k$. This leaves the $y_{k,i}$ for $k < h$ and $i \leq \min(g_k, g_{k+1})$ free to vary with the nest. To establish part (i)(b), let

$$\lambda_{(k)} = (e_k - g_k + 1 - y_{k,1}, \dots, e_k - g_k + r_k - y_{k,r_k}), \tag{32}$$

for $k = 1, \dots, h-1$, where $r_k = \min(g_k, g_{k+1})$. We also set $\lambda_{(0)} = \lambda_{(h)} = \varnothing$ and $(\lambda_{(k)})_i = 0$ for $i > r_k$.

Since the $y_{k,i}$ are strictly increasing and bounded by (31), $\lambda_{(k)}$ is a partition of length $\ell(\lambda_{(k)}) \leq \min(g_k, g_{k+1})$ with possible trailing zeroes. Since equality holds in (31) for the fixed east steps that end at sinks, we have

$$y_{k,i} = e_k - g_k + i - (\lambda_{(k)})_i \tag{33}$$

for all $1 \leq k \leq h$ and $i \leq g_k$.

The lattice point at $x = k - 1$ on π_i with the smallest y-coordinate is the left endpoint of the east step with y-coordinate $y_{k,i}$. If $k > 1$ and $i \leq g_{k-1}$, the point with the largest y-coordinate is the right endpoint of the east step with y-coordinate $y_{k-1,i}$. Otherwise, if $k = 1$ or $g_{k-1} < i \leq g_k$, this highest point is the i-th source, with y-coordinate $e_{k-1} - g_{k-1} + i$, if we take $g_0 = 0$ in the case $k = 1$. This shows that the y-coordinates of the points on π_i at $x = k - 1$ are given by the interval $I_{k,i}$ in all cases. These intervals clearly determine the partitions $\lambda_{(k)}$.

For part (ii), suppose we are given partitions $\lambda_{(1)}, \dots, \lambda_{(h-1)}$ with $\ell(\lambda_{(k)}) \leq \min(g_k, g_{k+1})$ such that (30) holds. By Remark 3.1.4 (iii), the sequence $\mathbf{g}$ is positive and unimodal with maximum equal to the number r of sources (or sinks) in the den, and for each $i = 1, \dots, r$ the set $\{k \in [h] \mid g_k \geq i\}$ is the non-empty interval $[k_0, k_1]$ such that the i-th source and its matching sink are at $x = k_0 - 1$ and $x = k_1$. For each i, we start by constructing an east end path π_i such that the intervals $I_{k,i}$ for $k \in [k_0, k_1]$ describe the lattice points on π_i.

Fix i and the corresponding interval $[k_0, k_1]$. The i-th source has y-coordinate $e_{k_0-1} - g_{k_0-1} + i$, if we set $g_0 = 0$ for $k_0 = 1$. Since k_0 is minimal with $i \leq g_{k_0}$, we have $i > g_{k_0-1}$ and therefore $(\lambda_{(k_0-1)})_i = 0$ since $\ell(\lambda_{(k_0-1)}) \leq g_{k_0-1}$. Hence, the upper endpoint of the interval $I_{k_0,i}$ is the y-coordinate of the i-th source. The sink matching the i-th source has y-coordinate $e_{k_1} - g_{k_1} + i$. Since k_1 is maximal with $i \leq g_{k_1}$, either $k_1 = h$ or $i > g_{k_1+1}$, and therefore $(\lambda_{(k_1)})_i = 0$. Hence, the lower endpoint of $I_{k_1,i}$ is the y-coordinate of the matching sink. For the rest, if $k_0 \leq k < k_1$, the lower endpoint of $I_{k,i}$ is equal to the upper endpoint of $I_{k+1,i}$. Hence, there exists a unique east end path π_i from the i-th source to the i-th sink with non-sink lattice points at $x = k-1$ given by the intervals $I_{k,i}$ for this i and $k_0 \leq k \leq k_1$.

By construction, the paths π_i defined in this way are nested with π_i below π_j for $i < j$. The upper endpoint of the interval $I_{k,i}$ is at most $e_{k-1} - g_{k-1} + g_k$, which is equal to d_{k-1} by the definition of g_k, so the paths π_i form a nest in the den. □

Example 3.2.2. Nests in the den in Figure 1 are parameterized by partitions $\lambda_{(1)}, \dots, \lambda_{(6)}$ of lengths at most $1, 2, 4, 3, 3, 2$, subject to the inequalities in (30). The nest π^0 on the right in Figure 2 corresponds to $\lambda_{(i)} = \varnothing$ for all $i = 1, \dots, 6$. The nest π on the left in Figure 2 corresponds to $\lambda_{(1)} = (1)$, $\lambda_{(2)} = (1)$, $\lambda_{(3)} = (1)$, $\lambda_{(4)} = (1, 1, 1)$, $\lambda_{(5)} = \lambda_{(6)} = \varnothing$.

3.3. **Combinatorial statistics associated with nests.** We now define statistics $a(\pi)$ and $\operatorname{dinv}_p(\pi)$ for each nest π in a den, closely related to the area and dinv statistics seen in the Loehr-Warrington conjecture and the various generalizations of the shuffle theorem.

Definition 3.3.1. Let π be a nest in a den $(h, p, \mathbf{d}, \mathbf{e})$ and let $\mathbf{g}$ be the sequence in (24), so there are g_k east steps from $x = k-1$ to $x = k$ in π. For $k = 1, \ldots, h$, let $y_{k,1} < \cdots < y_{k,g_k}$ be the y-coordinates of these east steps. We define

$$a(\pi) = \sum_{k=1}^{h} \sum_{i=1}^{g_k} (e_k - g_k + i - y_{k,i}). \tag{34}$$

Equivalently, by (33), $a(\pi) = |\lambda| = \sum_k |\lambda_{(k)}|$ in terms of the parameterization in Lemma 3.2.1.

If we let $\operatorname{area}(\pi_i)$ denote the number of lattice squares above π_i and below some fixed boundary—for instance, the number of lattice squares above π_i in the rectangle with corners at its source and sink—then both $a(\pi)$ and $\sum_i \operatorname{area}(\pi_i)$ have the form $(\text{constant} - \sum_{k,i} y_{k,i})$. Hence, $a(\pi)$ differs from $\sum_i \operatorname{area}(\pi_i)$ by a constant not depending on π.

If there is a nest π^0 in the den such that $a(\pi^0) = 0$, it must correspond via Lemma 3.2.1 to $\lambda_{(k)} = \varnothing$ for all k, or equivalently to $y_{k,i} = e_k - g_k + i$ for all k and i. Since this is always an upper bound on $y_{k,i}$, the path π_i in any nest π lies weakly below the path π_i^0, and $a(\pi)$ is equal to the sum of the areas $\operatorname{area}(\pi_i)$ between π_i and π_i^0. We can make this more precise as follows.

Corollary 3.3.2. *A den $(h, p, \mathbf{d}, \mathbf{e})$ has a nest π^0 such that $a(\pi^0) = 0$ if and only if $e_k \leq d_{k-1}$ for all $k = 1, \ldots, h$. Such a nest π^0 is unique. If it exists, then each path π_i in any nest π lies weakly below π_i^0, and $a(\pi)$ is equal to the sum of the areas* $\operatorname{area}(\pi_i) = |\pi_i^0/\pi_i|$ *enclosed between the paths π_i and π_i^0.*

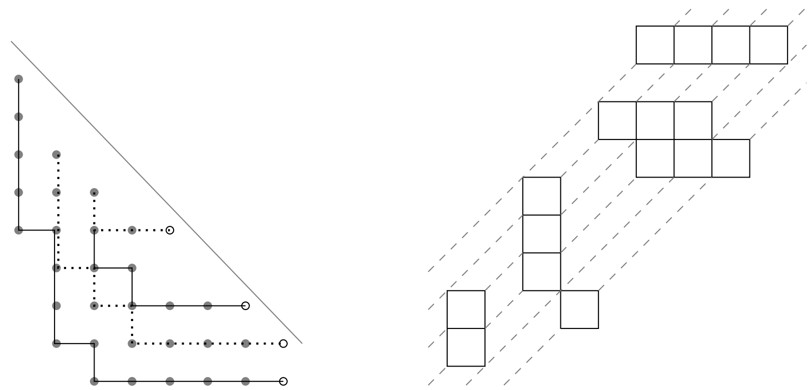
If $\mathbf{d}$ is weakly decreasing, which is always the case if $p > 0$ by condition (21)*, then the den is either abandoned (has no nests), or it has a nest π^0 as above.*

Proof. By the definition of $\mathbf{g}$ in (24), we have $e_k \leq d_{k-1}$ if and only if $e_k - g_k \leq e_{k-1} - g_{k-1}$, taking $g_0 = 0$. This is equivalent to (30) for all $\lambda_{(k)} = \varnothing$. The first paragraph then follows from the preceding observations and Lemma 3.2.1.

For the last part, assume that $\mathbf{d}$ is weakly decreasing. If $e_k > d_{k-1}$ for some k, it follows that $e_k > d_k$, so (k, e_k) is a sink. The east step ending at this sink in any nest would start at $(k-1, e_k)$, but this is not allowed, since $e_k > d_{k-1}$. Hence, the den is abandoned unless $e_k \leq d_{k-1}$ for all $k = 1, \ldots, h$. □

Example 3.3.3. The den in Figure 1 has a unique nest π^0 with $a(\pi^0) = 0$, shown on the right in Figure 2. For the nest π on the left in Figure 2, the areas between corresponding paths π_i^0 and π_i add up to $a(\pi) = 4 + 1 + 1 + 0 = 6$.

Definition 3.3.4. Let π be a nest in a den $(h, p, \mathbf{d}, \mathbf{e})$. We define $\operatorname{dinv}_p(\pi)$ to be the number of tuples (P, i, S, j), where P is a non-sink lattice point on π_i, S is a south step on π_j, P is strictly left of S, and the line of slope $-p$ through P passes through S (necessarily through the interior of S, since we assume p is irrational).

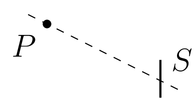


π $\qquad\qquad$ $\boldsymbol{\nu}(\pi)$

FIGURE 3. (i) The nest π drawn below a line $y + p\,x = s$ with $p \approx 1.04$ and $s \approx 9.80$. (ii) The tuple of skew diagrams $\boldsymbol{\nu}(\pi)$, arranged southwest to northeast with dashed lines showing boxes of equal content.

In effect, $\operatorname{dinv}_p(\pi)$ counts pairs P, S in π whose relative position is as indicated, with multiplicities if P or S lies on more than one path in the nest.

(35)



Example 3.3.5. In the den in Figure 1, take $p = 1 + \epsilon$ for a small $\epsilon > 0$. Then P and S contribute to $\operatorname{dinv}_p(\pi)$ if P is northwest of the upper end of S on the same diagonal of slope -1. In Figure 3 we have redrawn the nest π from Figure 2, displaying lattice points, as an aid to checking that $\operatorname{dinv}_p(\pi) = 22$.

3.4. **LLT polynomial associated with a nest.** Next we define a tuple of skew diagrams $\boldsymbol{\nu}(\pi)$ attached to each nest π in a den.

Given a den $(h, p, \mathbf{d}, \mathbf{e})$, fix a real number s such that the line $y + px = s$ passes weakly above all the heads and feet. For $i = 1, \ldots, h$, let $c_i = \{s - p\,(i-1)\}$ be the height of the gap between the line $y + p\,x = s$ and the highest lattice point weakly below it at $x = i - 1$, where $\{a\} = a - \lfloor a \rfloor$ denotes the fractional part of a real number a.

Let $\sigma \in S_h$ be the permutation such that $\sigma(1), \ldots, \sigma(h)$ are in the same relative order as $c_1, \ldots, c_h$, i.e., such that $\sigma(c_1, \ldots, c_h)$ is increasing. Note that the c_i are distinct, since p is irrational.

Definition 3.4.1. Given a den $(h, p, \mathbf{d}, \mathbf{e})$, fix s and define $c_1, \ldots, c_h$ and σ as above. For each nest π in the den, $\boldsymbol{\nu}(\pi)$ is the tuple of skew diagrams $(\nu_{(1)}, \ldots, \nu_{(h)})$ with $\nu_{(j)}$ defined as follows.

Let $k = \sigma^{-1}(j)$. By Remark 3.1.4 (iii), the paths in π which have a non-sink lattice point on the line $x = k - 1$ are $\pi_1, \ldots, \pi_{g_k}$, for g_k defined in (24). For $i = 1, \ldots, g_k$, let y_i and w_i be the maximum and minimum y-coordinates of lattice points on π_i at $x = k - 1$. Then we set

$$\nu_{(j)} = \beta/\alpha = (\beta_1, \dots, \beta_{g_k})/(\alpha_1, \dots, \alpha_{g_k}), \tag{36}$$

where $\alpha_i = \lfloor s - p\,(k-1) \rfloor - y_i + i$ and $\beta_i = \lfloor s - p\,(k-1) \rfloor - w_i + i$. Note that $(y_1, \dots, y_{g_k})$ and $(w_1, \dots, w_{g_k})$ are strictly increasing, with $y_r \geq w_r$, so β/α makes sense as a skew diagram.

Less formally, for each $k = 1, \dots, h$, if $O_k = (k-1, \lfloor s - p\,(k-1) \rfloor)$ is the highest lattice point at $x = k-1$ below the line $y + p\,x = s$, we construct a skew diagram by turning runs of south steps in paths in π at $x = k-1$ into rows of a skew diagram, placed so that the content of the box corresponding to a south step S is the distance between O_k and the south endpoint of S. Then $\boldsymbol{\nu}(\pi)$ is the list of these skew diagrams in increasing order of the gaps c_k. Note that this makes the reading order on boxes of $\boldsymbol{\nu}(\pi)$ correspond to the ordering of south steps in π by increasing distance below the line $y + p\,x = s$, with occurrences of the same south step S on two paths π_i, π_j ordered by $i < j$.

Example 3.4.2. In Figure 3, we have re-drawn the nest π from Example 3.3.5 and Figure 2 below a line $y + p\,x = s$ with $p = 1 + \epsilon$ for $\epsilon \approx .04$, and $s \approx 9.8$ chosen so the line passes a little above the source at $(0, 9)$ and the sinks at $(4, 5)$, $(6, 3)$ and $(7, 2)$. The gaps c_k between this line and the highest lattice points below it increase in the order $c_7 < \cdots < c_1$, giving $\sigma = w_0$, the longest permutation in S_7. Accordingly, the skew diagrams $\nu_{(1)}, \dots, \nu_{(7)}$ in $\boldsymbol{\nu}(\pi)$ are associated to south runs on the lines $x = k-1$ in the order $k = 7, 6, \dots, 1$.

The first three diagrams in $\boldsymbol{\nu}(\pi)$ are empty. The last four are plotted in Figure 3, arranged from southwest to northeast and positioned so that boxes of equal content are on the same diagonal line.

Remark 3.4.3. In terms of the parameterization in Lemma 3.2.1, the skew diagram $\nu_{(j)} = \beta/\alpha$ in (36) is given by $\alpha = (a^{g_k}) + \lambda_{(k-1)}$, $\beta = (b^{g_k}) + \lambda_{(k)}$, where $k = \sigma^{-1}(j)$, $a = \lfloor s - p\,(k-1) \rfloor - e_{k-1} + g_{k-1}$, $b = \lfloor s - p\,(k-1) \rfloor - e_k + g_k$.

Although the definition of $\boldsymbol{\nu}(\pi)$ involves an auxiliary choice of the line $y + p\,x = s$, one can check that when s varies, $\boldsymbol{\nu}(\pi)$ changes by rotations of the kind in Proposition 2.2.4. The LLT polynomial $\mathcal{G}_{\boldsymbol{\nu}(\pi)}(X; q)$ therefore depends only on π and the slope parameter p. We can make this more explicit as follows.

Let $\mathbf{S}(\pi)$ be the set of pairs (S, i) such that S is a south step on π_i, and let $\hat{c}(S)$ denote the vertical distance $s - (l + p\,k)$ between the line $y + p\,x = s$ and the south endpoint (k, l) of S. We say that an ordered pair of elements (S, i), $(S', j) \in \mathbf{S}(\pi)$ is an *attacking pair* if $0 < \hat{c}(S') - \hat{c}(S) < 1$. This means that S and S' are distinct, some line of slope $-p$ passes through them both, and they are ordered with $\hat{c}(S) < \hat{c}(S')$. The differences $\hat{c}(S') - \hat{c}(S)$ and the set of attacking pairs do not depend on s. Via the natural correspondence between $\mathbf{S}(\pi)$ and the set of boxes in $\boldsymbol{\nu}(\pi)$, attacking pairs in $\mathbf{S}(\pi)$ correspond to attacking pairs in $\boldsymbol{\nu}(\pi)$.

Definition 3.4.4. A *negative labeling* of a nest π is a map $N\colon \mathbf{S}(\pi) \to \mathbb{Z}_+$ that satisfies the conditions

(i) N is strictly increasing from north to south along each run of south steps in each π_i;
(ii) if (S, i) and $(S', i+1)$ are on the same vertical line with S' immediately above S, then $N(S, i) \leq N(S', i+1)$.

We define $\operatorname{inv}(N)$ to be the number of attacking pairs (S,i), (S',j) in $\mathbf{S}(\pi)$ such that $N(S,i) \geq N(S',j)$.

A *positive labeling* $P\colon \mathbf{S}(\pi) \to \mathbb{Z}_+$ is defined similarly, but with 'weakly increasing' in place of 'strictly increasing' in (i), with $P(S,i) < P(S',i+1)$ in place of $N(S,i) \leq N(S',i+1)$ in (ii), and with $P(S,i) > P(S',j)$ in place of $N(S,i) \geq N(S',j)$ when defining $\operatorname{inv}(P)$.

By construction, if we transfer labels from steps in $\mathbf{S}(\pi)$ to the corresponding boxes in $\boldsymbol{\nu}(\pi)$, positive or negative labelings P or N correspond to tableaux $T \in \mathrm{SSYT}_{\pm}(\boldsymbol{\nu}(\pi))$ with positive or negative letters. The definition of $\mathcal{G}_{\boldsymbol{\nu}}$ and Proposition 2.2.2 therefore yield

$$\mathcal{G}_{\boldsymbol{\nu}(\pi)}(X;q) = \sum_P q^{\operatorname{inv}(P)} \mathbf{x}^P, \tag{37}$$

$$\omega\, \mathcal{G}_{\boldsymbol{\nu}(\pi)}(X;q) = \sum_N q^{\operatorname{inv}(N)} \mathbf{x}^N, \tag{38}$$

where the sums are over positive and negative labelings P and N, respectively, and $\mathbf{x}^P = \prod_{(S,i)\in \mathbf{S}(\pi)} x_{P(S,i)}$, with $\mathbf{x}^N$ defined similarly.

3.5. **Main theorem.** We have now defined the ingredients needed to state our main combinatorial result.

Theorem 3.5.1. *Given a den $(h,p,\mathbf{d},\mathbf{e})$, with $\mathbf{g}$ as in (24), define a Catalanimal*

$$H = H\big(R_q, R_t, R_{qt}, ((d_0-e_1)^{g_1}, \ldots, (d_{h-1}-e_h)^{g_h})\big) \tag{39}$$

of length $l = |\mathbf{g}| = \sum_k g_k$, taking $R_q = R_t$ to be the set of positive roots α_{ij} $(i<j)$ such that i, j are in distinct blocks of the partition of $[l]$ into intervals of lengths g_k, and R_{qt} to be the subset of these roots with i, j in non-adjacent blocks.

Then the polynomial part of H is given by

$$H_{\mathrm{pol}}(\mathbf{z}) = \sum_\pi t^{a(\pi)} q^{\operatorname{dinv}_p(\pi)} \mathcal{G}_{\boldsymbol{\nu}(\pi)}(z_1,\ldots,z_l;q^{-1}), \tag{40}$$

where the sum is over all nests π in the given den, and $a(\pi)$, $\operatorname{dinv}_p(\pi)$ and $\mathcal{G}_{\boldsymbol{\nu}(\pi)}(X;q)$ are as defined in §§3.3–3.4.

There are several alternative ways to formulate the conclusion of Theorem 3.5.1. Using Proposition 2.3.2, we can connect it with the Schiffmann algebra, as follows. Note that the root sets in (39) satisfy $R_{qt} = [R_q, R_t]$, so the Catalanimal H is tame.

Corollary 3.5.2. *Given a den $(h,p,\mathbf{d},\mathbf{e})$, let $\zeta = \psi_{\widehat{\Gamma}}(H) \in \mathcal{E}^+$ be the Schiffmann algebra element represented by the tame Catalanimal H in (39). Then*

$$\zeta \cdot 1 = \sum_\pi t^{a(\pi)} q^{\operatorname{dinv}_p(\pi)} \omega\, \mathcal{G}_{\boldsymbol{\nu}(\pi)}(X;q^{-1}). \tag{41}$$

We can also reformulate (40) and (41) in terms of labeled nests using (37–38). The resulting identities are

$$H_{\mathrm{pol}}(\mathbf{z}) = \sum_{\pi,P} t^{a(\pi)} q^{\mathrm{dinv}_p(\pi)-\mathrm{inv}(P)} \mathbf{z}^P, \tag{42}$$

$$\zeta \cdot 1 = \sum_{\pi,N} t^{a(\pi)} q^{\mathrm{dinv}_p(\pi)-\mathrm{inv}(N)} \mathbf{x}^N, \tag{43}$$

where the sums are over nests π and positive or negative labelings P or N of π (Definition 3.4.4), with the positive labelings P in (42) having labels between 1 and l.

Corollary 3.5.3. *(i) The right hand sides of* (40) *through* (43) *are symmetric in q and t.* *(ii) The left hand sides of* (40) *through* (43) *are q,t Schur positive, i.e., they are linear combinations of Schur functions with coefficients in $\mathbb{N}[q,t]$.*

Proof. (i) The Catalanimal H in the theorem is symmetric in q and t by construction.

(ii) A priori, the coefficients are in $\mathbb{Z}[q^{\pm 1}, t]$, but (i) implies that they are in $\mathbb{Z}[q,t]$. It was shown in [12] that LLT polynomials $\mathcal{G}_{\boldsymbol{\nu}}(X;q)$ are q Schur positive. Hence, the coefficients are in $\mathbb{N}[q,t]$. □

Remark 3.5.4. If the den $(h, p, \mathbf{d}, \mathbf{e})$ has no nests, Theorem 3.5.1 implies that the left hand sides of (40) through (43) are zero.

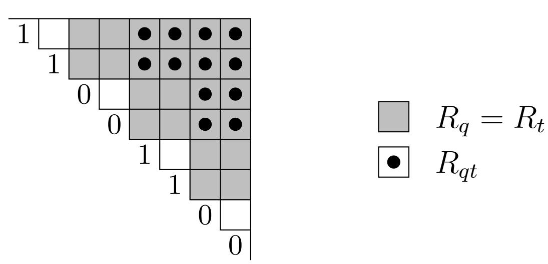
Example 3.5.5. To illustrate Theorem 3.5.1, we write everything out for the den defined by

$$h = 4, \quad p = \frac{1}{2} + \epsilon, \quad \mathbf{d} = (3,2,2,1,-1), \quad \mathbf{e} = (1,2,2,1,1). \tag{44}$$

This den has sources at $(0,2)$, $(0,3)$ and sinks at $(4,0)$, $(4,1)$. Its nests are pairs (π_1, π_2) of nested generalized Dyck paths, with π_1 from $(0,2)$ to $(4,0)$ and π_2 from $(0,3)$ to $(4,1)$, each path staying weakly below the line of slope $-1/2$ connecting its endpoints. For this den we have $\mathbf{g} = (2,2,2,2)$, since each nest consists of two paths from $x=0$ to $x=4$.

The Catalanimal $H = H(R_q, R_t, R_{qt}, \lambda)$ on the left hand side of (40) has length 8, with root sets and weight displayed below. Matrix position (i,j) in the diagram represents the root α_{ij}. The weight λ is written along the diagonal.

(45)

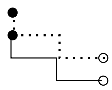


One can verify by expanding the raising operator series that the polynomial part of this Catalanimal is given by

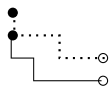
$$H_{\mathrm{pol}}(\mathbf{z}) = (q^3 t + q^2 t^2 + q\,t^3)\, s_{31}(\mathbf{z}) + (q^4 + q^3 t + 2\,q^2 t^2 + q\,t^3 + t^4)\, s_{22}(\mathbf{z}) + (q^3 + 2\,q^2 t + 2\,q\,t^2 + t^3)\, s_{211}(\mathbf{z}) + (q^2 + q\,t + t^2)\, s_{1111}(\mathbf{z}). \tag{46}$$

The following table displays each nest π in the den along with the corresponding term $t^{a(\pi)}q^{\mathrm{dinv}_p(\pi)}\mathcal{G}_{\boldsymbol{\nu}(\pi)}(\mathbf{z};q^{-1})$ on the right hand side of (40), with $\mathcal{G}_{\boldsymbol{\nu}(\pi)}(\mathbf{z};q^{-1})$ expanded as a linear combination of Schur functions $s_\lambda(\mathbf{z})$. The reader can verify that (46) is the sum of these terms.

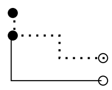
(47)



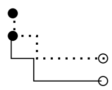
$$t^0\, q^5\, \big(q^{-1}s_{22}(\mathbf{z}) + q^{-2}s_{211}(\mathbf{z}) + q^{-3}s_{1111}(\mathbf{z})\big)$$



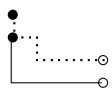
$$t^1\, q^3\, \big(s_{31}(\mathbf{z}) + s_{22}(\mathbf{z}) + 2\,q^{-1}s_{211}(\mathbf{z}) + q^{-2}s_{1111}(\mathbf{z})\big)$$



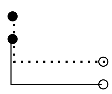
$$t^2\, q^2\, \big(s_{31}(\mathbf{z}) + s_{22}(\mathbf{z}) + q^{-1}s_{211}(\mathbf{z})\big)$$



$$t^2\, q^3\, \big(q^{-1}s_{22}(\mathbf{z}) + q^{-2}s_{211}(\mathbf{z}) + q^{-3}s_{1111}(\mathbf{z})\big)$$



$$t^3\, q^2\, \big(q^{-1}s_{31}(\mathbf{z}) + q^{-1}s_{22}(\mathbf{z}) + q^{-2}s_{211}(\mathbf{z})\big)$$



$$t^4\, q^0\, s_{22}(\mathbf{z})$$

3.6. **The single path case.** As a further class of examples, we examine more closely the instances of Theorem 3.5.1 and Corollary 3.5.2 for dens $(h, p, \mathbf{d}, \mathbf{e})$ with just one source and one sink, that is,

$$\mathbf{d} - \mathbf{e} = (1, 0, \ldots, 0, -1). \tag{48}$$

We assume that $h > 1$, as $h = 1$ gives trivial dens with at most one nest. We also note that if $p < 0$, then $d_i = e_i \leq e_h$ for $0 < i < h$. This again implies that the den has at most one nest, so we assume that $p \geq 0$.

Let $y + p\,x = s$ be the highest line of slope $-p$ that passes through one of the heads (= feet) (i, d_i) for $0 < i < h$. Conditions (21–22) hold if and only if all the heads for $0 < i < h$ lie in the band $s - 1 < y + p\,x \leq s$, and the source and sink at $x = 0$ and $x = h$ are above the lower boundary $y + p\,x = s - 1$ of this band. In other words, for $0 < i < h$, the head (i, d_i) is the highest lattice point below $y + p\,x = s$ on the line $x = i$, and the source at $x = 0$ and sink at $x = h$ are weakly above the highest lattice points on these respective lines.

Translating the picture vertically, we can assume that $e_h = 0$, i.e., the sink is on the x-axis. A nest π in this den is then a lattice path from $(0, d_0)$ to $(h, 0)$ that stays weakly below the line $y + p\,x = s$ except possibly for an initial south run along the y-axis and the final east step along the x-axis.

Let $r = s/p$ be the x-intercept of the line $y + p\,x = s$. If $r \geq h$, the sink $(h, 0)$ is weakly below the line, $(h, 1)$ is above the line, and every lattice path weakly below the line that ends at $(h, 0)$ ends with an east step. If $r < h$, the sink $(h, 0)$ is above the line, the point $(h - 1, 0)$ is below the line, so $h = \lfloor r \rfloor + 1$, and deleting the final east step in each nest π gives a path ending at $(\lfloor r \rfloor, 0)$. In either case, nests π correspond one-to-one with lattice paths from $(0, d_0)$ to $(\min(\lfloor r \rfloor, h), 0)$ that stay weakly below $y + p\,x = s$ except for a possible south run along the y-axis.

In this picture, $a(\pi)$ is the area between the path π and the highest such path π^0. In [6, Definition 5.4.1], $\operatorname{dinv}_p(\pi)$ was defined to be the number of p-balanced hooks whose arm and leg end on the path π; but this was also shown in the proof of [6, Proposition 5.4.4] to coincide with $\operatorname{dinv}_p(\pi)$ as defined here.

Since there is only one path, we have $g_i = 1$ for all i, so the root sets for the Catalanimal H in Theorem 3.5.1 are $R_q = R_t = R_+$ and $R_{qt} = [R_+, R_+]$, and its weight is given by $\lambda_i = d_{i-1} - e_i$, which is the number of south steps on $x = i - 1$ in the highest path π^0 under the line $y + p\,x = s$. If $r < h$, so $h = \lfloor r \rfloor + 1$, then $\lambda = \mathbf{b}$, where $\mathbf{b} = (b_1, \ldots, b_h)$ is as in [6, Theorem 5.5.1], H is equal to the function $\mathcal{H}_{\mathbf{b}}(\mathbf{z})$ in [6, Definition 3.7.1] (which is a Catalanimal), and we have $\psi_{\widehat{\Gamma}}(H) = D_{\mathbf{b}}$ in the notation of [6, §3.6] ($D_{\mathbf{b}}$ is a *Negut element* in $\mathcal{E}^+$). In general, $h \leq \lfloor r \rfloor + 1$, and H is a possibly shorter Catalanimal than $\mathcal{H}_{\mathbf{b}}$, with $\psi_{\widehat{\Gamma}}(H) = D_\lambda$, where λ is obtained by dropping some trailing zeroes from $\mathbf{b}$. By [6, Lemma 3.6.2], the trailing zeroes do not matter and we have $D_{\mathbf{b}} \cdot 1 = D_\lambda \cdot 1$.

In the single path case, Corollary 3.5.2 now reduces to the generalized shuffle theorem [6, Theorem 5.5.1] for paths under the line $y + p\,x = s$, including the more general version in [6, Remark 5.5.2] for paths extended along the y-axis.

4. The Loehr-Warrington conjecture and its (m, n) extension

In this section we construct a den such that the associated Catalanimal H in Theorem 3.5.1 represents the element $s_\mu[-MX^{m,n}]$ in the Schiffmann algebra, for any partition μ and coprime integers $m, n > 0$. In the case $n = 1$, Corollary 3.5.2 then yields a combinatorial formula for $\nabla^m s_\mu$, which we will show agrees with the one conjectured by Loehr and Warrington in [15].

4.1. LW dens.

Given a partition μ of length $\ell(\mu)$, we define the following for use in constructing its associated dens.

$$h(\mu) = \mu_1 + \ell(\mu) - 1 = \text{largest hook length in } \mu, \tag{49}$$

$$\delta_i(\mu) = \chi(\mu_1 - 1 - i \text{ is the content of the last box in some row of } \mu), \tag{50}$$

$$\varepsilon_i(\mu) = \chi(i \geq \mu_1) \tag{51}$$

for $0 \leq i \leq h(\mu)$, where $\chi(P) = 1$ if P is true, 0 if P is false. Note that the contents of boxes in μ range from $\mu_1 - h(\mu)$ to $\mu_1 - 1$. Since μ is a partition, for $i \leq h(\mu)$ we have $\varepsilon_i(\mu) = 1$ if and only if $\mu_1 - i$ is the content of the first box in some row of μ.

Definition 4.1.1. The *LW den* associated to a partition μ and a pair of coprime positive integers m, n is the den $(h, p, \mathbf{d}, \mathbf{e})$ defined as follows:

$$h = m\,h(\mu); \quad p = n/m - \epsilon, \tag{52}$$

where $\epsilon > 0$ is small;

$$
\begin{aligned}
d_i = e_i &= \lfloor n\, h(\mu) - i\, n/m \rfloor \text{ for } i \text{ not a multiple of } m, \\
d_{jm} &= n\, h(\mu) - n\, j + \delta_j(\mu) - 1, \\
e_{jm} &= n\, h(\mu) - n\, j + \varepsilon_j(\mu) - 1,
\end{aligned} \tag{53}
$$

for $0 \le i \le m\, h(\mu)$, $0 \le j \le h(\mu)$.

We will see below (Proposition 4.1.4) that the LW den is indeed a den.

Remark 4.1.2. (i) The p dependence in all constructions involving dens and nests comes from comparisons between p and finitely many rational numbers r. By saying that $\epsilon > 0$ is 'small' in (52), we mean that such comparisons give $p < r$ if $n/m \le r$, and $p > r$ if $n/m > r$.

(ii) Since $d_0 = n\, h(\mu) = (n/m)h$, the line $y + (n/m)x = d_0$ meets the coordinate axes at $(h, 0)$ and $(0, d_0)$. The quantity $\lfloor n\, h(\mu) - (n/m)i \rfloor$ is the y-coordinate of the highest lattice point on the line $x = i$ weakly below $y + (n/m)x = d_0$.

If i is not a multiple of m, the highest lattice point is strictly below $y + (n/m)x = d_0$, and $(i, d_i) = (i, e_i)$ is this point. For $i = jm$, the highest lattice point is on the line $y + (n/m)x = d_0$. In this case, (i, d_i) and (i, e_i) are each either on the line or one unit below, depending on the values of $\delta_j(\mu)$ and $\varepsilon_j(\mu)$. It follows that the sources and sinks in the LW den all lie on the bounding line $y + (n/m)x = d_0$, and that paths in every nest in this den stay weakly below the bounding line.

For $i = jm$, if $\delta_j(\mu) = \varepsilon_j(\mu) = 0$, the head and foot $(i, d_i) = (i, e_i)$ are both one unit below the bounding line. In this case there is no source or sink at $x = jm$, and the condition that paths in a nest lie weakly below the heads forbids the paths from touching the bounding line at $x = jm$. The other heads impose no further conditions. Every system of nested east end paths from the sources to the sinks, which stay weakly below the bounding line and do not touch it at the forbidden points, is therefore a nest in the LW den.

Example 4.1.3. For the partition $\mu = (4, 3, 3, 3, 2)$ we have $h(\mu) = 8$, $\boldsymbol{\delta} = (\delta_0(\mu), \dots, \delta_8(\mu)) = (1, 0, 1, 1, 1, 0, 1, 0, 0)$, and $\boldsymbol{\varepsilon} = (\varepsilon_0(\mu), \dots, \varepsilon_8(\mu)) = (0, 0, 0, 0, 1, 1, 1, 1, 1)$. In the simplest LW den, for $m = n = 1$, we get $\mathbf{d}$, $\mathbf{e}$ by adding $\boldsymbol{\delta} - \mathbf{1}$, $\boldsymbol{\varepsilon} - \mathbf{1}$ to the vector $(8, 7, \dots, 1, 0)$, giving

$$
(h, p, \mathbf{d}, \mathbf{e}) = \big(8,\ 1 - \epsilon,\ (8, 6, 6, 5, 4, 2, 2, 0, -1),\ (7, 6, 5, 4, 4, 3, 2, 1, 0)\big). \tag{54}
$$

For $(m, n) = (2, 1)$ we interleave the above $\mathbf{d}$ and $\mathbf{e}$ with the sequence $(7, 6, 5, 4, 3, 2, 1, 0)$ of numbers $\lfloor 8 - i/2 \rfloor$ for i odd, to get

$$
\begin{aligned}
(h, p, \mathbf{d}, \mathbf{e}) = \Big(16,\ \frac{1}{2} - \epsilon,\ &(8, 7, 6, 6, 6, 5, 5, 4, 4, 3, 2, 2, 2, 1, 0, 0, -1), \\
&(7, 7, 6, 6, 5, 5, 4, 4, 4, 3, 3, 2, 2, 1, 1, 0, 0)\Big).
\end{aligned} \tag{55}
$$

These two dens are plotted in Figure 4. In each den, we also display the nest π^0 such that $a(\pi^0) = 0$, for later reference.

Proposition 4.1.4. *The data $(h, p, \mathbf{d}, \mathbf{e})$ in Definition 4.1.1 define a valid den.*

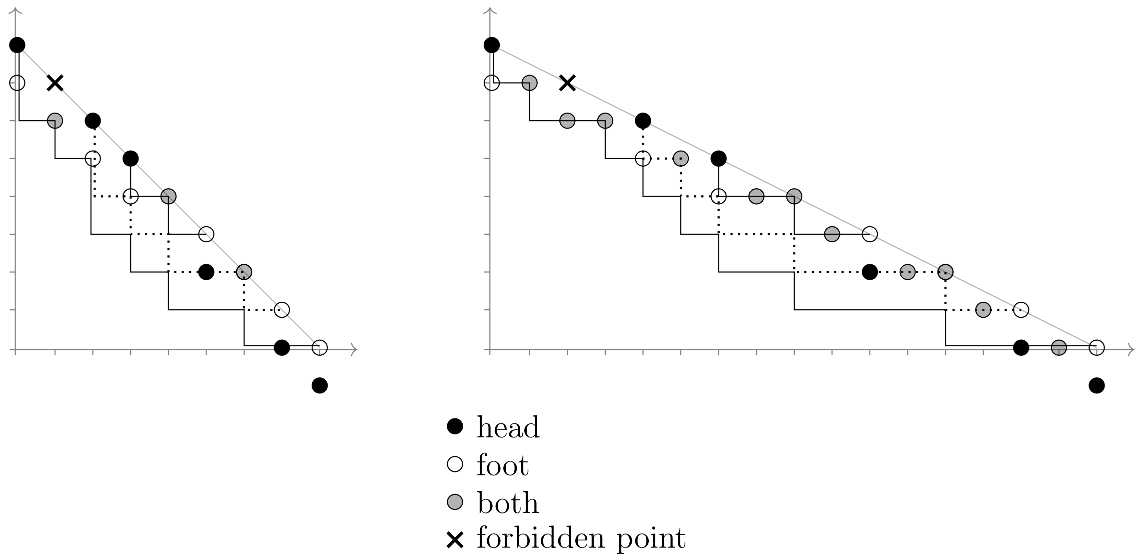


FIGURE 4. LW dens for $\mu = (4,3,3,3,2)$ and $m = n = 1$ (left), $m = 2$, $n = 1$ (right), with the highest nest π^0 in each den. Sources and sinks are the points on the bounding line that are heads or feet but not both.

Proof. For any μ we have $\delta_0(\mu) = 1$, $\varepsilon_0(\mu) = 0$, $\delta_{h(\mu)}(\mu) = 0$ (because $\mu_1 - 1 - h(\mu)$ is less than the content of any box in μ) and $\varepsilon_{h(\mu)}(\mu) = 1$. We also have $\sum_{i=0}^{h(\mu)} \delta_i(\mu) = \sum_{i=0}^{h(\mu)} \varepsilon_i(\mu) = \ell(\mu)$. This implies $d_0 - e_0 = 1 = e_h - d_h$ and $\sum_{i=0}^{h} d_i = \sum_{i=0}^{h} e_i$, verifying condition (23).

All heads and feet are between or on the lines $y + (n/m)x = d_0$ and $y + (n/m)x = d_0 - 1$, which implies $(d_i - d_j + 1)/(j - i) \geq n/m \geq (e_i - e_j - 1)/(j - i)$ for all $i < j$. Since all feet on the lower line are left of all feet on the upper line, the second inequality is strict. This implies conditions (21–22) for $p = n/m - \epsilon$. □

4.2. **An** (m, n) **Loehr-Warrington formula.** Our next theorem is a combinatorial formula for the symmetric function $s_\mu[-X^{m,n}] \cdot 1$. As we will see, this generalizes both the (km, kn)-shuffle theorem [3, 18], when $s_\mu = e_k$, and the Loehr-Warrington conjecture [15], when $n = 1$.

Lemma 4.2.1. *Let* $(h, p, \mathbf{d}, \mathbf{e})$ *be the LW den for* μ *and* m, n*.*

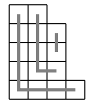
(i) With notation as in Definition 2.4.1, the sequence $\mathbf{g}$ *in* (24) *is the same as* $\gamma((\mu^\circ)^m)$*, or equivalently* $\gamma(\mu^\circ)$ *with each entry repeated* m *times.*

(ii) The Catalanimal H *in Theorem 3.5.1 is the same as the opposite Schur Catalanimal* $H^{m,n}_{(\mu^\circ)^m}$ *in Definition 2.4.1 and Theorem 2.4.2.*

Proof. The sources in the LW den have x-coordinate $i\,m$ for $\delta_i(\mu) = 1$, $\varepsilon_i(\mu) = 0$, and the sinks have x-coordinate $i\,m$ for $\delta_i(\mu) = 0$, $\varepsilon_i(\mu) = 1$. By Remark 3.1.4 (iii), the associated sequence $\mathbf{g}$ is therefore obtained by repeating each entry of the sequence $\gamma = (\gamma_1, \ldots, \gamma_{h(\mu)})$ m times, where $\gamma_{i+1} - \gamma_i = \delta_i(\mu) - \varepsilon_i(\mu)$ for $i = 0, \ldots, h(\mu) - 1$, and we set $\gamma_0 = 1$. To prove (i), we need to show that $\gamma = \gamma(\mu^\circ)$, which is the reverse of $\gamma(\mu)$.

Consider the decomposition of μ into hooks with corner on the main diagonal, as shown here for the partition $\mu = (4, 3, 3, 3, 2)$ in Example 4.1.3.

(56)



From the definitions we see that $\delta_i(\mu) = 1$ and $\varepsilon_i(\mu) = 0$ if and only if $\mu_1 - i - 1$ is the content of the rightmost box in the arm of a main diagonal hook. This is also the condition on $i \in [0, h(\mu)]$ to have $\gamma_{i+1}(\mu^\circ) - \gamma_i(\mu^\circ) = 1$, if we define $\gamma_i(\mu^\circ) = 0$ for $i < 1$ or $i > h(\mu)$.

Noting that a diagonal of content $c \leq 0$ contains the top of the leg in a main diagonal hook if and only if the diagonal of content $c-1$ does not contain the box at the end of a row of μ, we see that $\delta_i(\mu) = 0$ and $\varepsilon_i(\mu) = 1$ if and only if $\mu_1 - i$ is the content of the highest box in a leg. This is the condition on i to have $\gamma_{i+1}(\mu^\circ) - \gamma_i(\mu^\circ) = -1$.

When neither of these two conditions hold, we have $\gamma_{i+1}(\mu^\circ) = \gamma_i(\mu^\circ)$. Hence, $\gamma_{i+1}(\mu^\circ) - \gamma_i(\mu^\circ) = \delta_i(\mu) - \varepsilon_i(\mu)$ for all i, and therefore $\gamma = \gamma(\mu^\circ)$, proving (i).

Part (i) implies that the Catalanimals H and $H^{m,n}_{(\mu^\circ)^m}$ have the same root sets. The weight λ for $H^{m,n}_{(\mu^\circ)^m}$, given by (19), is constant on blocks of lengths $g_1, \ldots, g_h$. We need to verify that its value on the i-th block is $d_{i-1} - e_i$.

We can write (53) in the form

$$d_i = n\, h(\mu) - \lceil i\, n/m \rceil + \chi(m|i)(\delta_{i/m}(\mu) - 1) \tag{57}$$

$$e_i = n\, h(\mu) - \lceil i\, n/m \rceil + \chi(m|i)(\varepsilon_{i/m}(\mu) - 1). \tag{58}$$

Combining (57–58) with the definition of $\mathbf{b}(m, n)$ in (15) gives

$$d_{i-1} - e_i = \mathbf{b}(m, n)_{\mathrm{mod}_m(i)} + \chi(m|i-1)(\delta_{(i-1)/m}(\mu) - 1) + \chi(m|i)(1 - \varepsilon_{i/m}(\mu)). \tag{59}$$

To compare this with (19), note that the content c on the i-th diagonal from northwest to southeast in $(\mu^\circ)^m$ has $c \equiv i \pmod{m}$ by construction, so the term $\mathbf{b}(m, n)_{\mathrm{mod}_m(i)}$ here agrees with $\mathbf{b}(m, n)_{\mathrm{mod}_m(c)}$ in (19).

The i-th diagonal in $(\mu^\circ)^m$ always contains the first box in a row of $(\mu^\circ)^m$ if $i \not\equiv 1 \pmod{m}$. If $i = j\, m + 1$, the i-th diagonal contains the first box in a row of $(\mu^\circ)^m$ if and only if the $(j+1)$-st diagonal in μ° contains the first box in a row of μ°, that is, if and only if $\delta_j(\mu) = 1$. Hence,

$$\begin{aligned} &\chi(i\text{-th diagonal in } (\mu^\circ)^m \text{ contains the first box in a row}) \\ &\qquad = 1 + \chi(m|i-1)(\delta_{(i-1)/m}(\mu) - 1). \end{aligned} \tag{60}$$

Similarly, the i-th diagonal in $(\mu^\circ)^m$ always contains the last box in a row of $(\mu^\circ)^m$ if $i \not\equiv 0 \pmod{m}$. If $i = j\, m$, the i-th diagonal contains the last box in a row of $(\mu^\circ)^m$ if and only if the j-th diagonal in μ° contains the last box in a row of μ°, that is, if and only if $j \geq \mu_1$. Hence,

$$\begin{aligned} &\chi(i\text{-th diagonal in } (\mu^\circ)^m \text{ contains the last box in a row}) \\ &\qquad = 1 + \chi(m|i)(\varepsilon_{i/m}(\mu) - 1). \end{aligned} \tag{61}$$

Using (60) and (61) we see that (59) agrees with (19). □

Our (m,n) Loehr-Warrington formula is now given by the following theorem, which follows from Proposition 2.3.3, Theorem 2.4.2, Corollary 3.5.2, and Lemma 4.2.1.

Theorem 4.2.2. *For every partition μ and coprime positive integers m, n, we have the identity*

$$s_\mu[-MX^{m,n}] \cdot 1 = (-1)^{p(\mu)}(q\,t)^{p(\mu)+m\,n'(\gamma(\mu))} \sum_{\pi} t^{a(\pi)} q^{\mathrm{dinv}_p(\pi)} \omega\, \mathcal{G}_{\boldsymbol{\nu}(\pi)}(X; q^{-1}), \tag{62}$$

where $p(\mu)$ is the magic number (not to be confused with $p = n/m - \epsilon$), $n'(\gamma(\mu))$ is given by (18)*, and the sum is over nests π in the LW den for μ and m, n.*

For $n = 1$, the expression in (62) *is equal to $\nabla^m s_\mu(X)$.*

Remark 4.2.3. In the case $\mu = (1^k)$, giving $s_\mu = e_k$, we have $p(\mu) = n'(\gamma(\mu)) = 0$. As explained in [6, §6], Theorem 4.2.2 is then equivalent to the (km, kn)-shuffle theorem conjectured by Bergeron et al. [3] and proven by Mellit [18]. This case is also included in the path case discussed in §3.6, above.

4.3. **Comparison with the original Loehr-Warrington formula.** Loehr and Warrington [15, Conjecture 2.4] conjectured a combinatorial formula for $\nabla^m s_\mu$, which in their notation reads

$$\nabla^m s_\mu(X) = \mathrm{sgn}(\mu) \sum_{(G,R) \in \mathrm{LNDP}^m_\mu} t^{\mathrm{area}(G,R)} q^{\mathrm{dinv}(G,R)} \mathbf{x}_R. \tag{63}$$

We prove their conjecture (after correcting a mistake in the definition of $\mathrm{dinv}(G,R)$—see below) by verifying that this formula agrees with Theorem 4.2.2 for $n = 1$, when we express the latter in terms of labeled nests as in (43), namely

$$\nabla^m s_\mu(X) = (-1)^{p(\mu)}(q\,t)^{p(\mu)+m\,n'(\gamma(\mu))} \sum_{\pi, N} t^{a(\pi)} q^{\mathrm{dinv}_p(\pi) - \mathrm{inv}(N)} \mathbf{x}^N. \tag{64}$$

We now work out some of the details needed to see that (63) and (64) agree.

The notation $(G,R) \in \mathrm{LNDP}^m_\mu$ stands for a system of labeled nested m-Dyck paths which, after reflecting about a horizontal line, corresponds to a pair (π, N), where π is a nest in the LW den for μ and m, n with $n = 1$, and N is a negative labeling of π as in (38).

In more detail, the proof of Lemma 4.2.1 shows that the sources and sinks in the LW den are located on the bounding line at positions $x = jm$, where $\mu_1 - 1 - j$ is the content of the box at either end of a main diagonal hook in μ. Loehr and Warrington start and end their Dyck paths at corresponding positions constructed from a dissection of μ into border strips ([15, Fig. 1]). To see that these agree, observe that the heights, widths and content ranges of the border strips for any μ match those of the main diagonal hooks, as in this example:

(65) 

This also shows that the statistic $\mathrm{spin}(\mu^*) = \mathrm{adj}(\mu)$ in [15, §2.2], defined as the sum $\sum_B(\mathrm{width(B)} - 1)$ over border strips B in μ, is equal to the magic number $p(\mu)$, since the latter is the number of boxes strictly below the main diagonal, and thus the sum of the arm lengths of the main diagonal hooks.

Loehr and Warrington introduce additional Dyck paths of length zero which serve to forbid the other paths from touching the bounding line at certain points. These correspond to the points where nests in the LW den cannot touch the bounding line, as in Remark 4.1.2 (ii). The rest of Remark 4.1.2 (ii) then shows that nests π in the LW den correspond to systems of m-Dyck paths G for $(G, R) \in \mathrm{LNDP}^m_\mu$.

Deciphering the notation in [15] further, the indices (a, u) of entries $g_a^{(u)}$, $r_a^{(u)}$ in (G, R) correspond to pairs (S, i), where S is a south step in a path π_i in the nest π. When (a, u) corresponds to (S, i), the index u is a strictly increasing function of i. The integer $g_a^{(u)}$ is equal to m times the vertical distance from the north endpoint of S to the bounding line $y + (1/m)x = d_0$ for the LW den. The labels $r_a^{(u)}$ are subject to the same conditions as our labels $N(S, i)$.

To finish reconciling (63) with (64), we need to show the following.

Lemma 4.3.1. *For (G, R) corresponding to (π, N), the combinatorial statistics in* [15] *are related to ours by*

$$\operatorname{sgn}(\mu) \stackrel{\text{def}}{=} (-1)^{\operatorname{spin}(\mu^*)} = (-1)^{p(\mu)} \tag{66}$$

$$\operatorname{area}(G, R) = p(\mu) + m\, n'(\gamma(\mu)) + a(\pi) \tag{67}$$

$$\operatorname{dinv}(G, R) = p(\mu) + m\, n'(\gamma(\mu)) + \operatorname{dinv}_p(\pi) - \operatorname{inv}(N), \tag{68}$$

where $p = 1/m - \epsilon$.

Proof. For (66), we already observed that $\operatorname{spin}(\mu^*) = p(\mu)$.

For (67), the definition of $\operatorname{area}(G, R)$ corresponds to the sum of the areas $|\rho/\pi_i|$, where ρ is the highest lattice path from $(0, d_0) = (0, h(\mu))$ to $(h, 0) = (m\, h(\mu), 0)$ that stays weakly below the bounding line $y + (1/m)x = d_0$. To verify (67) it therefore suffices to show that for the nest π^0 with $a(\pi^0) = 0$, we have $\sum_i |\rho/\pi^0_i| = p(\mu) + mn'(\gamma(\mu))$.

Now $\sum_i |\rho/\pi^0_i| = \sum_E a_E$, where the sum is over east steps E in π^0, and a_E is the vertical distance between E and the east step weakly above it in ρ. Recall that g_k as defined in (24) is the number of east steps from $x = k-1$ to $x = k$ in any nest. For $k = jm$ with $1 \leq j < \mu_1$, the point at $x = k$ on the bounding line for the LW den is either a source or a forbidden point, and thus is not the right endpoint of an east step in π^0. For these values of k, the numbers a_E for east steps E from $x = k - 1$ to $x = k$ are $1, \ldots, g_k$. For other values of k the a_E are $0, 1, \ldots, g_k - 1$. It may be instructive to verify this with the examples in Figure 4.

It follows that

$$\sum_i |\rho/\pi^0_i| = \sum_E a_E = \sum_{k=1}^{mh(\mu)} \binom{g_k}{2} + \sum_{j=1}^{\mu_1 - 1} g_{jm}. \tag{69}$$

The first sum on the right is $m\, n'(\gamma(\mu))$ by Lemma 4.2.1(i). Since g_{jm} is the number of boxes on the diagonal of content $\mu_1 - j$ in μ, the second sum is $p(\mu)$.

For (68), the statistic $\operatorname{dinv}(G, R)$ is defined by [15, (11)], except that the expression $\chi(a \leq b)$ in the middle sum there should read $\chi((a < b) \vee ((a = b) \wedge (u < v)))$, as in the last sum. After correcting this mistake and exchanging indices (a, u) and (b, v) in the first two sums, we can rewrite [15, (11)] in the form

$$
\begin{aligned}
(70)\quad \operatorname{dinv}(G,R) = {} & \\
& \operatorname{adj}(\mu) + \sum \chi(0 < g_b^{(v)} - g_a^{(u)} \leq m)\, \chi(a \geq b)\, \chi(r_a^{(u)} < r_b^{(v)}) \\
& + \sum \chi(0 \leq g_b^{(v)} - g_a^{(u)} < m)\, \chi(a < b)\, \chi(r_a^{(u)} < r_b^{(v)}) \\
& + \sum \chi(0 \leq g_b^{(v)} - g_a^{(u)} < m)\, \chi(a = b \wedge u < v)\, \chi(r_a^{(u)} < r_b^{(v)}) \\
& + \sum |(g_b^{(v)} - g_a^{(u)} + [0, m-1]) \cap [1, m-1]|\, \chi((a > b) \vee (a = b \wedge u > v)),
\end{aligned}
$$

where the sums are over all pairs of valid indices (a,u), (b,v). Let (S,i), (S',j) be the south steps on paths in π corresponding to (a,u) and (b,v). The condition

$$
(71)\qquad \big((0 < g_b^{(v)} - g_a^{(u)} \leq m) \wedge (a \geq b)\big) \vee \big((0 \leq g_b^{(v)} - g_a^{(u)} < m) \wedge (a < b)\big)
$$

holds if and only if (S,i) and (S',j) form an attacking pair in $\mathbf{S}(\pi)$, as defined in §3.4, for $p = 1/m - \epsilon$. We leave it as an exercise for the reader to verify this, with the hint that if $g_b^{(v)} = g_a^{(u)}$, then S is strictly to the left of S' if and only if $a < b$, while if $g_b^{(v)} - g_a^{(u)} = m$, then S is strictly to the right of S' if and only if $a \geq b$. The first two sums in (70) therefore count attacking pairs that do not contribute to $\operatorname{inv}(N)$, that is, they add up to

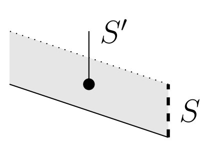
$$
(72)\qquad \mathcal{A}(\pi) - \operatorname{inv}(N),
$$

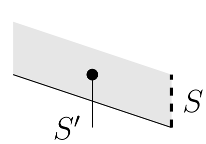
where $\mathcal{A}(\pi)$ is the number of attacking pairs in $\mathbf{S}(\pi)$, or equivalently in $\boldsymbol{\nu}(\pi)$.

Turning to the third sum in (70), If $a = b$ and $g_a^{(u)} \leq g_b^{(v)}$, then S and S' have the same y-coordinates, with S' weakly to the left of S. If S' is strictly to the left of S, then nesting implies $v < u$. Hence, if $u < v$, then $S = S'$ is a shared south step on paths π_i and π_j with $i < j$. In this case, the conditions on the labeling imply $r_a^{(u)} < r_b^{(v)}$. The third sum in (70) therefore reduces to the number of pairs $\{(S,i),(S,j)\}$ of shared south steps in π. We denote this number by $\operatorname{ss}(\pi)$.

Now we consider the last sum in (70). The terms with $a = b$, $g_a^{(u)} = g_b^{(v)}$ and $u > v$ contribute $m-1$ for each pair of shared south steps, giving $(m-1)\operatorname{ss}(\pi)$.

The remaining terms are zero if $|g_b^{(v)} - g_a^{(u)}| \geq m$. Otherwise, they correspond to pairs (S,i), (S',j) with $S \neq S'$ and some line of slope $-1/m$ passing through the interiors of both S and S'. If S' is to the right of S, this implies $b \geq a$, with $u < v$ if $a = b$, by nesting. Hence, S' is to the left of S. To describe the contribution from such a term geometrically, let B_S be the region bounded on the right by S and above and below by lines of slope $-1/m$ through the endpoints of S, with open boundaries above and on the right, and a closed boundary below. Then S' has an endpoint in B_S, and the contribution from the corresponding term is given by

(73) $r-1$, for 

r, for 

,

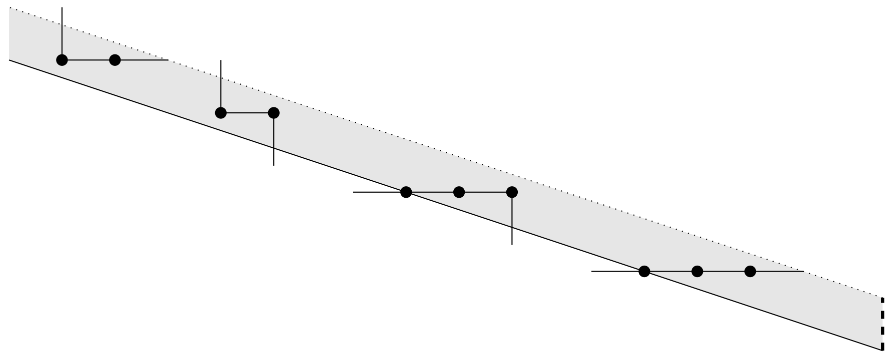


FIGURE 5. The possibilities for components of the intersection of a path π_j with B_S, illustrated with $m = 3$, $p = 1/3 - \epsilon$.

where $r = m - |g_b^{(v)} - g_a^{(u)}|$ is the integer such that the line segment $S' \cap B_S$ has length r/m. Note that we require the upper endpoint of S' in the second picture to be in the interior of B_S (although if it were on the boundary, it would contribute zero anyway).

Let $\delta(\pi)$ denote the sum of the contributions in (73) for all pairs (S, i), (S', j) with S and S' positioned as shown. Then the last sum in (70) is equal to $\delta(\pi) + (m-1)\,\mathrm{ss}(\pi)$.

Combining $\mathrm{adj}(\mu) = p(\mu)$ with the above, we obtain

$$\mathrm{dinv}(G, R) = p(\mu) + m\,\mathrm{ss}(\pi) + \delta(\pi) + \mathcal{A}(\pi) - \mathrm{inv}(N). \tag{74}$$

The following lemma now completes the verification of (68). □

Lemma 4.3.2. *For $p = 1/m - \epsilon$ and π a nest in the LW den for μ and m, n with $n = 1$, we have*

$$\mathrm{dinv}_p(\pi) = m\ \mathrm{ss}(\pi) - m\,n'(\gamma(\mu)) + \mathcal{A}(\pi) + \delta(\pi). \tag{75}$$

Proof. We evaluate $\mathrm{dinv}_p(\pi) - \mathcal{A}(\pi) - \delta(\pi)$.

For $p = 1/m - \epsilon$, each unordered pair $\{(S, i), (S', j)\}$ with S, S' positioned as in (73) forms an attacking pair in $\mathbf{S}(\pi)$ when ordered with the smaller of $\hat{c}(S)$, $\hat{c}(S')$ first. Every attacking pair has this form, so $\mathcal{A}(\pi)$ is the number of such unordered pairs.

Given a pair (S, i) in $\mathbf{S}(\pi)$ and a path π_j, consider the connected components of $\pi_j \cap B_S$. These are of four possible types, depicted in Figure 5, depending on whether they enter and exit B_S along the upper or the lower boundaries.

For $p = 1/m - \epsilon$, $\mathrm{dinv}_p(\pi)$ counts pairs (S, i), (P, j), where P is a point on π_j that lies in B_S. Using this, the description of $\mathcal{A}(\pi)$ above, and the definition of $\delta(\pi)$, one can check that each component of $\pi_j \cap B_S$ contributes $-m$ to $\mathrm{dinv}_p(\pi) - \mathcal{A}(\pi) - \delta(\pi)$ for components that cross B_S from top to bottom, m for components that cross from bottom to top, and zero for components of the other two types.

Since all sources are on the bounding line, the leftmost component (if any) of $\pi_j \cap B_S$ enters from above. From left to right, the components that cross B_S alternate between the second and fourth types shown in Figure 5, possibly with components of the other types in between.

If $i \le j$, so π_j is equal to or nested above π_i, or if $i > j$ and the paths π_i and π_j share the south step S, then $\pi_j \cap B_S$ is either empty, or its last component exits B_S on the upper boundary. In this case, crossings from top to bottom cancel those from bottom to top, giving a net contribution of zero to $\operatorname{dinv}_p(\pi) - \mathcal{A}(\pi) - \delta(\pi)$.

Otherwise, if $i > j$ and S is not on π_j, there is one more crossing from top to bottom than from bottom to top, for a net contribution of $-m$. This shows that $\operatorname{dinv}_p(\pi) - \mathcal{A}(\pi) - \delta(\pi)$ is equal to $-m$ times the number of tuples (S, i, j) with S on π_i and $i > j$, plus $m\,\mathrm{ss}(\pi)$. The number of such tuples (S, i, j) is the sum over all $i > j$ of the number of south steps on π_i.

For the LW den, the number of south steps on π_i is the length of the i-th main diagonal hook in μ, from southwest to northeast, and there are $i-1$ indices $j < i$. If we write $i-1$ in each box on the i-th main diagonal hook, the sum of these numbers is therefore the number of tuples (S, i, j). But these numbers sum to $\binom{\gamma}{2}$ on a content diagonal of length γ, so the sum of them all is $n'(\gamma(\mu))$. This gives

$$\operatorname{dinv}_p(\pi) - \mathcal{A}(\pi) - \delta(\pi) = m\ \mathrm{ss}(\pi) - m\, n'(\gamma(\mu)), \tag{76}$$

as desired. □

5. LLT series and semi-symmetric Hall-Littlewood polynomials

Using a strategy similar to that in [6], we will prove Theorem 3.5.1 by taking the polynomial part of an infinite series identity between the full Catalanimal on the left hand side and a sum of LLT series on the right. LLT series associated to a reductive group G and Levi subgroup $L \subseteq G$ were defined in [12]; for $G = GL_l$, they are series versions of the LLT polynomials $\mathcal{G}_{\boldsymbol{\nu}}(X; q)$. Since [12] is unpublished, in [6, §4] we gave a self-contained treatment (with some improvements) of the case when $G = GL_l$ and $L = T$ is the torus, which corresponds to $\mathcal{G}_{\boldsymbol{\nu}}(X; q)$ when $\boldsymbol{\nu}$ is a tuple of single-row skew diagrams. Building on [6], we extend this here to the case of any Levi subgroup $L = GL_{r_1} \times \cdots \times GL_{r_k} \subseteq GL_l$ and any $\mathcal{G}_{\boldsymbol{\nu}}(X; q)$.

5.1. **Hecke algebra and root system preliminaries.** We set $\Bbbk = \mathbb{Q}(q, t)$ as in §2. The algebra of Laurent polynomials $\Bbbk[z_1^{\pm 1}, \ldots, z_l^{\pm}]$ is the group algebra of the weight lattice of GL_l, with monomials $\mathbf{z}^\lambda = z_1^{\lambda_1} \cdots z_l^{\lambda_l}$ corresponding to weights $\lambda \in \mathbb{Z}^l$. As in §2, we denote the roots by $\alpha_{ij} = \varepsilon_i - \varepsilon_j$. For simple roots we abbreviate this to $\alpha_i = \alpha_{i,i+1}$.

The Weyl group S_l acts by permuting the variables, with Coxeter generators (simple reflections) given by the transpositions $s_i = (i \leftrightarrow i+1)$. Given $w \in S_l$, we let $\ell(w)$ denote the length of a reduced factorization $w = s_{i_1} \cdots s_{i_\ell}$; this is also the number of inversions in w. The longest element of S_l or any finite Coxeter group is denoted w_0. Usually it will be clear from the context what group w_0 belongs to; otherwise we indicate it with a superscript such as w_0^l.

The Demazure-Lusztig operators

$$T_i = q\, s_i + (1-q) \frac{1}{1 - \mathbf{z}^{-\alpha_i}} (s_i - 1) \tag{77}$$

generate an action of the Hecke algebra $\mathcal{H}(S_l)$ on $\Bbbk[z_1^{\pm 1}, \ldots, z_l^{\pm}]$. We have normalized them so that $(T_i - q)(T_i + 1) = 0$. As usual, for $w \in S_l$, we set $T_w = T_{i_1} \cdots T_{i_\ell}$, where $w = s_{i_1} \cdots s_{i_\ell}$ is a reduced factorization.

We use an overbar $\overline{\cdot}$ to signify inverting the variables q, t, z_i; thus

$$\overline{T_i} = q^{-1}\, s_i + (1-q^{-1}) \frac{1}{1-\mathbf{z}^{\alpha_i}} (s_i - 1). \tag{78}$$

One can then check that

$$\overline{T_i} = T_i^{-1}; \quad \text{hence} \quad \overline{T_w} = T_{w^{-1}}^{-1}. \tag{79}$$

Given a composition $\mathbf{r} = (r_1, \ldots, r_k)$ of l, we denote the corresponding Levi subgroup of GL_l and its Weyl group (which is a Young subgroup of S_l) by

$$GL_{\mathbf{r}} = GL_{r_1} \times \cdots \times GL_{r_k} \subseteq GL_l, \tag{80}$$

$$S_{\mathbf{r}} = S_{r_1} \times \cdots \times S_{r_k} \subseteq S_l. \tag{81}$$

Note that $R_+(GL_{\mathbf{r}})$ is the set of positive roots $\alpha_{ij} \in R_+(GL_l)$ such that i, j are in the same block of the partition of $[l]$ into intervals of lengths $r_1, \ldots, r_k$.

Here we have implicitly taken $\mathbf{r}$ to be a strict composition with all entries $r_i > 0$. If $\mathbf{r}$ is a *weak composition* with entries $r_i = 0$ allowed, we define $GL_{\mathbf{r}} = GL_{\mathbf{s}}$ where $\mathbf{s} = (r_{i_1}, \ldots, r_{i_j})$ is the subsequence of non-zero entries in $\mathbf{r}$.

The Levi subgroup $GL_{\mathbf{r}}$ has the same weight lattice $\mathbb{Z}^l$ as GL_l. A weight λ is dominant (resp. dominant and regular) for $GL_{\mathbf{r}}$ iff $\lambda_i \geq \lambda_{i+1}$ (resp. $\lambda_i > \lambda_{i+1}$) for all i such that $\alpha_i \in R_+(GL_{\mathbf{r}})$, or equivalently such that $s_i \in S_{\mathbf{r}}$. We denote the set of dominant weights by $X^+(GL_{\mathbf{r}})$ and the set of regular dominant weights by $X^{++}(GL_{\mathbf{r}})$.

We write $\rho_{\mathbf{r}}$ (or just ρ if $\mathbf{r} = (l)$) for a weight such that

$$\langle \alpha_i^\vee, \rho_{\mathbf{r}} \rangle \overset{\text{def}}{=} (\rho_{\mathbf{r}})_i - (\rho_{\mathbf{r}})_{i+1} = 1 \quad \text{for every simple root } \alpha_i \in R_+(GL_{\mathbf{r}}). \tag{82}$$

Such a weight is unique up to adding an $S_{\mathbf{r}}$ invariant weight. When we use this notation, the choice of $\rho_{\mathbf{r}}$ will either be fixed or make no difference.

We define the semi-symmetrization operator for $GL_{\mathbf{r}}$ by means of the following lemma.

Lemma 5.1.1. *For any composition* $\mathbf{r}$ *of* l*, there is an operator* $\boldsymbol{\delta}_{\mathbf{r}}$ *on* $\Bbbk[z_1^{\pm 1}, \ldots, z_l^{\pm 1}]$ *given by either of two equivalent formulas*

$$\boldsymbol{\delta}_{\mathbf{r}} = \frac{1}{\prod_{\alpha \in R_+(GL_{\mathbf{r}})} (1 - \mathbf{z}^{-\alpha})} \sum_{w \in S_{\mathbf{r}}} (-1)^{\ell(w)} w \tag{83}$$

$$= \frac{q^{\ell(w_0^{\mathbf{r}})}}{\prod_{\alpha \in R_+(GL_{\mathbf{r}})} (1 - q\, \mathbf{z}^{-\alpha})} \sum_{w \in S_{\mathbf{r}}} (-q)^{-\ell(w)} T_w. \tag{84}$$

Proof. Fixing a choice of $\rho_{\mathbf{r}}$, formula (83) can also be written

$$\boldsymbol{\delta}_{\mathbf{r}} = \mathbf{z}^{\rho_{\mathbf{r}}}\, \boldsymbol{\sigma}_{\mathbf{r}}\, \mathbf{z}^{-\rho_{\mathbf{r}}}, \tag{85}$$

where $\boldsymbol{\sigma}_{\mathbf{r}}$ is the Weyl symmetrization operator for $GL_{\mathbf{r}}$ in (10). In particular, the operator $\boldsymbol{\delta}_{\mathbf{r}}$ defined by (83) acts on $\Bbbk[z_1^{\pm 1}, \ldots, z_l^{\pm 1}]$. Let $\boldsymbol{\delta}_{\mathbf{r}}^q$ denote the operator defined by (84). We are to prove that $\boldsymbol{\delta}_{\mathbf{r}}^q = \boldsymbol{\delta}_{\mathbf{r}}$.

Let $V = \Bbbk[z_1^{\pm 1}, \ldots, z_l^{\pm 1}]$. For each i such that $s_i \in S_{\mathbf{r}}$, let $W_i = V^{s_i} = \{f \in V \mid s_i f = f\}$ be the subspace of s_i invariant functions. Let $A_{\mathbf{r}} = \sum_{w \in S_{\mathbf{r}}} (-1)^{\ell(w)} w$ denote the antisymmetrization operator. It is a general property of Coxeter group representations that $\sum_i W_i$ is an invariant subspace and that $V / \sum_i W_i$ carries the sign representation, thus every $w \in S_{\mathbf{r}}$ acts on $V/\sum_i W_i$ as $(-1)^{\ell(w)}$, and $A_{\mathbf{r}}$ acts as the scalar $|S_{\mathbf{r}}|$. This implies, first, that $A_{\mathbf{r}}$ is surjective on $V/\sum_i W_i$, that is, the space of all antisymmetric functions spans $V/\sum_i W_i$, and, second, that if $V' \subseteq V$ is any subspace such that $A_{\mathbf{r}}V'$ is the space of all antisymmetric functions, then V' spans $V/\sum_i W_i$, that is, $V = V' + \sum_i W_i$. In particular, this holds with $V' = \mathbf{z}^{\rho_{\mathbf{r}}} V^{S_{\mathbf{r}}}$, since every antisymmetric $f \in V$ has the form $f = A_{\mathbf{r}}\, \mathbf{z}^{\rho_{\mathbf{r}}} g$, where $g \in V$ is $S_{\mathbf{r}}$ invariant.

Since $T_i f = q\, f$ for $f \in W_i$, both operators $\boldsymbol{\delta}_{\mathbf{r}}$ and $\boldsymbol{\delta}^q_{\mathbf{r}}$ kill the subspaces W_i. Both operators also commute with multiplication by any $S_{\mathbf{r}}$ invariant function g. Hence, to prove $\boldsymbol{\delta}^q_{\mathbf{r}} = \boldsymbol{\delta}_{\mathbf{r}}$ it suffices to show that $\boldsymbol{\delta}^q_{\mathbf{r}} \mathbf{z}^{\rho_{\mathbf{r}}} = \boldsymbol{\delta}_{\mathbf{r}} \mathbf{z}^{\rho_{\mathbf{r}}}$. By (85), we have $\boldsymbol{\delta}_{\mathbf{r}} \mathbf{z}^{\rho_{\mathbf{r}}} = \mathbf{z}^{\rho_{\mathbf{r}}}$. Meanwhile, $\boldsymbol{\delta}^q_{\mathbf{r}} \mathbf{z}^{\rho_{\mathbf{r}}} = \mathbf{z}^{\rho_{\mathbf{r}}}$ is equivalent to the well-known identity [1]

$$\sum_{w \in S_{\mathbf{r}}} (-q)^{-\ell(w)} T_w\, \mathbf{z}^{\rho_{\mathbf{r}}} = q^{-\ell(w_0^{\mathbf{r}})}\, \mathbf{z}^{\rho_{\mathbf{r}}} \prod_{\alpha \in R_+(GL_{\mathbf{r}})} (1 - q\, \mathbf{z}^{-\alpha}). \tag{86}$$ □

Remark 5.1.2. For i such that $s_i \in S_{\mathbf{r}}$, the Hecke algebra antisymmetrization operator $A^q_{\mathbf{r}} = \sum_{w \in S_{\mathbf{r}}} (-q)^{-\ell(w)} T_w$ in (86) can be factored in the form $B \cdot (T_i - q)$ for an element B of the Hecke algebra, and therefore satisfies $A^q_{\mathbf{r}} T_i = -A^q_{\mathbf{r}}$. More generally, this implies $A^q_{\mathbf{r}} T_w^{\pm 1} = (-1)^{\ell(w)} A^q_{\mathbf{r}}$, for all $w \in S_{\mathbf{r}}$, and consequently also $\boldsymbol{\delta}_{\mathbf{r}}\, T_w^{\pm 1} = (-1)^{\ell(w)} \boldsymbol{\delta}_{\mathbf{r}}$.

5.2. **Semi-symmetric Hall-Littlewood polynomials.** As in [6, (72)], we define non-symmetric Hall-Littlewood polynomials for GL_l by

$$E_\lambda(\mathbf{z}; q) = q^{-\ell(w)} T_w\, \mathbf{z}^{\lambda_+}, \tag{87}$$

where $\lambda = w(\lambda_+)$ with $w \in S_l$ and λ_+ dominant. If λ_+ has non-trivial stabilizer, w is not unique, but the formula does not depend on the choice. For $\sigma \in S_l$, we also define twisted versions

$$E^\sigma_\lambda(\mathbf{z}; q) = q^{|\mathrm{Inv}(\sigma^{-1}) \cap \mathrm{Inv}(\lambda + \epsilon\rho)|}\, \overline{T_\sigma}\, E_{\sigma^{-1}(\lambda)}(\mathbf{z}; q) \tag{88}$$

$$F^\sigma_\lambda(\mathbf{z}; q) = \overline{E^{\sigma w_0}_{-\lambda}(\mathbf{z}; q)}, \tag{89}$$

where $\mathrm{Inv}((a_1, \ldots, a_l)) = \{(i < j) \mid a_i > a_j\}$, and ϵ is small, so $\mathrm{Inv}(\sigma^{-1}) = \{(i < j) \mid \sigma^{-1}(i) > \sigma^{-1}(j)\}$ and $\mathrm{Inv}(\lambda + \epsilon\rho) = \{(i < j) \mid \lambda_i \geq \lambda_j\}$. Note that for $\sigma = 1$, $E^\sigma_\lambda(\mathbf{z}; q)$ reduces to the untwisted $E_\lambda(\mathbf{z}; q)$.

From the definition, one can verify the recurrence [6, (76)]

$$E^\sigma_\lambda = \begin{cases} q^{-\chi(\lambda_i \leq \lambda_{i+1})}\, T_i\, E^{s_i\sigma}_{s_i\lambda}, & s_i\sigma > \sigma, \\ q^{\chi(\lambda_i \geq \lambda_{i+1})}\, T_i^{-1}\, E^{s_i\sigma}_{s_i\lambda}, & s_i\sigma < \sigma. \end{cases} \tag{90}$$

The E^σ_λ are determined by this recurrence and the initial condition $E^\sigma_\lambda = \mathbf{z}^\lambda$ for all σ if λ is a dominant weight.

Definition 5.2.1. (i) Given a composition $\mathbf{r} = (r_1, \ldots, r_k)$ and a permutation $\sigma \in S_k$, let $l = |\mathbf{r}| = r_1 + \cdots + r_k$ and define $\widehat{\sigma} \in S_l$ to be the permutation that carries intervals of lengths $\sigma^{-1}(\mathbf{r}) = (r_{\sigma(1)}, \ldots, r_{\sigma(k)})$ to intervals of lengths $\mathbf{r}$ in the order given by σ. More precisely,

$$\widehat{\sigma}(r_{\sigma(1)} + \cdots + r_{\sigma(i-1)} + j) = r_1 + \cdots + r_{\sigma(i)-1} + j \tag{91}$$

for $i = 1, \ldots, k$ and $j = 1, \ldots, r_{\sigma(i)}$.

(ii) Given $\mathbf{r}$ and σ as in (i), and $\mu \in X^{++}(GL_{\mathbf{r}})$ a regular dominant weight for $GL_{\mathbf{r}}$, we define *semi-symmetric Hall-Littlewood polynomials*

$$E^{\sigma}_{\mathbf{r},\mu}(\mathbf{z};q) = \boldsymbol{\delta}_{\mathbf{r}}\, E^{\widehat{\sigma}}_{\mu}(\mathbf{z};q), \qquad F^{\sigma}_{\mathbf{r},\mu}(\mathbf{z};q) = \boldsymbol{\delta}_{\mathbf{r}}\, F^{\widehat{\sigma}}_{\mu}(\mathbf{z};q). \tag{92}$$

Remark 5.2.2. (i) For simplicity, we have suppressed $\mathbf{r}$ from the notation for $\widehat{\sigma}$. Although this should not usually cause confusion, one should note that $\widehat{\sigma}^{\,-1}$ is not given by $\widehat{\sigma^{-1}}$ for the same $\mathbf{r}$, but rather by $\widehat{\sigma^{-1}}$ defined relative to the composition $\sigma^{-1}(\mathbf{r})$.

For example, if $\sigma = (2,3,1)$ in one-line notation, and $\mathbf{r} = (1,4,3)$, then $\sigma^{-1} = (3,1,2)$ and $\sigma^{-1}(\mathbf{r}) = (4,3,1)$. Partitioning the set $\{1,\ldots,8\}$ into intervals $I_1 = \{1\}$, $I_2 = \{2,3,4,5\}$, $I_3 = \{6,7,8\}$ of lengths $\mathbf{r}$ and intervals $J_1 = \{1,2,3,4\}$, $J_2 = \{5,6,7\}$, $J_3 = \{8\}$ of lengths $\sigma^{-1}(\mathbf{r})$, the permutation $\widehat{\sigma} = (2,3,4,5,6,7,8,1)$ carries J_1, J_2, J_3 to $I_2 = I_{\sigma(1)}$, $I_3 = I_{\sigma(2)}$, $I_1 = I_{\sigma(3)}$. The inverse permutation $\widehat{\sigma}^{-1}$ that carries I_1, I_2, I_3 back to J_3, J_1, J_2 is $\widehat{\sigma^{-1}}$ defined relative to the composition $\sigma^{-1}(\mathbf{r}) = (4,3,1)$ that gives the intervals J_i, rather than the original composition $\mathbf{r} = (1,4,3)$ that gave the intervals I_i.

(ii) If $\mathbf{r}$ is a weak composition, so $GL_{\mathbf{r}} = GL_{\mathbf{s}}$ where $\mathbf{s} = (r_{i_1},\ldots,r_{i_j})$ is the subsequence of non-zero entries in $\mathbf{r}$, then the definitions of $E^{\sigma}_{\mathbf{r},\mu}(\mathbf{z};q)$ and $F^{\sigma}_{\mathbf{r},\mu}(\mathbf{z};q)$ reduce to $E^{\sigma}_{\mathbf{r},\mu}(\mathbf{z};q) = E^{\tau}_{\mathbf{s},\mu}(\mathbf{z};q)$, $F^{\sigma}_{\mathbf{r},\mu}(\mathbf{z};q) = F^{\tau}_{\mathbf{s},\mu}(\mathbf{z};q)$ where $\tau \in S_j$ is the permutation such that $\tau^{-1}(1),\ldots,\tau^{-1}(j)$ are in the same relative order as $\sigma^{-1}(i_1),\ldots,\sigma^{-1}(i_j)$.

Example 5.2.3. (i) At $q = 1$, we have $E^{\sigma}_{\mu}(\mathbf{z};1) = \mathbf{z}^{\mu}$ for any σ. Using (85), it follows that

$$\mathbf{z}^{-\rho_{\mathbf{r}}} E^{\sigma}_{\mathbf{r},\lambda+\rho_{\mathbf{r}}}(\mathbf{z};1) = \chi_{\lambda}(GL_{\mathbf{r}}) \tag{93}$$

if $\lambda \in X^{+}(GL_{\mathbf{r}})$ is a dominant weight for $GL_{\mathbf{r}}$, where $\chi_{\lambda}(GL_{\mathbf{r}})$ is the irreducible $GL_{\mathbf{r}}$ character with highest weight λ.

(ii) For $\mathbf{r} = (l)$, we must have $\sigma = 1 \in S_1$. If λ is a dominant weight for GL_l, then $E_{\lambda+\rho}(\mathbf{z};q) = \mathbf{z}^{\lambda+\rho}$, which implies $\mathbf{z}^{-\rho} E^{1}_{(l),\lambda+\rho}(\mathbf{z};q) = \chi_{\lambda}$, independent of q. Note that this is quite different from the usual symmetric Hall-Littlewood polynomial $P_{\lambda}(\mathbf{z};q)$.

We develop some initial properties of these polynomials for later use, beginning with expressions for the semi-symmetric polynomials $F^{\sigma}_{\mathbf{r}}(\mathbf{z};q)$ in terms of $E^{\sigma}_{\mathbf{r}}(\mathbf{z};q)$.

Lemma 5.2.4. *Given $\mathbf{r} = (r_1,\ldots,r_k)$, $\sigma \in S_k$ and $\mu \in X^{++}(GL_{\mathbf{r}})$, we have*

$$F^{\sigma}_{\mathbf{r},\mu}(\mathbf{z},q) = \mathbf{z}^{\rho_{\mathbf{r}} - w_0^{\mathbf{r}}(\rho_{\mathbf{r}})}\, \overline{E^{\sigma w_0^k}_{\mathbf{r},-w_0^{\mathbf{r}}(\mu)}(\mathbf{z};q)} \tag{94}$$

$$= \mathbf{z}^{\rho_{\mathbf{r}} - w_0^{\mathbf{r}}(\rho_{\mathbf{r}})} w_0\big(E^{w_0^k\sigma}_{w_0^k(\mathbf{r}),w_0 w_0^{\mathbf{r}}(\mu)}(\mathbf{z};q^{-1})\big). \tag{95}$$

Note that $\rho_{\mathbf{r}} - w_0^{\mathbf{r}}(\rho_{\mathbf{r}}) = \sum_{\alpha\in R_+(GL_{\mathbf{r}})} \alpha$ does not depend on the choice of $\rho_{\mathbf{r}}$.

Proof. From (83) we find $\overline{\boldsymbol{\delta}_{\mathbf{r}}} = (-1)^{\ell(w_0^{\mathbf{r}})}\mathbf{z}^{w_0^{\mathbf{r}}(\rho_{\mathbf{r}})-\rho_{\mathbf{r}}}\boldsymbol{\delta}_{\mathbf{r}}$. Applying $\boldsymbol{\delta}_{\mathbf{r}}$ on both sides of the definition $F^{\widehat{\sigma}}_{\mu} = \overline{E^{\widehat{\sigma}w_0}_{-\mu}}$ therefore gives

$$F^{\sigma}_{\mathbf{r},\mu} = (-1)^{\ell(w_0^{\mathbf{r}})}\mathbf{z}^{\rho_{\mathbf{r}} - w_0^{\mathbf{r}}(\rho_{\mathbf{r}})}\,\overline{\boldsymbol{\delta}_{\mathbf{r}} E^{\widehat{\sigma}w_0}_{-\mu}}. \tag{96}$$

Since $\widehat{\sigma}w_0$ is maximal in its coset $S_{\mathbf{r}}\widehat{\sigma}w_0$, and $-\mu$ is regular and anti-dominant for $GL_{\mathbf{r}}$, it follows by repeated use of the recurrence (90) that $E^{\widehat{\sigma}w_0}_{-\mu} = \overline{T_{w_0^{\mathbf{r}}}} E^{w_0^{\mathbf{r}}\widehat{\sigma}w_0}_{-w_0^{\mathbf{r}}(\mu)}$. Now $\widehat{\sigma w_0^k} = w_0^{\mathbf{r}}\widehat{\sigma}w_0$, and $\boldsymbol{\delta}_{\mathbf{r}} T_i^{\pm 1} = -\boldsymbol{\delta}_{\mathbf{r}}$ for $s_i \in S_{\mathbf{r}}$ by Remark 5.1.2, hence $\overline{\boldsymbol{\delta}_{\mathbf{r}} T_{w_0^{\mathbf{r}}}} = (-1)^{\ell(w_0^{\mathbf{r}})} \boldsymbol{\delta}_{\mathbf{r}}$. Combining these, the right hand side of (96) becomes $\mathbf{z}^{\rho_{\mathbf{r}} - w_0^{\mathbf{r}}(\rho_{\mathbf{r}})} \boldsymbol{\delta}_{\mathbf{r}} E_{-w_0^{\mathbf{r}}(\mu)}^{\widehat{\sigma w_0^k}} = \mathbf{z}^{\rho_{\mathbf{r}} - w_0^{\mathbf{r}}(\rho_{\mathbf{r}})} \overline{E_{\mathbf{r}, -w_0^{\mathbf{r}}(\mu)}^{\sigma w_0^k}}$, giving (94).

To prove (95), we use the identity

$$\overline{E_{-\lambda}^{\tau}(\mathbf{z}; q)} = w_0\big(E_{w_0(\lambda)}^{w_0 \tau w_0}(\mathbf{z}; q^{-1})\big). \tag{97}$$

Like the equivalent identity [5, (109)], one can prove (97) by verifying that after applying w_0, both sides are characterized by the recurrence (90) with the variables reversed and inverted. We also observe that (83) implies $\overline{\boldsymbol{\delta}_{\mathbf{r}}} w_0 = w_0 \boldsymbol{\delta}_{\mathbf{r}'}$, where $\mathbf{r}' = w_0^k(\mathbf{r}) = (r_k, \ldots, r_1)$, and that $w_0(\widehat{\sigma w_0^k}) w_0 = \widehat{w_0^k \sigma}$, where $\widehat{w_0^k \sigma}$ is defined with respect to $\mathbf{r}'$ rather than $\mathbf{r}$. Then (95) follows from (94) and

$$\overline{E_{\mathbf{r}, -w_0^{\mathbf{r}}(\mu)}^{\sigma w_0^k}(\mathbf{z}; q)} = \overline{\boldsymbol{\delta}_{\mathbf{r}} E_{-w_0^{\mathbf{r}}(\mu)}^{\widehat{\sigma w_0^k}}(\mathbf{z}; q)} = w_0\big(\boldsymbol{\delta}_{\mathbf{r}'} E_{w_0 w_0^{\mathbf{r}}(\mu)}^{\widehat{w_0^k \sigma}}(\mathbf{z}; q^{-1})\big) = w_0\big(E_{\mathbf{r}', w_0 w_0^{\mathbf{r}}(\mu)}^{w_0^k \sigma}(\mathbf{z}; q^{-1})\big)\,. \tag{98}$$

□

By [6, Corollary 4.3.1], E_{λ}^{σ} has the monic and triangular form

$$E_{\lambda}^{\sigma}(\mathbf{z}; q) = \mathbf{z}^{\lambda} + \sum_{\mu < \lambda} c_{\lambda, \mu}(q)\, \mathbf{z}^{\mu} \tag{99}$$

with respect to a suitable partial ordering $<$ on the weight lattice $\mathbb{Z}^l$. If μ is a regular weight for $GL_{\mathbf{r}}$, then $\mathbf{z}^{-\rho_{\mathbf{r}}} \boldsymbol{\delta}_{\mathbf{r}}(\mathbf{z}^{\mu}) = \pm \chi_{\nu}(GL_{\mathbf{r}})$, where $\nu + \rho_{\mathbf{r}}$ is the unique dominant (and regular) weight in the orbit $S_{\mathbf{r}} \cdot \mu$. The ordering $<$ has the property that the dominant weight for $GL_{\mathbf{r}}$ in any $S_{\mathbf{r}}$ orbit is the unique minimal element in that orbit; hence $\nu + \rho_{\mathbf{r}} \leq \mu$ in this case. If μ is not regular for $GL_{\mathbf{r}}$, then $\boldsymbol{\delta}_{\mathbf{r}}(\mathbf{z}^{\mu}) = 0$.

For every $\lambda \in X^+(GL_{\mathbf{r}})$ it now follows from (99) that $\mathbf{z}^{-\rho_{\mathbf{r}}} E_{\mathbf{r}, \lambda + \rho_{\mathbf{r}}}^{\sigma}$ has the form

$$\mathbf{z}^{-\rho_{\mathbf{r}}} E_{\mathbf{r}, \lambda + \rho_{\mathbf{r}}}^{\sigma}(\mathbf{z}; q) = \chi_{\lambda}(GL_{\mathbf{r}}) + \sum_{\substack{\nu \in X^+(GL_{\mathbf{r}}) \\ \nu + \rho_{\mathbf{r}} < \lambda + \rho_{\mathbf{r}}}} a_{\lambda, \nu}(q)\, \chi_{\nu}(GL_{\mathbf{r}}). \tag{100}$$

More precisely, given the choice of $\rho_{\mathbf{r}}$, (100) holds for all $\lambda \in X^+(GL_{\mathbf{r}})$, although the coefficients $a_{\lambda, \nu}(q)$ and the set of weights ν that occur depend on $\rho_{\mathbf{r}}$. In particular, for any fixed choice of $\rho_{\mathbf{r}}$, it follows that

$$\{\mathbf{z}^{-\rho_{\mathbf{r}}} E_{\mathbf{r}, \mu}^{\sigma} \mid \mu \in X^{++}(GL_{\mathbf{r}})\} \tag{101}$$

is a basis of $\Bbbk[z_1^{\pm 1}, \ldots, z_l^{\pm}]^{S_{\mathbf{r}}}$. Then (94) implies that

$$\{\mathbf{z}^{-\rho_{\mathbf{r}}} F_{\mathbf{r}, \mu}^{\sigma} \mid \mu \in X^{++}(GL_{\mathbf{r}})\} \tag{102}$$

is also a basis. Note that $\Bbbk[z_1^{\pm 1}, \ldots, z_l^{\pm 1}]^{S_{\mathbf{r}}}$ is the algebra of virtual $GL_{\mathbf{r}}$ characters with coefficients in $\Bbbk$.

Remark 5.2.5. The coefficients $c_{\lambda, \mu}(q)$ in (99), and therefore also $a_{\lambda, \nu}(q)$ in (100), are in $\mathbb{Z}[q^{-1}]$. Hence, $E_{\mathbf{r}, \mu}^{\sigma}(\mathbf{z}; q)$ and $F_{\mathbf{r}, \mu}^{\sigma}(\mathbf{z}; q)$ have coefficients in $\mathbb{Z}[q^{-1}]$ and $\mathbb{Z}[q]$, respectively, and (101–102) are free module bases over these coefficient rings in place of $\Bbbk$.

Lemma 5.2.6. *Given* $\mathbf{r}$, σ *and* $\widehat{\sigma}$ *as in Definition 5.2.1,* $\mu \in X^{++}(GL_{\mathbf{r}})$*, and* $w \in S_{\mathbf{r}}$*, we have*

$$E^{\widehat{\sigma}}_{w(\mu)} = q^{-\ell(w)} T_w E^{\widehat{\sigma}}_{\mu}, \tag{103}$$

$$F^{\widehat{\sigma}}_{w(\mu)} = \overline{T_w} F^{\widehat{\sigma}}_{\mu}. \tag{104}$$

Proof. The case $\sigma = 1$ of (103) follows from the definition (87) of E_μ, because $\mu \in X^{++}(GL_{\mathbf{r}})$ implies that if $v \in S_l$ is such that $\mu = v(\mu_+)$, and $w \in S_{\mathbf{r}}$, then $w \cdot v$ is a reduced factorization of wv. For general σ, let $v = \widehat{\sigma}^{-1} w \widehat{\sigma}$, and note that $v \in S_{\sigma^{-1}(\mathbf{r})}$. Since $\widehat{\sigma}^{-1}$ is minimal in both of its cosets $S_{\sigma^{-1}(\mathbf{r})}\, \widehat{\sigma}^{-1}$ and $\widehat{\sigma}^{-1}\, S_{\mathbf{r}}$, each side of $\widehat{\sigma}^{-1} \cdot w = v \cdot \widehat{\sigma}^{-1}$ is a reduced factorization, giving $T_{\widehat{\sigma}^{-1}} T_w = T_v T_{\widehat{\sigma}^{-1}}$, or equivalently $T_w \overline{T_{\widehat{\sigma}}} = \overline{T_{\widehat{\sigma}}} T_v$. We also have $\ell(v) = \ell(w)$. Then

$$\begin{aligned} q^{-\ell(w)}\, T_w\, E^{\widehat{\sigma}}_{\mu} = q^{e-\ell(w)}\, T_w \overline{T_{\widehat{\sigma}}}\, E_{\widehat{\sigma}^{-1}(\mu)} &= q^e\, \overline{T_{\widehat{\sigma}}}\, q^{-\ell(v)}\, T_v\, E_{\widehat{\sigma}^{-1}(\mu)} \\ &= q^e\, \overline{T_{\widehat{\sigma}}}\, E_{v\widehat{\sigma}^{-1}(\mu)} = q^e\, \overline{T_{\widehat{\sigma}}}\, E_{\widehat{\sigma}^{-1} w(\mu)}, \end{aligned} \tag{105}$$

where $e = |\mathrm{Inv}(\widehat{\sigma}^{-1}) \cap \mathrm{Inv}(\mu + \epsilon\rho)|$. For the third equality, we used $\widehat{\sigma}^{-1}(\mu) \in X^{++}(GL_{\sigma^{-1}(\mathbf{r})})$ and the $\sigma = 1$ case. Since $\widehat{\sigma}^{-1}$ is increasing on intervals of lengths $r_1, \ldots, r_k$, we have $e = |\mathrm{Inv}(\widehat{\sigma}^{-1}) \cap \mathrm{Inv}(w(\mu) + \epsilon\rho)|$ for any $w \in S_{\mathbf{r}}$. The last formula in (105) therefore reduces to $E^{\widehat{\sigma}}_{w(\mu)}$.

For (104), let $u = w_0 \widehat{\sigma}^{-1} w \widehat{\sigma} w_0$ (here $w_0 = w_0^l \in S_l$), and note that $u \in S_{w_0^k \sigma^{-1}(\mathbf{r})}$. Since $w_0 \widehat{\sigma}^{-1}$ is maximal in both of its cosets $w_0 \widehat{\sigma}^{-1} S_{\mathbf{r}}$ and $S_{w_0^k \sigma^{-1}(\mathbf{r})} w_0 \widehat{\sigma}^{-1}$, the factorizations $(w_0 \widehat{\sigma}^{-1} w^{-1}) \cdot w$ and $u \cdot (u^{-1} w_0 \widehat{\sigma}^{-1})$ are reduced, giving $T_w T^{-1}_{w_0 \widehat{\sigma}^{-1}} = T^{-1}_{w_0 \widehat{\sigma}^{-1} w^{-1}} = T^{-1}_{u^{-1} w_0 \widehat{\sigma}^{-1}} = T^{-1}_{w_0 \widehat{\sigma}^{-1}} T_u$, or equivalently $T_w \overline{T_{\widehat{\sigma} w_0}} = \overline{T_{\widehat{\sigma} w_0}} T_u$. Then

$$\begin{aligned} T_w\, E^{\widehat{\sigma} w_0}_{-\mu} = q^d\, T_w\, \overline{T_{\widehat{\sigma} w_0}}\, E_{-w_0 \widehat{\sigma}^{-1}(\mu)} &= q^d\, \overline{T_{\widehat{\sigma} w_0}}\, T_u\, E_{-w_0 \widehat{\sigma}^{-1}(\mu)} \\ &= q^{d+\ell(u)}\, \overline{T_{\widehat{\sigma} w_0}}\, E_{-u w_0 \widehat{\sigma}^{-1}(\mu)} = q^{d+\ell(u)}\, \overline{T_{\widehat{\sigma} w_0}}\, E_{-w_0 \widehat{\sigma}^{-1} w(\mu)}, \end{aligned} \tag{106}$$

where $d = |\mathrm{Inv}(w_0 \widehat{\sigma}^{-1}) \cap \mathrm{Inv}(-\mu + \epsilon\rho)|$. For the third equality, we used the $\sigma = 1$ case of (103) with $-w_0 \widehat{\sigma}^{-1}(\mu) \in X^{++}(GL_{w_0^k \sigma^{-1}(\mathbf{r})})$. Since $w_0 \widehat{\sigma}^{-1}$ is decreasing on intervals of lengths $r_1, \ldots, r_k$, changing $-\mu$ to $-w(\mu)$ in the formula for d creates $\ell(w) = \ell(u)$ new inversions, giving $d + \ell(u) = |\mathrm{Inv}(w_0 \widehat{\sigma}^{-1}) \cap \mathrm{Inv}(-w(\mu) + \epsilon\rho)|$. The last formula in (106) now reduces to $E^{\widehat{\sigma} w_0}_{-w(\mu)}$, showing that $E^{\widehat{\sigma} w_0}_{-w(\mu)} = T_w\, E^{\widehat{\sigma} w_0}_{-\mu}$. Taking $\overline{\,\cdot\,}$ on both sides gives (104). □

Finally, since $\boldsymbol{\delta}_{\mathbf{r}}$ and T_i commute with multiplication by $z_1 \cdots z_l$, we have the identities

$$(z_1 \cdots z_l)^m\, E^{\sigma}_{\mathbf{r},\mu}(\mathbf{z}; q) = E^{\sigma}_{\mathbf{r},(m^l)+\mu}(\mathbf{z}; q), \qquad (z_1 \cdots z_l)^m\, F^{\sigma}_{\mathbf{r},\mu}(\mathbf{z}; q) = F^{\sigma}_{\mathbf{r},(m^l)+\mu}(\mathbf{z}; q). \tag{107}$$

5.3. **Orthogonality.** For $f \in \Bbbk[z_1^{\pm 1}, \ldots, z_l^{\pm 1}]^{S_{\mathbf{r}}}$, let $\langle 1_{GL_{\mathbf{r}}} \rangle\, f$ denote the coefficient of the trivial character when f is expanded in terms of irreducible $GL_{\mathbf{r}}$ characters. The formula

$$\langle 1_{GL_{\mathbf{r}}} \rangle\, \boldsymbol{\sigma}_{\mathbf{r}}(f) = \langle \mathbf{z}^0 \rangle\, f \cdot \prod\nolimits_{\alpha \in R_+(GL_{\mathbf{r}})} (1 - \mathbf{z}^\alpha) \tag{108}$$

holds for any $f \in \Bbbk[z_1^{\pm 1}, \ldots, z_l^{\pm 1}]$, as can be verified by reducing to the case $f = \mathbf{z}^\lambda$, for which both sides become $(-1)^{\ell(w)}$ if $\lambda + \rho_{\mathbf{r}} = w(\rho_{\mathbf{r}})$ for $w \in S_{\mathbf{r}}$, or zero otherwise. If f is $S_{\mathbf{r}}$

invariant, then $\boldsymbol{\sigma}_{\mathbf{r}}(f) = f\boldsymbol{\sigma}_{\mathbf{r}}(1) = f$, and we obtain

$$\langle 1_{GL_{\mathbf{r}}} \rangle\, f = \langle \mathbf{z}^0 \rangle\, f \cdot \prod\nolimits_{\alpha \in R_+(GL_{\mathbf{r}})} (1 - \mathbf{z}^{\alpha}). \tag{109}$$

We define a symmetric inner product on $\Bbbk[z_1^{\pm 1}, \ldots, z_l^{\pm 1}]^{S_{\mathbf{r}}}$ by

$$\langle f, g \rangle_q^{\mathbf{r}} \overset{\text{def}}{=} \langle 1_{GL_{\mathbf{r}}} \rangle\, f\, g \prod\nolimits_{\alpha \in R_+ \setminus R_+(GL_{\mathbf{r}})} \frac{1 - \mathbf{z}^{\alpha}}{1 - q^{-1}\mathbf{z}^{\alpha}}. \tag{110}$$

For $\mathbf{r} = (1^l)$ this reduces to the inner product

$$\langle f, g \rangle_q = \langle \mathbf{z}^0 \rangle\, f\, g \prod\nolimits_{\alpha \in R_+} \frac{1 - \mathbf{z}^{\alpha}}{1 - q^{-1}\mathbf{z}^{\alpha}} \tag{111}$$

in [6, Proposition 4.3.2]. For general $\mathbf{r}$, (109) implies that $\langle f, g \rangle_q^{\mathbf{r}}$ and $\langle f, g \rangle_q$ are related by

$$\langle f, g \rangle_q^{\mathbf{r}} = \langle\, f,\, g \cdot \prod\nolimits_{\alpha \in R_+(GL_{\mathbf{r}})} (1 - q^{-1}\mathbf{z}^{\alpha})\, \rangle_q. \tag{112}$$

We remark that (110) and (111) are to be interpreted by expanding the factors $(1 - q^{-1}\mathbf{z}^{\alpha})^{-1} = 1 + q^{-1}\mathbf{z}^{\alpha} + \cdots$ as geometric series, yielding a power series in q^{-1} over $\mathbb{Z}[z_1^{\pm}, \ldots, z_l^{\pm 1}]$, which is $S_{\mathbf{r}}$ invariant in the case of (110). Upon taking the coefficient $\langle 1_{GL_{\mathbf{r}}} \rangle$ or $\langle \mathbf{z}^0 \rangle$, only finitely many terms in the series survive. In the case of (111), this is clear, and for (110) it then follows from (112).

Proposition 5.3.1. *Given* $\mathbf{r}$ *and* σ *as in Definition 5.2.1, and any choice of* $\rho_{\mathbf{r}}$ *as in* (82), *we have dual bases of* $\Bbbk[z_1^{\pm 1}, \ldots, z_l^{\pm 1}]^{S_{\mathbf{r}}}$

$$\Big\langle \mathbf{z}^{-\rho_{\mathbf{r}}} E^{\sigma}_{\mathbf{r},\lambda},\, \overline{\mathbf{z}^{-\rho_{\mathbf{r}}} F^{\sigma}_{\mathbf{r},\mu}} \Big\rangle_q^{\mathbf{r}} = \delta_{\lambda,\mu} \qquad (\lambda, \mu \in X^{++}(GL_{\mathbf{r}})). \tag{113}$$

Proof. We have already seen that the two sets $\{\mathbf{z}^{-\rho_{\mathbf{r}}} E^{\sigma}_{\mathbf{r},\lambda} \mid \lambda \in X^{++}(GL_{\mathbf{r}})\}$ and $\{\overline{\mathbf{z}^{-\rho_{\mathbf{r}}} F^{\sigma}_{\mathbf{r},\mu}} \mid \mu \in X^{++}(GL_{\mathbf{r}})\}$ are bases of $\Bbbk[z_1^{\pm 1}, \ldots, z_l^{\pm 1}]^{S_{\mathbf{r}}}$. We also note that $\langle \mathbf{z}^{-\rho_{\mathbf{r}}} E^{\sigma}_{\mathbf{r},\lambda}, \overline{\mathbf{z}^{-\rho_{\mathbf{r}}} F^{\sigma}_{\mathbf{r},\mu}} \rangle_q^{\mathbf{r}}$ is independent of the choice of $\rho_{\mathbf{r}}$, since $\langle f, g \rangle_q^{\mathbf{r}}$ is a function of fg. The case $\mathbf{r} = (1^l)$ is [6, Proposition 4.3.2]. We will use this result to prove the general case.

For $\mu \in X^{++}(GL_{\mathbf{r}})$, define

$$E^{\sigma,-}_{\mathbf{r},\mu} = \big(q^{-\ell(w_0^{\mathbf{r}})} \prod\nolimits_{\alpha \in R_+(GL_{\mathbf{r}})} (1 - q\mathbf{z}^{-\alpha})\big) E^{\sigma}_{\mathbf{r},\mu} = \sum_{w \in S_{\mathbf{r}}} (-q)^{-\ell(w)} T_w E^{\widehat{\sigma}}_{\mu}, \tag{114}$$

$$F^{\sigma,-}_{\mathbf{r},\mu} = \big(q^{-\ell(w_0^{\mathbf{r}})} \prod\nolimits_{\alpha \in R_+(GL_{\mathbf{r}})} (1 - q\mathbf{z}^{-\alpha})\big) F^{\sigma}_{\mathbf{r},\mu} = \sum_{w \in S_{\mathbf{r}}} (-q)^{-\ell(w)} T_w F^{\widehat{\sigma}}_{\mu}. \tag{115}$$

Using Lemma 5.2.6, we can also write

$$E^{\sigma,-}_{\mathbf{r},\mu} = \sum_{w \in S_{\mathbf{r}}} (-1)^{\ell(w)} E^{\widehat{\sigma}}_{w(\mu)}. \tag{116}$$

Defining $A^q_{\mathbf{r}} = \sum_{w \in S_{\mathbf{r}}} (-q)^{-\ell(w)} T_w$ as in Remark 5.1.2, we have

$$\sum_{w \in S_{\mathbf{r}}} (-q)^{-\ell(w)} T_w = A^q_{\mathbf{r}} = A^q_{\mathbf{r}} \cdot (-1)^{\ell(w_0^{\mathbf{r}})} T^{-1}_{w_0^{\mathbf{r}}} = q^{-\ell(w_0^{\mathbf{r}})} \sum_{w \in S_{\mathbf{r}}} (-q)^{\ell(w)} \overline{T_w}, \tag{117}$$

and Lemma 5.2.6 then implies

$$F^{\sigma,-}_{\mathbf{r},\mu} = q^{-\ell(w_0^{\mathbf{r}})} \sum_{w\in S_{\mathbf{r}}} (-q)^{\ell(w)} F^{\widehat{\sigma}}_{w(\mu)}. \tag{118}$$

Macdonald's identity [16, Theorem (2.8)] for $GL_{\mathbf{r}}$ gives

$$W_{\mathbf{r}}(q) \stackrel{\text{def}}{=} \sum_{w\in S_{\mathbf{r}}} q^{\ell(w)} = \sum_{w\in S_{\mathbf{r}}} w\left(\prod\nolimits_{\alpha\in R_+(GL_{\mathbf{r}})} \frac{1-q\,\mathbf{z}^{-\alpha}}{1-\mathbf{z}^{-\alpha}}\right) = \boldsymbol{\sigma}_{\mathbf{r}}\big(\prod\nolimits_{\alpha\in R_+(GL_{\mathbf{r}})}(1-q\,\mathbf{z}^{-\alpha})\big). \tag{119}$$

Combining this with (108), if f is $S_{\mathbf{r}}$ invariant, we find

$$\begin{aligned} W_{\mathbf{r}}(q)\,\langle 1_{GL_{\mathbf{r}}}\rangle\, f &= \langle 1_{GL_{\mathbf{r}}}\rangle\, \boldsymbol{\sigma}_{\mathbf{r}}\big(f\prod\nolimits_{\alpha\in R_+(GL_{\mathbf{r}})}(1-q\,\mathbf{z}^{-\alpha})\big) \\ &= \langle \mathbf{z}^0\rangle\, f\cdot \prod\nolimits_{\alpha\in R_+(GL_{\mathbf{r}})} \big((1-\mathbf{z}^{\alpha})(1-q\,\mathbf{z}^{-\alpha})\big). \end{aligned} \tag{120}$$

Now we calculate

$$\begin{aligned} &W_{\mathbf{r}}(q)\big\langle\, \mathbf{z}^{-\rho_{\mathbf{r}}} E^{\sigma}_{\mathbf{r},\lambda},\, \overline{\mathbf{z}^{-\rho_{\mathbf{r}}} F^{\sigma}_{\mathbf{r},\mu}}\,\big\rangle^{\mathbf{r}}_q \\ &\quad = W_{\mathbf{r}}(q)\,\langle 1_{GL_{\mathbf{r}}}\rangle\, E^{\sigma}_{\mathbf{r},\lambda}\, \overline{F^{\sigma}_{\mathbf{r},\mu}} \prod\nolimits_{\alpha\in R_+\setminus R_+(GL_{\mathbf{r}})} \frac{1-\mathbf{z}^{\alpha}}{1-q^{-1}\mathbf{z}^{\alpha}} \\ &\quad = \langle \mathbf{z}^0\rangle\, E^{\sigma}_{\mathbf{r},\lambda}\, \overline{F^{\sigma}_{\mathbf{r},\mu}} \prod\nolimits_{\alpha\in R_+(GL_{\mathbf{r}})} \big((1-q^{-1}\mathbf{z}^{\alpha})(1-q\,\mathbf{z}^{-\alpha})\big) \prod\nolimits_{\alpha\in R_+} \frac{1-\mathbf{z}^{\alpha}}{1-q^{-1}\,\mathbf{z}^{\alpha}} \\ &\quad = \big\langle\, E^{\sigma}_{\mathbf{r},\lambda} \prod\nolimits_{\alpha\in R_+(GL_{\mathbf{r}})}(1-q\,\mathbf{z}^{-\alpha}),\, \overline{F^{\sigma}_{\mathbf{r},\mu} \prod\nolimits_{\alpha\in R_+(GL_{\mathbf{r}})}(1-q\,\mathbf{z}^{-\alpha})}\,\big\rangle_q \\ &\quad = \langle\, q^{\ell(w_0^{\mathbf{r}})} E^{\sigma,-}_{\mathbf{r},\lambda},\, \overline{q^{\ell(w_0^{\mathbf{r}})} F^{\sigma,-}_{\mathbf{r},\mu}}\,\rangle_q \\ &\quad = \big\langle \sum_{w\in S_{\mathbf{r}}} (-1)^{\ell(w)} E^{\widehat{\sigma}}_{w(\lambda)},\, q^{\ell(w_0^{\mathbf{r}})} \sum_{w\in S_{\mathbf{r}}} (-q)^{-\ell(w)} \overline{F^{\widehat{\sigma}}_{w(\mu)}}\,\big\rangle_q, \end{aligned} \tag{121}$$

where we canceled $q^{\pm\ell(w_0^{\mathbf{r}})}$ in the penultimate line and used (116) and (118) to get the last line. By [6, Proposition 4.3.2], the functions $E^{\widehat{\sigma}}_{\nu}$ and $\overline{F^{\widehat{\sigma}}_{\nu}}$ are dual bases for $\langle -,-\rangle_q$. Hence, the last line in (121) simplifies to $W_{\mathbf{r}}(q)\,\delta_{\lambda,\mu}$, and the result follows. □

5.4. **LLT series.** Generalizing [6, Definition 4.4.1], we now define LLT series associated to GL_l and any Levi subgroup $GL_{\mathbf{r}}$.

Definition 5.4.1. Given a composition $\mathbf{r} = (r_1,\ldots,r_k)$ of l, a permutation $\sigma\in S_k$, and weights $\alpha,\beta\in X^{++}(GL_{\mathbf{r}})$, the *LLT series* $\mathcal{L}^{\sigma}_{\mathbf{r},\beta/\alpha}(\mathbf{z};q)$ is the infinite formal linear combination of irreducible GL_l characters with coefficients defined by

$$\langle\chi_\lambda\rangle\, \mathcal{L}^{\sigma^{-1}}_{\mathbf{r},\beta/\alpha}(\mathbf{z};q^{-1}) = \langle E^{\sigma}_{\mathbf{r},\beta}(\mathbf{z};q)\rangle\, \chi_\lambda\cdot E^{\sigma}_{\mathbf{r},\alpha}(\mathbf{z};q) \tag{122}$$

in terms of the basis $\{E^{\sigma}_{\mathbf{r},\mu} \mid \mu\in X^{++}(GL_{\mathbf{r}})\}$ of the space $\mathbf{z}^{\rho_{\mathbf{r}}}\Bbbk[z_1^{\pm1},\ldots,z_l^{\pm1}]^{S_{\mathbf{r}}}$ (note that this space is independent of the choice of $\rho_{\mathbf{r}}$ and closed under multiplication by GL_l characters).

Remark 5.4.2. (i) The elements $E^{\sigma,-}_{\mathbf{r},\mu}(\mathbf{z};q)$ in (114) form a basis of the space of Laurent polynomials antisymmetric with respect to the action of the Hecke algebra $\mathcal{H}(S_{\mathbf{r}})$. In terms of this basis, an alternative formulation equivalent to (122) is

$$\langle \chi_\lambda \rangle\, \mathcal{L}^{\sigma^{-1}}_{\mathbf{r},\beta/\alpha}(\mathbf{z}; q^{-1}) = \langle E^{\sigma,-}_{\mathbf{r},\beta}(\mathbf{z}; q) \rangle\, \chi_\lambda \cdot E^{\sigma,-}_{\mathbf{r},\alpha}(\mathbf{z}; q). \tag{123}$$

(ii) By Remark 5.2.5, the right hand side of (122) belongs to $\mathbb{Z}[q^{-1}]$. The q^{-1} on the left hand side of (122) serves to give $\mathcal{L}^{\sigma}_{\mathbf{r},\beta/\alpha}(\mathbf{z}; q)$ coefficients in $\mathbb{Z}[q]$, while the indexing with σ^{-1} instead of σ allows us to formulate the connection between LLT series and LLT polynomials more naturally in §5.5.

The next proposition gives a formula for $\mathcal{L}^{\sigma}_{\mathbf{r},\beta/\alpha}(\mathbf{z}; q)$, generalizing [6, Proposition 4.4.2]. To state it we need the q-symmetrization operator

$$\mathbf{H}^{\mathbf{r}}_q(f) = \boldsymbol{\sigma}\Big(\frac{f(\mathbf{z})}{\prod_{\alpha\in R_+\setminus R_+(GL_{\mathbf{r}})}(1 - q\,\mathbf{z}^\alpha)} \Big). \tag{124}$$

Here $f(z_1, \dots, z_l)$ is a Laurent polynomial, and (124) is to be interpreted as a formal infinite linear combination of irreducible GL_l characters by expanding each factor $(1 - q\,\mathbf{z}^\alpha) = 1 + q\,\mathbf{z}^\alpha + \cdots$ as a geometric series before applying the Weyl symmetrization operator $\boldsymbol{\sigma}$.

Although we won't use it, we mention that when f is $S_{\mathbf{r}}$ invariant, $\mathbf{H}^{\mathbf{r}}_q$ is a q-analog of induction from $GL_{\mathbf{r}}$ characters to GL_l characters. When f is a product of Schur functions $\prod_i s_{\lambda_{(i)}}(Z_i)$ in blocks of variables $Z_1 = z_1, \dots, z_{r_1}$, $Z_2 = z_{r_1+1}, \dots, z_{r_1+r_2}$, etc., $\mathbf{H}^{\mathbf{r}}_q(f)$ is a q-analog of $\prod_i s_{\lambda_{(i)}}(\mathbf{z})$, whose Schur expansion yields the generalized Kostka polynomials studied by Shimozono, Weyman and Zabrocki in [21, 22].

Proposition 5.4.3. *For $\mathbf{r}$, σ, α, β as in Definition 5.4.1, we have*

$$\mathcal{L}^{\sigma}_{\mathbf{r},\beta/\alpha}(\mathbf{z}; q) = \mathbf{H}^{w_0^k(\mathbf{r})}_q\big(w_0 (F^{\sigma^{-1}}_{\mathbf{r},\beta}\, \overline{E^{\sigma^{-1}}_{\mathbf{r},\alpha}}) \big). \tag{125}$$

Proof. By Proposition 5.3.1,

$$\begin{aligned} \langle \chi_\lambda \rangle\, \mathcal{L}^{\sigma}_{\mathbf{r},\beta/\alpha}(\mathbf{z}; q) &= \langle\, \mathbf{z}^{\rho_{\mathbf{r}}} F^{\sigma^{-1}}_{\mathbf{r},\beta}(\overline{\mathbf{z}}; q),\ \mathbf{z}^{-\rho_{\mathbf{r}}} \chi_\lambda \cdot E^{\sigma^{-1}}_{\mathbf{r},\alpha}(\mathbf{z}; q^{-1}) \,\rangle^{\mathbf{r}}_{q^{-1}} \\ &= \langle \mathbf{z}^0 \rangle\, \chi_\lambda\, F^{\sigma^{-1}}_{\mathbf{r},\beta}(\overline{\mathbf{z}}; q)\, E^{\sigma^{-1}}_{\mathbf{r},\alpha}(\mathbf{z}; q^{-1})\, \frac{\prod_{\alpha\in R_+}(1 - \mathbf{z}^\alpha)}{\prod_{\alpha\in R_+\setminus R_+(GL_{\mathbf{r}})}(1 - q\,\mathbf{z}^\alpha)}. \end{aligned} \tag{126}$$

We can invert the variables z_i and apply w_0 without changing the constant term, so the above is equal to

$$\langle \mathbf{z}^0 \rangle\, \overline{\chi_\lambda}\, w_0\big(F^{\sigma^{-1}}_{\mathbf{r},\beta}(\mathbf{z}; q)\, \overline{E^{\sigma^{-1}}_{\mathbf{r},\alpha}(\mathbf{z}; q)} \big)\, \frac{\prod_{\alpha\in R_+}(1 - \mathbf{z}^\alpha)}{\prod_{\alpha\in R_+\setminus R_+(GL_{w_0^k(\mathbf{r})})}(1 - q\,\mathbf{z}^\alpha)}. \tag{127}$$

Using (108) for GL_l, this is the same as

$$\langle 1_{GL_l} \rangle\, \overline{\chi_\lambda}\, \boldsymbol{\sigma}\Bigg(\frac{w_0(F^{\sigma^{-1}}_{\mathbf{r},\beta}\, \overline{E^{\sigma^{-1}}_{\mathbf{r},\alpha}})}{\prod_{\alpha\in R_+\setminus R_+(GL_{w_0^k(\mathbf{r})})}(1 - q\,\mathbf{z}^\alpha)} \Bigg) = \langle \chi_\lambda \rangle\, \mathbf{H}^{w_0^k(\mathbf{r})}_q\big(w_0 (F^{\sigma^{-1}}_{\mathbf{r},\beta}\, \overline{E^{\sigma^{-1}}_{\mathbf{r},\alpha}}) \big). \tag{128}$$

□

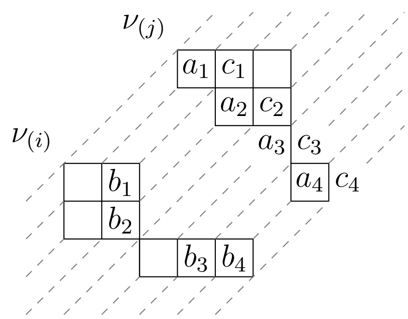


FIGURE 6. Examples of σ-triples. The dashed lines show boxes of equal content in $\nu_{(i)}$ and $\nu_{(j)}$, with $i < j$. If $\sigma(i) < \sigma(j)$, then (a_1, b_1, c_1) and (a_3, b_3, c_3) are σ-triples. If $\sigma(i) > \sigma(j)$, then (a_2, b_2, c_2) and (a_4, b_4, c_4) are σ-triples. Triples (a_3, b_3, c_3) and (a_4, b_4, c_4) illustrate the point that a and/or c may be just outside a (possibly empty) row of $\nu_{(j)}$.

5.5. **Relation between LLT series and LLT polynomials.** Here we derive an identity relating the polynomial part $\mathcal{L}^{\sigma}_{\mathbf{r},\beta/\alpha}(\mathbf{z}; q)_{\mathrm{pol}}$ of an LLT series to an LLT polynomial $\mathcal{G}_{\boldsymbol{\nu}}(X; q)$, extending the treatment of the case $\mathbf{r} = (1^l)$ in [6, §4.5].

Definition 5.5.1. Let $\boldsymbol{\nu} = \boldsymbol{\beta}/\boldsymbol{\alpha} = (\beta_{(1)}/\alpha_{(1)}, \ldots, \beta_{(k)}/\alpha_{(k)})$ be a tuple of skew diagrams, and let $\sigma \in S_k$ be a permutation. A *σ-triple* in $\boldsymbol{\beta}/\boldsymbol{\alpha}$ is an ordered triple of boxes (a, b, c) such that

(i) b is a box of $\nu_{(i)} = \beta_{(i)}/\alpha_{(i)}$ for some i;
(ii) a is either in or immediately left of a row of $\nu_{(j)}$ and c is either in or immediately right of the same row, for some $j > i$;
(iii) a and c are adjacent with a left of c; and
(iv) b has the same content as c if $\sigma(i) < \sigma(j)$, or the same content as a if $\sigma(i) > \sigma(j)$.

More precisely, (ii) and (iii) mean that if $\alpha_{(j)} = (\alpha_1, \ldots, \alpha_m)$ and $\beta_{(j)} = (\beta_1, \ldots, \beta_m)$, then we have $a = (x, y)$ and $c = (x + 1, y)$ for some $1 \le y \le m$ and $\alpha_y \le x \le \beta_y$. In particular, a and c can be the boxes left and right of an empty row with $\beta_y = \alpha_y$. The set of triples thus depends on the presentation of $\boldsymbol{\nu}$ as $\boldsymbol{\beta}/\boldsymbol{\alpha}$ and not just on the set of boxes in $\boldsymbol{\nu}$.

Strictly speaking, the indices i and j are part of the data of a triple, in keeping with our understanding that the set of boxes of $\boldsymbol{\nu}$ is the disjoint union of the sets of boxes of the $\nu_{(i)}$. Figure 6 illustrates the definition.

Definition 5.5.2. (i) Given a tuple of skew diagrams $\boldsymbol{\beta}/\boldsymbol{\alpha} = (\beta_{(1)}/\alpha_{(1)}, \ldots, \beta_{(k)}/\alpha_{(k)})$ and $\sigma \in S_k$, an *increasing σ-triple* in a negative tableau $T \in \mathrm{SSYT}_-(\boldsymbol{\beta}/\boldsymbol{\alpha})$ is a σ-triple (a, b, c) such that $T(a) < T(b) < T(c)$, with the convention $T(a) = -\infty$, $T(c) = \infty$ if a or c is not a box of $\boldsymbol{\beta}/\boldsymbol{\alpha}$.

(ii) We define the generating function

$$N^{\sigma}_{\boldsymbol{\beta}/\boldsymbol{\alpha}}(X; q) = \sum_{T \in \mathrm{SSYT}_-(\boldsymbol{\beta}/\boldsymbol{\alpha})} q^{h_\sigma(T)} \mathbf{x}^T, \tag{129}$$

where $h_\sigma(T)$ is the number of increasing σ-triples in T.

Although it is not obvious a priori, the next proposition implies that $N^\sigma_{\boldsymbol{\beta}/\boldsymbol{\alpha}}(X;q)$ is symmetric.

Proposition 5.5.3. *We have the identity*

$$N^\sigma_{\boldsymbol{\beta}/\boldsymbol{\alpha}}(X;q) = q^{h_\sigma(\boldsymbol{\beta}/\boldsymbol{\alpha})}\omega\,\mathcal{G}_{\sigma(\boldsymbol{\beta}/\boldsymbol{\alpha})}(X;q^{-1}), \tag{130}$$

where $h_\sigma(\boldsymbol{\beta}/\boldsymbol{\alpha})$ is the number of σ-triples in $\boldsymbol{\beta}/\boldsymbol{\alpha}$.

Proof. Let $\boldsymbol{\nu}$ be the tuple of skew diagrams given by $\boldsymbol{\beta}/\boldsymbol{\alpha}$ and note that $\sigma(\boldsymbol{\nu})$ is the rearrangement of $\boldsymbol{\nu}$ with $\sigma(\boldsymbol{\nu})_{(\sigma(i))} = \nu_{(i)}$. For $a \in \nu_{(i)}$, let $\sigma(a)$ denote the corresponding box of $\sigma(\boldsymbol{\nu})_{(\sigma(i))}$. We also use this notation for boxes adjacent to $\nu_{(i)}$, which may occur in a σ-triple.

For a tableau $T \in \mathrm{SSYT}_-(\boldsymbol{\nu})$ let $\sigma(T)$ denote its image in $\mathrm{SSYT}_-(\sigma(\boldsymbol{\nu}))$, defined by $\sigma(T)(\sigma(a)) = T(a)$.

By Proposition 2.2.2, the right hand side of (130) can be written

$$\sum_{T\in \mathrm{SSYT}_-(\boldsymbol{\beta}/\boldsymbol{\alpha})} q^{h_\sigma(\boldsymbol{\beta}/\boldsymbol{\alpha})-\mathrm{inv}(\sigma(T))}\mathbf{x}^{\sigma(T)}. \tag{131}$$

Since $\mathbf{x}^{\sigma(T)} = \mathbf{x}^T$, (130) will follow if we show that

$$h_\sigma(T) = h_\sigma(\boldsymbol{\beta}/\boldsymbol{\alpha}) - \mathrm{inv}(\sigma(T)). \tag{132}$$

Consider the image $(a',b',c') = (\sigma(a),\sigma(b),\sigma(c))$ of a σ-triple (a,b,c) in $\boldsymbol{\beta}/\boldsymbol{\alpha}$. The definition of σ-triple implies that (a',b') is an attacking pair in $\sigma(\boldsymbol{\nu})$ if both boxes a and b are in $\boldsymbol{\nu}$, and similarly for (b',c'). One also sees that every attacking pair in $\sigma(\boldsymbol{\nu})$ belongs in this way to the image of a unique triple. Since a,c are in the same row, we have $T(a) < T(c)$ for every negative tableau T. This holds even if a or c is not in $\boldsymbol{\nu}$, by the convention that $T(a) = -\infty$, $T(c) = \infty$ in these cases. Hence, at most one of the pairs (a',b'), (b',c') is an attacking inversion in $\sigma(T)$, since we would have $T(a) \geq T(b) \geq T(c)$ if they both were. Moreover, (a,b,c) is an increasing σ-triple if and only if neither (a',b') nor (b',c') is an attacking inversion. Hence, the number of increasing σ-triples is the total number of σ-triples minus the number of attacking pairs in $\sigma(T)$. ☐

Let $\mathbf{r} = (r_1,\dots,r_k)$ be a composition of l, and let $\alpha,\beta \in X^{++}(GL_{\mathbf{r}})$ be dominant regular weights for $GL_{\mathbf{r}}$ such that $\alpha_i \leq \beta_i$ for all $1 \leq i \leq l$. To these data we associate a tuple of skew diagrams $\boldsymbol{\beta}/\boldsymbol{\alpha} = (\beta_{(1)}/\alpha_{(1)},\dots,\beta_{(k)}/\alpha_{(k)})$ by defining

$$\begin{aligned}(\alpha_{(i)})_j &= \alpha_{r_1+\cdots+r_{i-1}+j} + j,\\ (\beta_{(i)})_j &= \beta_{r_1+\cdots+r_{i-1}+j} + j\end{aligned} \tag{133}$$

for $1 \leq i \leq k$ and $1 \leq j \leq r_i$. In other words, $(\alpha_{(1)}|\cdots|\alpha_{(k)}) = \alpha - \rho_{\mathbf{r}}$ and $(\beta_{(1)}|\cdots|\beta_{(k)}) = \beta - \rho_{\mathbf{r}}$, where $(\cdot|\cdots|\cdot)$ denotes concatenation and $\rho_{\mathbf{r}} = -((1,\dots,r_1)|\cdots|(1,\dots,r_k))$.

This construction has the combinatorially natural feature that the contents of the boxes in the j-th row of $\beta_{(i)}/\alpha_{(i)}$ are $\alpha_m+1,\dots,\beta_m$, where $m = r_1+\cdots+r_{i-1}+j$ is the index corresponding to the j-th position in the i-th block of the partition of $[l]$ into intervals of lengths r_i.

Theorem 5.5.4. *Given a composition* $\mathbf{r} = (r_1, \ldots, r_k)$ *of* l*,* $\sigma \in S_k$*, and weights* $\alpha, \beta \in X^{++}(GL_{\mathbf{r}})$*, we have*

$$\mathcal{L}^{\sigma}_{\mathbf{r},\beta/\alpha}(\mathbf{z};q)_{\mathrm{pol}} = \begin{cases} q^{h_\sigma(\boldsymbol{\beta}/\boldsymbol{\alpha})}\mathcal{G}_{\sigma(\boldsymbol{\beta}/\boldsymbol{\alpha})}(z_1,\ldots,z_l;q^{-1}) & \text{if } \alpha_i \leq \beta_i \text{ for all } 1 \leq i \leq l, \\ 0 & \text{otherwise,} \end{cases} \tag{134}$$

where $\boldsymbol{\beta}/\boldsymbol{\alpha}$ *is the tuple associated to* $\mathbf{r}$*,* α*,* β *by the construction above.*

The proof will be based on Proposition 5.5.3 and the following lemma.

Lemma 5.5.5. *For* $\alpha, \beta \in X^{++}(GL_{\mathbf{r}})$ *and* $\sigma \in S_k$*, we have*

$$\langle E^{\sigma,-}_{\mathbf{r},\beta}(\mathbf{z};q)\rangle\, e_m(\mathbf{z})\, E^{\sigma,-}_{\mathbf{r},\alpha}(\mathbf{z};q) = \begin{cases} q^{-|\mathrm{Inv}(\beta+\epsilon\,\tau)\setminus\mathrm{Inv}(\alpha+\epsilon\,\tau)|} & \text{if } \beta-\alpha = \varepsilon_I \text{ for } I\subseteq[l],\ |I|=m, \\ 0 & \text{otherwise,} \end{cases}$$

where e_m *is an elementary symmetric function,* $\tau = \widehat{\sigma}^{\,-1}$*, and* ε_I *is the* 0-1 *vector with* 1*'s in positions* $i \in I$*.*

Proof. Using (116), the coefficient $\langle E^{\sigma,-}_{\mathbf{r},\beta}\rangle f$ of $E^{\sigma,-}_{\mathbf{r},\beta}$ in any $S_{\mathbf{r}}$-antisymmetric function f is equal to $\langle E^{\widehat{\sigma}}_{\beta}\rangle f$. Applying this with $f = e_m\, E^{\sigma,-}_{\mathbf{r},\alpha}$,

$$\langle E^{\sigma,-}_{\mathbf{r},\beta}\rangle\, e_m\, E^{\sigma,-}_{\mathbf{r},\alpha} = \langle E^{\widehat{\sigma}}_{\beta}\rangle\, e_m\, E^{\sigma,-}_{\mathbf{r},\alpha} = \sum_{w\in S_{\mathbf{r}}} (-1)^{\ell(w)} \langle E^{\widehat{\sigma}}_{\beta}\rangle\, e_m\, E^{\widehat{\sigma}}_{w(\alpha)}. \tag{135}$$

By [6, Lemma 4.5.1] the coefficient $\langle E^{\widehat{\sigma}}_{\beta}\rangle\, e_m\, E^{\widehat{\sigma}}_{w(\alpha)}$ vanishes unless $\beta = w(\alpha)+\varepsilon_I$ for some I. Since $w \in S_{\mathbf{r}}$ and $\alpha \in X^{++}(GL_{\mathbf{r}})$, $w(\alpha)$ is not dominant for $GL_{\mathbf{r}}$ if $w \neq 1$. In that case there is an index i such that $s_i \in S_{\mathbf{r}}$ and $w(\alpha)_i < w(\alpha)_{i+1}$. Since $\beta \in X^{++}(GL_{\mathbf{r}})$ we then have $\beta_i - w(\alpha)_i - (\beta_{i+1} - w(\alpha)_{i+1}) \geq 2$, and therefore $\beta - w(\alpha)$ is not of the form ε_I. This shows that the terms for $w \neq 1$ on the right hand side of (135) vanish, leaving

$$\langle E^{\sigma,-}_{\mathbf{r},\beta}\rangle\, e_m\, E^{\sigma,-}_{\mathbf{r},\alpha} = \langle E^{\widehat{\sigma}}_{\beta}\rangle\, e_m\, E^{\widehat{\sigma}}_{\alpha}. \tag{136}$$

The lemma now follows from [6, Lemma 4.5.1]. □

Proof of Theorem 5.5.4. Let $L^{\sigma}_{\mathbf{r},\beta/\alpha}(X;q)$ be the unique linear combination of Schur functions $s_\lambda(X)$ with $\ell(\lambda) \leq l$ that specializes in l variables $z_1, \ldots, z_l$ to $L^{\sigma}_{\mathbf{r},\beta/\alpha}(\mathbf{z};q) = \mathcal{L}^{\sigma}_{\mathbf{r},\beta/\alpha}(\mathbf{z};q)_{\mathrm{pol}}$. We will prove that

$$L^{\sigma}_{\mathbf{r},\beta/\alpha}(X;q) = \begin{cases} q^{h_\sigma(\boldsymbol{\beta}/\boldsymbol{\alpha})}\mathcal{G}_{\sigma(\boldsymbol{\beta}/\boldsymbol{\alpha})}(X;q^{-1}) & \text{if } \alpha_i \leq \beta_i \text{ for all } 1 \leq i \leq l, \\ 0 & \text{otherwise.} \end{cases} \tag{137}$$

Clearly this implies (134) (actually, (134) and (137) are equivalent, by Corollary 2.2.3).

By (123), we have

$$\langle\, s_\lambda,\, L^{\sigma}_{\mathbf{r},\beta/\alpha}(X;q^{-1})\,\rangle = \langle E^{\sigma^{-1},-}_{\mathbf{r},\beta}\rangle\, s_\lambda(\mathbf{z})\, E^{\sigma^{-1},-}_{\mathbf{r},\alpha}, \tag{138}$$

where $\langle -, - \rangle$ is the Hall inner product on symmetric functions. Note that (138) holds even if $\ell(\lambda) > l$, since both sides are zero in that case. By linearity, (138) therefore holds with any symmetric function f in place of s_λ. In particular, taking $f = e_\mu = e_{\mu_1}\cdots e_{\mu_n}$, we have

$$\langle\, e_\mu,\, L^{\sigma}_{\mathbf{r},\beta/\alpha}(X;q^{-1})\,\rangle = \langle E^{\sigma^{-1},-}_{\mathbf{r},\beta}\rangle\, e_\mu(\mathbf{z})\, E^{\sigma^{-1},-}_{\mathbf{r},\alpha}. \tag{139}$$

Evaluating this last expression by using Lemma 5.5.5 to multiply by e_{μ_1} through e_{μ_n} in succession gives a sum over chains of weights

$$\alpha = \alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(n)} = \beta \qquad (\alpha^{(i)} \in X^{++}(GL_{\mathbf{r}})) \tag{140}$$

such that for each $i = 1, \ldots, n$ we have $\alpha^{(i)} = \alpha^{(i-1)} + \varepsilon_{I_i}$ for some index set $I_i \subseteq [l]$ of size $|I_i| = \mu_i$. In particular, this shows that $L^{\sigma}_{\mathbf{r},\beta/\alpha}(X;q) = 0$ if the condition $\alpha_i \leq \beta_i$ for all i does not hold, so we assume from now on that it does.

To the weights in (140) we can now associate a chain of tuples of skew diagrams

$$\varnothing = \boldsymbol{\alpha}^{(0)}/\boldsymbol{\alpha} \subseteq \boldsymbol{\alpha}^{(1)}/\boldsymbol{\alpha} \subseteq \cdots \subseteq \boldsymbol{\alpha}^{(n)}/\boldsymbol{\alpha} = \boldsymbol{\beta}/\boldsymbol{\alpha} \tag{141}$$

by the construction in (133). The condition on the $\alpha^{(i)}$ means that each $\boldsymbol{\alpha}^{(i)}/\boldsymbol{\alpha}^{(i-1)}$ is a tuple of vertical strips of size $|\boldsymbol{\alpha}^{(i)}/\boldsymbol{\alpha}^{(i-1)}| = \mu_i$. In other words, $\boldsymbol{\alpha}^{(i)}/\boldsymbol{\alpha}^{(i-1)}$ is the set of boxes a with $T(a) = \bar{i}$ in a negative tableau $T \in \mathrm{SSYT}_-(\boldsymbol{\beta}/\boldsymbol{\alpha})$ of weight $\mathbf{x}^T = \mathbf{x}^{\mu}$.

From Lemma 5.5.5, the term in (139) corresponding to T, or to the weight sequence (140), is $q^{-h(T)}$, where

$$h(T) = \sum_{i=1}^{n} |\mathrm{Inv}(\alpha^{(i)} + \epsilon\,\tau) \setminus \mathrm{Inv}(\alpha^{(i-1)} + \epsilon\,\tau)|, \tag{142}$$

with $\tau = (\widehat{\sigma^{-1}})^{-1}$. We claim that $h(T) = h_\sigma(T)$ is the number of increasing σ-triples in T. Granting the claim, we then have

$$\langle e_\mu,\, L^{\sigma}_{\mathbf{r},\beta/\alpha}(X;q)\rangle = \sum_{\substack{T\in \mathrm{SSYT}_-(\boldsymbol{\beta}/\boldsymbol{\alpha})\\ \mathbf{x}^T=\mathbf{x}^\mu}} q^{h_\sigma(T)}. \tag{143}$$

By definition, the sum on the right is the coefficient $\langle \mathbf{x}^\mu\rangle\, N^{\sigma}_{\boldsymbol{\beta}/\boldsymbol{\alpha}} = \langle e_\mu,\, \omega\, N^{\sigma}_{\boldsymbol{\beta}/\boldsymbol{\alpha}}\rangle$. Using Proposition 5.5.3, this implies

$$L^{\sigma}_{\mathbf{r},\beta/\alpha}(X;q) = \omega\, N^{\sigma}_{\boldsymbol{\beta}/\boldsymbol{\alpha}}(X;q) = q^{h_\sigma(\boldsymbol{\beta}/\boldsymbol{\alpha})}\mathcal{G}_{\sigma(\boldsymbol{\beta}/\boldsymbol{\alpha})}(X;q^{-1}). \tag{144}$$

It remains only to verify that $h(T) = h_\sigma(T)$. Let $[l] = J_1 \coprod \cdots \coprod J_k$ be the partition of $[l]$ into intervals of lengths $|J_j| = r_j$. The weights $\alpha^{(i)}$ are strictly decreasing on each block J_j, so $|\mathrm{Inv}(\alpha^{(i)} + \epsilon\,\tau) \setminus \mathrm{Inv}(\alpha^{(i-1)} + \epsilon\,\tau)|$ only counts inversions between distinct blocks.

Now, $\tau = (\widehat{\sigma^{-1}})^{-1}$ carries the blocks J_j to intervals of lengths $\sigma(\mathbf{r})$ in the order given by σ; in other words, for $s \in J_j$, $s' \in J_{j'}$ with $j < j'$, we have $\tau(s) < \tau(s')$ if and only if $\sigma(j) < \sigma(j')$. Thus, if $\alpha^{(i)}_s = \alpha^{(i)}_{s'}$, we have $(s, s') \in \mathrm{Inv}(\alpha^{(i)} + \epsilon\,\tau)$ if and only if $\sigma(j) > \sigma(j')$.

By construction, $\alpha^{(i)}_s$ is the content of the last box in the row of $\boldsymbol{\alpha}^{(i)}/\boldsymbol{\alpha}$ corresponding to the index s, or of the box immediately left of an empty row. For $s \in J_j$, $s' \in J_{j'}$ with $j < j'$, it follows that $(s, s') \in \mathrm{Inv}(\alpha^{(i)} + \epsilon\,\tau) \setminus \mathrm{Inv}(\alpha^{(i-1)} + \epsilon\,\tau)$ if and only if $\boldsymbol{\alpha}^{(i)}/\boldsymbol{\alpha}^{(i-1)}$ has a box b in the row corresponding to s, and one of the following two conditions holds, where a is the last box in the row of $\boldsymbol{\alpha}^{(i-1)}/\boldsymbol{\alpha}$ corresponding to s', or the box immediately to the left if this row is empty:

(i) $\sigma(j) < \sigma(j')$ and $c(b) = c(a) + 1$ and the box c with content $c(a) + 1 = c(b)$ in the same row as a is not in $\boldsymbol{\alpha}^{(i)}/\boldsymbol{\alpha}$; or

(ii) $\sigma(j) > \sigma(j')$ and $c(b) = c(a)$ and the box c with content $c(a) + 1 = c(b) + 1$ in the same row as a is not in $\boldsymbol{\alpha}^{(i)}/\boldsymbol{\alpha}$.

These conditions are equivalent to (a, b, c) forming an increasing σ-triple in T with $T(b) = \bar{i}$. Since a triple is determined by the box b and the index of the row containing a and c, we see that $|\mathrm{Inv}(\alpha^{(i)} + \epsilon\,\tau) \setminus \mathrm{Inv}(\alpha^{(i-1)} + \epsilon\,\tau)|$ counts σ-triples in T such that $T(b) = \bar{i}$. Summing over i yields $h(T) = h_\sigma(T)$, as claimed. □

6. Cauchy identity and winding permutations

As in [6], the infinite series form of our main theorem will follow by combining a Cauchy identity for Hall-Littlewood polynomials—semi-symmetric Hall-Littlewood polynomials $E^\sigma_{\mathbf{r},\mu}$, $F^\sigma_{\mathbf{s},\mu}$, in this case—with an identity that allows us to change the 'twist' σ when the latter has a special form. In this section we establish the two identities that we need.

6.1. Cauchy identity.

Our next theorem generalizes the Cauchy identity for non-symmetric Hall-Littlewood polynomials [6, Theorem 5.1.1]. A new feature that appears in the semi-symmetric case is that different compositions $\mathbf{r}$, $\mathbf{s}$ may govern the blocks of variables in the functions $E^\sigma_{\mathbf{r},\mu}(\mathbf{x}, q)$ and $F^\sigma_{\mathbf{s},\mu}(\mathbf{y}, q)$ that play a role in the identity, subject to some conditions which we now define.

Definition 6.1.1. Given $\sigma \in S_k$, a sequence $(m_1, \ldots, m_k) \in \mathbb{Z}^k$ is *σ-almost decreasing* if

$$m_i \geq m_j - \chi(\sigma^{-1}(i) > \sigma^{-1}(j)) \quad \text{for all } i < j, \tag{145}$$

and *σ-almost increasing* if

$$m_i \leq m_j + \chi(\sigma^{-1}(i) < \sigma^{-1}(j)) \quad \text{for all } i < j. \tag{146}$$

If $\mathbf{r} = (r_1, \ldots, r_k)$ is a strict composition, then any choice of $\rho_\mathbf{r}$ satisfying (82) determines two sequences $(M_1, \ldots, M_k)$ and $(m_1, \ldots, m_k)$ such that $\rho_\mathbf{r}$ is the concatenation of blocks $(M_i, M_i - 1, \ldots, m_i)$ of length r_i. We extend this to weak compositions as follows.

Convention 6.1.2. Let $\mathbf{r} = (r_1, \ldots, r_k)$ be a weak composition. Whenever we choose $\rho_\mathbf{r}$ satisfying (82), we also choose sequences of integers $(M_1, \ldots, M_k)$ and $(m_1, \ldots, m_k)$ such that for $r_i > 0$, the corresponding block of $\rho_\mathbf{r}$ is $(M_i, M_i - 1, \ldots, m_i)$, and for $r_i = 0$ we have $M_i < m_i$. We refer to M_i and m_i as the *block maxima and minima of $\rho_\mathbf{r}$*, including any artificial maxima and minima $M_i < m_i$ that we may have ascribed to empty blocks.

Theorem 6.1.3. *Suppose we are given weak compositions $\mathbf{r} = (r_1, \ldots, r_k)$, $\mathbf{s} = (s_1, \ldots, s_k)$, a permutation $\sigma \in S_k$, and a choice of $\rho_\mathbf{r}$, $\rho_\mathbf{s}$ with associated block maxima and minima, in keeping with Convention 6.1.2. Assume that $\rho_\mathbf{r}$ and $\rho_\mathbf{s}$ have the same block maxima M_i, and let m_i and n_i be their respective block minima. Assume further that $(m_1, \ldots, m_k)$ is σ-almost decreasing and $(n_1, \ldots, n_k)$ is σ-almost increasing.*

Then, using notation explained below, we have the identity

$$\frac{\prod_{i<j} \Omega[-q\,t\,X_i Y_j]}{\prod_{i\leq j} \Omega[-t\,X_i Y_j]} = \sum_\lambda t^{|\lambda|}\, \mathbf{x}^{-\rho_\mathbf{r}} E^\sigma_{\mathbf{r},\lambda+\rho_\mathbf{r}}(\mathbf{x}; q^{-1})\, \mathbf{y}^{-\rho_\mathbf{s}} F^\sigma_{\mathbf{s},\lambda+\rho_\mathbf{s}}(\mathbf{y}; q). \tag{147}$$

The variables on the right hand side are $\mathbf{x} = x_1, \ldots, x_r$, $\mathbf{y} = y_1, \ldots, y_s$, where $r = |\mathbf{r}| = \sum_i r_i$ and $s = |\mathbf{s}| = \sum_i s_i$. The index λ ranges over tuples of partitions $(\lambda_{(1)}, \ldots, \lambda_{(k)})$ such that

$\ell(\lambda_{(i)}) \leq \min(r_i, s_i)$, *with* $|\lambda| = \sum_i |\lambda_{(i)}|$ *denoting the sum of all the parts. In the expression* $\lambda + \rho_{\mathbf{r}}$, *we interpret* λ *as a weight in* $X^+(GL_{\mathbf{r}})$ *by padding* $\lambda_{(i)}$ *with zeroes to length* r_i *and concatenating;* $\lambda + \rho_{\mathbf{s}}$ *is interpreted in a similar way.*

The expressions involving Ω *on the left hand side are as defined in §2.1, with plethystic alphabets* $X_1, \ldots, X_k$ *constructed from blocks of length* r_i *among the variables* $\mathbf{x}$, *by the rule*

$$X_i = x_{i,1} + \cdots + x_{i,r_i}, \quad \text{where } x_{i,j} = x_{r_1+\cdots+r_{i-1}+j}. \tag{148}$$

The alphabets $Y_1, \ldots, Y_k$ *are similarly constructed from blocks of length* s_i *among the* $\mathbf{y}$ *variables.*

Remark 6.1.4. (i) The left hand side of (147) expands to

$$\frac{\prod_{i<j} \Omega[-q\,t\,X_i Y_j]}{\prod_{i\leq j} \Omega[-t\,X_i Y_j]} = \frac{\prod_{i<j} \prod_{a=1}^{r_i} \prod_{b=1}^{s_j} (1 - q\,t\,x_{i,a} y_{j,b})}{\prod_{i\leq j} \prod_{a=1}^{r_i} \prod_{b=1}^{s_j} (1 - t\,x_{i,a} y_{j,b})}. \tag{149}$$

(ii) If $\mathbf{r} = \mathbf{s} = (1^l)$ and $\rho_{\mathbf{r}} = \rho_{\mathbf{s}}$ is constant, the theorem reduces to [6, Theorem 5.1.1]. A little more generally, if $\mathbf{r} = \mathbf{s} = (1^l)$, the hypotheses on the block maxima and minima are satisfied when $\rho_{\mathbf{r}} = \rho_{\mathbf{s}}$ has the form $\sigma(1, 1, \ldots, 1, 0, 0, \ldots, 0) + (\text{constant})$.

(iii) If $k = 1$, so $\mathbf{r} = (r)$, $\mathbf{s} = (s)$ and $\sigma = 1 \in S_1$, we have $\mathbf{x}^{-\rho_{\mathbf{r}}} E^{\sigma}_{\mathbf{r},\lambda+\rho_{\mathbf{r}}}(\mathbf{x}; q^{-1}) = s_\lambda(x_1, \ldots, x_r)$ and $\mathbf{y}^{-\rho_{\mathbf{s}}} F^{\sigma}_{\mathbf{s},\lambda+\rho_{\mathbf{s}}}(\mathbf{y}; q) = s_\lambda(y_1, \ldots, y_s)$ by Example 5.2.3 (ii) and (94). In this case the theorem reduces to the classical Cauchy identity for Schur functions.

(iv) Adding a constant vector to $\rho_{\mathbf{r}}$ or $\rho_{\mathbf{s}}$ does not change the conclusion, so the hypothesis that $\rho_{\mathbf{r}}$ and $\rho_{\mathbf{s}}$ have the same block maxima can be weakened to having block maxima that differ by a constant. The hypotheses with this weakening seem to be essentially as general as possible.

(v) The artificial maxima and minima ascribed to any zero-length blocks in $\rho_{\mathbf{r}}$ or $\rho_{\mathbf{s}}$ have no effect on the identity (147). Nevertheless, to conclude that the identity holds, we require that such maxima and minima can be chosen satisfying the hypotheses of the theorem.

Before proving Theorem 6.1.3, we develop a series of lemmas.

Lemma 6.1.5. *Given* $\sigma \in S_l$, *let* $m, n \in \mathbb{Z}^l$ *be* σ*-almost decreasing sequences such that* $m \leq n$ *coordinate-wise, and* $m \neq n$. *Then there is an index* $k \in [l]$ *such that* $m_k < n_k$, *and* $m + \varepsilon_k$ *is* σ*-almost decreasing, where* ε_k *is the* k*-th unit coordinate vector.*

Proof. In fact, let k be the smallest index such that $m_k < n_k$. In positions $i < j$, condition (145) for $m + \varepsilon_k$ is the same as the condition on m if $k \notin \{i, j\}$, and is weaker than the condition on m if $i = k$. If $j = k$, then since $m_i = n_i$ for $i < k$ by assumption, the condition on $m + \varepsilon_k$ in positions $i < j$ is weaker than the condition on n. □

Lemma 6.1.6. *Given* $\sigma \in S_l$ *and* $1 \leq k < l$, *if* $m = (m_1, \ldots, m_l)$ *is* σ*-almost decreasing, then the sequence*

$$m' = (m_1, \ldots, m_{k-1}, m'_k, m'_{k+1}, m_{k+2}, \ldots, m_l) \tag{150}$$

is $(s_k\sigma)$*-almost decreasing, where*

$$\begin{aligned} m'_k &= \max\left(m_k - \chi(\sigma^{-1}(k+1) > \sigma^{-1}(k)),\, m_{k+1}\right) \\ m'_{k+1} &= m_k. \end{aligned} \tag{151}$$

Proof. We have $m'_i = m_{s_k(i)}$ for $i \neq k$, so if $i < j$ and $k \notin \{i,j\}$, then $s_k(i) < s_k(j)$ and the condition $m'_i \geq m'_j - \chi((s_k\sigma)^{-1}(i) > (s_k\sigma)^{-1}(j))$ becomes $m_{s_k(i)} \geq m_{s_k(j)} - \chi(\sigma^{-1}s_k(i) > \sigma^{-1}s_k(j))$, which holds by hypothesis.

For $i = k$ and $j > k+1$, the condition becomes $m'_k \geq m_j - \chi(\sigma^{-1}(k+1) > \sigma^{-1}(j))$, which follows from $m_{k+1} \geq m_j - \chi(\sigma^{-1}(k+1) > \sigma^{-1}(j))$ and $m'_k \geq m_{k+1}$. For $(i,j) = (k,k+1)$ the condition becomes $m'_k \geq m'_{k+1} - \chi(\sigma^{-1}(k+1) > \sigma^{-1}(k))$, which follows from the definition of m'_k and m'_{k+1}.

For $i < k = j$, the condition becomes $m_i \geq m'_k - \chi(\sigma^{-1}(i) > \sigma^{-1}(k+1))$. By the definition of m'_k, this is the conjunction of $m_i \geq m_{k+1} - \chi(\sigma^{-1}(i) > \sigma^{-1}(k+1))$ and $m_i \geq m_k - \chi(\sigma^{-1}(k+1) > \sigma^{-1}(k)) - \chi(\sigma^{-1}(i) > \sigma^{-1}(k+1))$. The first of these holds by hypothesis. The second follows from the hypothesis $m_i \geq m_k - \chi(\sigma^{-1}(i) > \sigma^{-1}(k))$ and the inequality

$$\chi(\sigma^{-1}(i) > \sigma^{-1}(k)) \leq \chi(\sigma^{-1}(i) > \sigma^{-1}(k+1)) + \chi(\sigma^{-1}(k+1) > \sigma^{-1}(k)), \tag{152}$$

which is logically equivalent to the transitive law $(\sigma^{-1}(i) < \sigma^{-1}(k+1)) \wedge (\sigma^{-1}(k+1) < \sigma^{-1}(k)) \Rightarrow (\sigma^{-1}(i) < \sigma^{-1}(k))$. □

Lemma 6.1.7. *The Demazure-Lusztig operators T_k in (77) have the following properties, where $(r,s) \geq (a,b)$ means $r \geq a$ and $s \geq b$.*
(i) If $(r,s) \geq (c,c+1)$, then every term $z_k^u z_{k+1}^v$ in $T_k(z_k^r z_{k+1}^s)$ has $(u,v) \geq (c+1,c)$.
(ii) If $(r,s) \geq (c+1,c)$, then every term $z_k^u z_{k+1}^v$ in $T_k^{-1}(z_k^r z_{k+1}^s)$ has $(u,v) \geq (c,c+1)$.
(iii) If $(r,s) \geq (c,c)$, then every term $z_k^u z_{k+1}^v$ in $T_k(z_k^r z_{k+1}^s)$ or $T_k^{-1}(z_k^r z_{k+1}^s)$ has $(u,v) \geq (c,c)$.
(iv) If $r \geq s$, then $q^{-1}T_k(z_k^r z_{k+1}^s) = z_k^s z_{k+1}^r + O(z_k^{s+1})$.
(v) If $r > s$, then $T_k^{-1}(z_k^r z_{k+1}^s) = z_k^s z_{k+1}^r + O(z_k^{s+1})$.

Proof. All the properties follow from the explicit formulas

$$\begin{aligned}
T_k(z_k^r z_{k+1}^s) &= \begin{cases} q\, z_k^s z_{k+1}^r + (q-1)(z_k^r z_{k+1}^s + z_k^{r-1} z_{k+1}^{s+1} + \ldots + z_k^{s+1} z_{k+1}^{r-1}) & r > s, \\ q\, z_k^s z_{k+1}^r & r = s, \\ z_k^s z_{k+1}^r + (1-q)(z_k^{s-1} z_{k+1}^{r+1} + z_k^{s-2} z_{k+1}^{r+2} + \ldots + z_k^{r+1} z_{k+1}^{s-1}) & r < s, \end{cases} \\
T_k^{-1}(z_k^r z_{k+1}^s) &= \begin{cases} q^{-1}\, z_k^s z_{k+1}^r + (q^{-1}-1)(z_k^r z_{k+1}^s + z_k^{r+1} z_{k+1}^{s-1} + \ldots + z_k^{s-1} z_{k+1}^{r+1}) & r < s, \\ q^{-1}\, z_k^s z_{k+1}^r & r = s, \\ z_k^s z_{k+1}^r + (1-q^{-1})(z_k^{s+1} z_{k+1}^{r-1} + z_k^{s+2} z_{k+1}^{r+2} + \ldots + z_k^{r-1} z_{k+1}^{s+1}) & r > s. \end{cases}
\end{aligned} \tag{153}$$

□

Lemma 6.1.8. *Let $\sigma \in S_l$, $1 \leq k < l$, and $m, m' \in \mathbb{Z}^l$ be as in Lemma 6.1.6, and suppose $\nu \in \mathbb{Z}^l$ is such that $\nu \geq m'$ coordinate-wise. Let $V_m = \mathbb{Z}[q^{\pm 1}]\{\mathbf{z}^\mu \mid \mu \geq m\}$. Then the operator T_k satisfies*
(i) $T_k(\mathbf{z}^\nu) \in V_m$ if $\sigma^{-1}(k) < \sigma^{-1}(k+1)$;
(ii) $T_k^{-1}(\mathbf{z}^\nu) \in V_m$ if $\sigma^{-1}(k) > \sigma^{-1}(k+1)$.

Proof. Since T_k acts only on the variables z_k, z_{k+1}, we need only consider the exponents μ_k, μ_{k+1} of terms $\mathbf{z}^\mu$ occurring in $T_k^{\pm 1}(\mathbf{z}^\nu)$.

In case (i), we have $(\nu_k, \nu_{k+1}) \geq (\max(m_k - 1, m_{k+1}), m_k)$ with $m_{k+1} \leq m_k$ and want to show that $(\mu_k, \mu_{k+1}) \geq (m_k, m_{k+1})$ for every term $\mathbf{z}^\mu$ in $T_k(\mathbf{z}^\nu)$. If $m_{k+1} < m_k$, this follows from Lemma 6.1.7 (i) with $c = m_k - 1$. If $m_k = m_{k+1}$, it follows from Lemma 6.1.7 (iii).

In case (ii), we have $(\nu_k, \nu_{k+1}) \geq (\max(m_k, m_{k+1}), m_k)$ with $m_{k+1} \leq m_k + 1$ and want to show that $(\mu_k, \mu_{k+1}) \geq (m_k, m_{k+1})$ for every term $\mathbf{z}^\mu$ in $T_k^{-1}(\mathbf{z}^\nu)$. If $m_{k+1} = m_k + 1$, this follows from Lemma 6.1.7 (ii) with $c = m_k$. If $m_{k+1} \leq m_k$, it follows from Lemma 6.1.7 (iii). □

Lemma 6.1.9. *Given $\sigma \in S_l$ and $m, \lambda \in \mathbb{Z}^l$, if m is σ-almost decreasing and $\lambda \geq m$ coordinate-wise, then for every term $\mathbf{z}^\mu$ with non-zero coefficient in $E^\sigma_\lambda(\mathbf{z}; q)$, we have $\mu \geq m$ coordinate-wise.*

Proof. If λ is dominant, then $\mathbf{z}^\lambda$ is the only term and the result is a tautology. Otherwise, pick an index k such that $\lambda_k < \lambda_{k+1}$. Then the recurrence (90) gives

$$E^\sigma_\lambda = \begin{cases} q^{-1}\, T_k\, E^{s_k\sigma}_{s_k(\lambda)}, & \sigma^{-1}(k) < \sigma^{-1}(k+1), \\ T_k^{-1}\, E^{s_k\sigma}_{s_k(\lambda)}, & \sigma^{-1}(k) > \sigma^{-1}(k+1). \end{cases} \tag{154}$$

Let m' be the $(s_k\sigma)$-almost decreasing sequence given by Lemma 6.1.6 for this σ, m and k. Then $m'_k \leq \max(m_k, m_{k+1}) \leq \max(\lambda_k, \lambda_{k+1}) = \lambda_{k+1}$. Since m' and $s_k(m)$ agree in all but the k-th position, this shows that $s_k(\lambda) \geq m'$. By induction on $|\mathrm{Inv}(-\lambda)|$, we can assume that all terms $\mathbf{z}^\nu$ in $E^{s_k\sigma}_{s_k(\lambda)}$ satisfy $\nu \geq m'$. The result now follows from Lemma 6.1.8. □

Remark 6.1.10. Suppose λ itself is σ-almost decreasing. Then Lemma 6.1.9 with $m = \lambda$ implies that $\mu \geq \lambda$ coordinate-wise for every term $\mathbf{z}^\mu$ in E^σ_λ. Since E^σ_λ is homogeneous of degree $|\lambda|$, $\mu \geq \lambda$ implies $\mu = \lambda$. Hence, $E^\sigma_\lambda = \mathbf{z}^\lambda$. In fact, it can be shown that $E^\sigma_\lambda = \mathbf{z}^\lambda$ if and only if λ is σ-almost decreasing.

Lemma 6.1.11. *Given σ, m and λ as in Lemma 6.1.9, suppose that $m + \varepsilon_j$ is also σ-almost decreasing, where ε_j is the j-th unit coordinate vector. Then the coefficient of $z_j^{m_j}$ in E^σ_λ is given by*

$$\langle z_j^{m_j} \rangle\, E^\sigma_\lambda(\mathbf{z}; q) = \begin{cases} E^\tau_{\hat\lambda}(z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_l; q) & \lambda_j = m_j, \\ 0 & \lambda_j > m_j, \end{cases} \tag{155}$$

where $\hat\lambda = (\lambda_1, \dots, \lambda_{j-1}, \lambda_{j+1}, \dots, \lambda_l)$ and $\tau \in S_{l-1}$ is the permutation such that $\tau^{-1}(1), \dots, \tau^{-1}(l-1)$ are in the same relative order as $\sigma^{-1}(1), \dots, \sigma^{-1}(j-1), \sigma^{-1}(j+1), \dots, \sigma^{-1}(l)$.

Proof. If $\lambda_j > m_j$, we have $\lambda \geq m + \varepsilon_j$. Since $m + \varepsilon_j$ is assumed to be σ-almost decreasing, the result follows from Lemma 6.1.9 in this case. Now assume that $\lambda_j = m_j$.

If λ is dominant, so is $\hat\lambda$. Then both sides of (155) reduce to $z_1^{\lambda_1} \cdots z_{j-1}^{\lambda_{j-1}} z_{j+1}^{\lambda_{j+1}} \cdots z_l^{\lambda_l}$. If λ is not dominant, we proceed by induction on $|\mathrm{Inv}(-\lambda)|$, again using the recurrence (154) for some index k such that $\lambda_k < \lambda_{k+1}$.

If $k \notin \{j-1, j\}$, then using Lemma 6.1.6 for this σ, m and k, and using it again with $m + \varepsilon_j$ in place of m, we get a sequence m' such that both m' and $m' + \varepsilon_j$ are $(s_k\sigma)$-almost decreasing. We also have $s_k(\lambda) \geq m'$, as in the proof of Lemma 6.1.9. Then we have

$$\langle z_j^{m_j} \rangle\, E^{s_k\sigma}_{s_k(\lambda)}(\mathbf{z}; q) = E^{s_{k'}\tau}_{s_{k'}(\hat\lambda)}(z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_l; q) \tag{156}$$

by induction, where $k' = k$ if $k < j-1$ and $k' = k-1$ if $k > j$. Note that $s_{k'}\tau$ and $s_{k'}(\hat\lambda)$ are to $s_k\sigma$ and $s_k(\lambda)$ as τ and $\hat\lambda$ are to σ and λ.

Letting $\hat{\mathbf{z}}$ denote the variables with z_j omitted, the recurrence for $E^\tau_{\hat\lambda}$ takes the form

$$E^\tau_{\hat\lambda}(\hat{\mathbf{z}};q) = \begin{cases} q^{-1}\, T_k\, E^{s_{k'}\tau}_{s_{k'}(\hat\lambda)}(\hat{\mathbf{z}};q), & \sigma^{-1}(k) < \sigma^{-1}(k+1), \\ T_k^{-1}\, E^{s_{k'}\tau}_{s_{k'}(\hat\lambda)}(\hat{\mathbf{z}};q), & \sigma^{-1}(k) > \sigma^{-1}(k+1). \end{cases} \tag{157}$$

To see this, observe that the variables z_k, z_{k+1} on which T_k acts are in positions k', $k'+1$ in $\hat{\mathbf{z}}$, and that $\tau^{-1}(k') < \tau^{-1}(k'+1)$ if and only if $\sigma^{-1}(k) < \sigma^{-1}(k+1)$. For $k \notin \{j-1, j\}$, taking the coefficient of $z_j^{m_j}$ commutes with T_k, so (154), (156), and (157) imply $\langle z_j^{m_j}\rangle\, E^\sigma_\lambda(\mathbf{z};q) = E^\tau_{\hat\lambda}(\hat{\mathbf{z}};q)$, as desired.

Since $m+\varepsilon_j$ is σ-almost decreasing, we have $\lambda_{j-1} \geq m_{j-1} \geq m_j = \lambda_j$, so we never have $k = j-1$. This leaves the case $k = j$. In this case, let

$$m' = (m_1, \ldots, m_{j-1}, m'_j, m'_{j+1}, m_{j+2}, \ldots, m_l), \tag{158}$$

where

$$\begin{aligned} m'_j &= m_j + \chi(\sigma^{-1}(j) > \sigma^{-1}(j+1)), \\ m'_{j+1} &= m_j. \end{aligned} \tag{159}$$

Then $s_j(\lambda) \geq m'$, since $\lambda_{j+1} > \lambda_j$ implies $\lambda_{j+1} \geq m_j + 1 \geq m'_j$. Provided that m' and $m' + \varepsilon_{j+1}$ are $(s_j\sigma)$-almost decreasing, the inductive hypothesis implies

$$\langle z_{j+1}^{m'_{j+1}}\rangle\, E^{s_j\sigma}_{s_j(\lambda)}(\mathbf{z};q) = E^\tau_{\hat\lambda}(z_1, \ldots, z_j, z_{j+2}, \ldots, z_l; q), \tag{160}$$

since deleting position $j+1$ from $s_j\sigma$ and $s_j(\lambda)$ gives the same τ and $\hat\lambda$ as deleting position j from σ and λ. We now verify that m' and $m'+\varepsilon_{j+1}$ are indeed $(s_j\sigma)$-almost decreasing. We then complete the proof by showing that $\langle z_j^{m_j}\rangle E^\sigma_\lambda(\mathbf{z};q) = s_j\big(\langle z_{j+1}^{m_j}\rangle E^{s_j\sigma}_{s_j(\lambda)}(\mathbf{z};q)\big)$, which is equal to $E^\tau_{\hat\lambda}(\hat{\mathbf{z}};q)$ by (160) and the fact that $m_j = m'_{j+1}$.

For $j \notin \{r,s\}$, the conditions on entries in positions $r < s$ for m' and $m'+\varepsilon_{j+1}$ to be $(s_j\sigma)$-almost decreasing reduce to the conditions in positions $s_j(r) < s_j(s)$ for m and $m+\varepsilon_j$ to be σ-almost decreasing.

In positions $r < j$, the required condition for both m' and $m'+\varepsilon_{j+1}$ is $m_r \geq m'_j - \chi(\sigma^{-1}(r) > \sigma^{-1}(j+1))$. This follows from $m_r \geq m_j + 1 - \chi(\sigma^{-1}(r) > \sigma^{-1}(j))$, which holds because $m+\varepsilon_j$ is σ-almost decreasing, and $1 - \chi(\sigma^{-1}(r) > \sigma^{-1}(j)) \geq \chi(\sigma^{-1}(j) > \sigma^{-1}(j+1)) - \chi(\sigma^{-1}(r) > \sigma^{-1}(j+1))$, which is logically equivalent to $(\sigma^{-1}(r) < \sigma^{-1}(j+1)) \wedge (\sigma^{-1}(j) > \sigma^{-1}(j+1)) \Rightarrow (\sigma^{-1}(r) < \sigma^{-1}(j))$.

In positions j and $s > j+1$ the required condition is $m'_j \geq m_s - \chi(\sigma^{-1}(j+1) > \sigma^{-1}(s))$. This follows from $m_j \geq m_s - \chi(\sigma^{-1}(j) > \sigma^{-1}(s))$, which holds because m is σ-almost decreasing, and $\chi(\sigma^{-1}(j) > \sigma^{-1}(s)) \leq \chi(\sigma^{-1}(j) > \sigma^{-1}(j+1)) + \chi(\sigma^{-1}(j+1) > \sigma^{-1}(s))$, which is logically equivalent to $(\sigma^{-1}(j) < \sigma^{-1}(j+1)) \wedge (\sigma^{-1}(j+1) < \sigma^{-1}(s)) \Rightarrow (\sigma^{-1}(j) < \sigma^{-1}(s))$.

Finally, in positions j, $j+1$, the condition on $m'+\varepsilon_{j+1}$, which is stronger than the one on m', is $m'_j \geq m'_{j+1} + 1 - \chi(\sigma^{-1}(j+1) > \sigma^{-1}(j))$. This holds with equality by the definition of m'_j, m'_{j+1}.

We have left to prove that $\langle z_j^{m_j}\rangle E^\sigma_\lambda(\mathbf{z};q) = s_j\big(\langle z_{j+1}^{m_j}\rangle E^{s_j\sigma}_{s_j(\lambda)}(\mathbf{z};q)\big)$. We do this by using the expression for E^σ_λ in terms of $E^{s_j\sigma}_{s_j(\lambda)}$ given by the recurrence (154) with $k = j$.

By Lemma 6.1.9, every term $\mathbf{z}^\nu$ of $E^{s_j\sigma}_{s_j(\lambda)}$ satisfies $\nu \geq m'$.

In the case $\sigma^{-1}(j) < \sigma^{-1}(j+1)$, this gives $(\nu_j, \nu_{j+1}) \geq (m_j, m_j)$. If $\nu_{j+1} = m_j$, then we have $\langle z_j^{m_j}\rangle\, q^{-1}T_j\mathbf{z}^\nu = s_j(\langle z_{j+1}^{m_j}\rangle\, \mathbf{z}^\nu)$ by Lemma 6.1.7 (iv). If $\nu_{j+1} > m_j$, then $\langle z_j^{m_j}\rangle T_j\mathbf{z}^\nu = 0 = \langle z_{j+1}^{m_j}\rangle \mathbf{z}^\nu$ by Lemma 6.1.7 (i), and we again have $\langle z_j^{m_j}\rangle\, q^{-1}T_j\mathbf{z}^\nu = s_j(\langle z_{j+1}^{m_j}\rangle\, \mathbf{z}^\nu)$.

In the case $\sigma^{-1}(j) > \sigma^{-1}(j+1)$, we have $(\nu_j, \nu_{j+1}) \geq (m_j+1, m_j)$. If $\nu_{j+1} = m_j$, then $\langle z_j^{m_j}\rangle\, T_j^{-1}\mathbf{z}^\nu = s_j(\langle z_{j+1}^{m_j}\rangle\, \mathbf{z}^\nu)$ by Lemma 6.1.7 (v). If $\nu_{j+1} > m_j$, then $\langle z_j^{m_j}\rangle T_j\mathbf{z}^\nu = 0 = \langle z_{j+1}^{m_j}\rangle \mathbf{z}^\nu$ by Lemma 6.1.7 (iii), again giving $\langle z_j^{m_j}\rangle\, T_j^{-1}\mathbf{z}^\nu = s_j(\langle z_{j+1}^{m_j}\rangle\, \mathbf{z}^\nu)$.

In either case, (154) yields $\langle z_j^{m_j}\rangle E^\sigma_\lambda = s_j\big(\langle z_{j+1}^{m_j}\rangle E^{s_j\sigma}_{s_j(\lambda)}\big)$, as desired. ☐

The next two lemmas will be stated and proved for a composition $\mathbf{r}$ that we implicitly assume is strict. However, both lemmas generalize immediately to weak compositions by Remark 5.2.2 (ii).

Lemma 6.1.12. *Given a composition $\mathbf{r} = (r_1, \ldots, r_k)$, a permutation $\sigma \in S_k$, and a choice of $\rho_\mathbf{r}$, let $\mathbf{r}' = w_0^k(\mathbf{r})$ and $\rho_{\mathbf{r}'} = w_0 w_0^{\mathbf{r}}(\rho_\mathbf{r})$; i.e., the blocks of $\rho_{\mathbf{r}'}$ are those of $\rho_\mathbf{r}$ in reverse order. Then for any tuple of partitions $\lambda = (\lambda_{(1)}, \ldots, \lambda_{(k)})$ such that $\ell(\lambda_{(i)}) \leq r_i$, we have*

$$\mathbf{z}^{-\rho_\mathbf{r}} F^\sigma_{\mathbf{r},\lambda+\rho_\mathbf{r}}(\mathbf{z}; q) = w_0\big(\mathbf{z}^{-\rho_{\mathbf{r}'}} E^{w_0^k\sigma}_{\mathbf{r}',\lambda'+\rho_{\mathbf{r}'}}(\mathbf{z}; q^{-1})\big), \tag{161}$$

with notation $\lambda + \rho_\mathbf{r}$ and $\lambda' + \rho_{\mathbf{r}'}$ as in Theorem 6.1.3, and $\lambda' = (\lambda_{(k)}, \ldots, \lambda_{(1)})$.

Proof. Follows from (95). ☐

Lemma 6.1.13. *Given a composition $\mathbf{r} = (r_1, \ldots, r_k)$ and a permutation $\sigma \in S_k$, fix a choice of $\rho_\mathbf{r}$ with σ-almost decreasing block minima. Then the Laurent polynomials $\mathbf{z}^{-\rho_\mathbf{r}} E^\sigma_{\mathbf{r},\lambda+\rho_\mathbf{r}}(\mathbf{z}; q)$ for weights $\lambda \in \mathbb{N}^{|\mathbf{r}|} \cap X^+(GL_\mathbf{r})$ are polynomials in $\mathbf{z}$, and form a basis of the ring $\Bbbk[\mathbf{z}]^{S_\mathbf{r}}$ of $S_\mathbf{r}$ invariant polynomials.*

Remark 6.1.14. The condition $\lambda \in \mathbb{N}^{|\mathbf{r}|} \cap X^+(GL_\mathbf{r})$ means that λ is a concatenation of partitions $\lambda_{(1)}, \ldots, \lambda_{(k)}$ of lengths $\ell(\lambda_{(i)}) \leq r_i$, where $\lambda_{(i)}$ is padded with zeroes to length r_i.

Proof of Lemma 6.1.13. Let $\widehat{m} = (m_1^{r_1}, \ldots, m_k^{r_k})$ be the concatenation of constant blocks $(m_i^{r_i})$, where $m_1, \ldots, m_k$ are the block minima of $\rho_\mathbf{r}$. Then $\widehat{m}$ is $\widehat{\sigma}$-almost decreasing and $\lambda + \rho_\mathbf{r} \geq \widehat{m}$ for every $\lambda \in \mathbb{N}^{|\mathbf{r}|} \cap X^+(GL_\mathbf{r})$, so Lemma 6.1.9 implies that $\mathbf{z}^{-\widehat{m}} E^{\widehat{\sigma}}_{\lambda+\rho_\mathbf{r}}(\mathbf{z}; q)$ is a polynomial in $\mathbf{z}$. Note that $\rho'_\mathbf{r} = \rho_\mathbf{r} - \widehat{m}$ is the weight satisfying (82) whose block minima are equal to zero. Since $\boldsymbol{\delta}_\mathbf{r}$ commutes with multiplication by the $S_\mathbf{r}$ invariant monomial $\mathbf{z}^{\widehat{m}}$, we have $\mathbf{z}^{-\rho_\mathbf{r}} E^\sigma_{\mathbf{r},\lambda+\rho_\mathbf{r}} = \mathbf{z}^{-\rho'_\mathbf{r}} \boldsymbol{\delta}_\mathbf{r}\, \mathbf{z}^{-\widehat{m}} E^{\widehat{\sigma}}_{\lambda+\rho_\mathbf{r}}$.

We now check that the $S_\mathbf{r}$ invariant Laurent polynomial $\mathbf{z}^{-\rho'_\mathbf{r}} \boldsymbol{\delta}_\mathbf{r}\, \mathbf{z}^{-\widehat{m}} E^{\widehat{\sigma}}_{\lambda+\rho_\mathbf{r}}$ is in fact a polynomial in $\mathbf{z}$. Consider a monomial $\mathbf{z}^\nu$ appearing in $\mathbf{z}^{-\widehat{m}} E^{\widehat{\sigma}}_{\lambda+\rho_\mathbf{r}}(\mathbf{z}; q)$, which must satisfy $\nu \in \mathbb{N}^{|\mathbf{r}|}$ by the previous paragraph. From (85) we obtain

$$\mathbf{z}^{-\rho'_\mathbf{r}} \boldsymbol{\delta}_\mathbf{r}(\mathbf{z}^\nu) = \begin{cases} \pm\chi_{\nu_+ - \rho'_\mathbf{r}}(GL_\mathbf{r}) & \text{if } \nu \text{ is } GL_\mathbf{r} \text{ regular,} \\ 0 & \text{otherwise,} \end{cases} \tag{162}$$

where ν_+ is the $GL_\mathbf{r}$ dominant weight in the $S_\mathbf{r}$ orbit of ν. If $\nu \in \mathbb{N}^{|\mathbf{r}|}$ is $GL_\mathbf{r}$ regular, then $\nu_+ \in \mathbb{N}^{|\mathbf{r}|} \cap X^{++}(GL_\mathbf{r})$ satisfies $\nu_+ \geq \rho'_\mathbf{r}$ coordinate-wise, so $\chi_{\nu_+ - \rho'_\mathbf{r}}$ is a polynomial character.

Since the polynomial characters $\chi_\lambda(GL_\mathbf{r})$ for $\lambda \in \mathbb{N}^{|\mathbf{r}|} \cap X^+(GL_\mathbf{r})$ are a basis of $\Bbbk[\mathbf{z}]^{S_\mathbf{r}}$, it follows from (100) that the polynomials $\mathbf{z}^{-\rho_\mathbf{r}} E^\sigma_{\mathbf{r},\lambda+\rho_\mathbf{r}}$ are also a basis. ☐

Corollary 6.1.15. *If, instead, the block minima of $\rho_{\mathbf{r}}$ are σ-almost increasing, Lemma 6.1.13 holds with $\mathbf{z}^{-\rho_{\mathbf{r}}} F^{\sigma}_{\mathbf{r},\lambda+\rho_{\mathbf{r}}}(\mathbf{z};q)$ in place of $\mathbf{z}^{-\rho_{\mathbf{r}}} E^{\sigma}_{\mathbf{r},\lambda+\rho_{\mathbf{r}}}(\mathbf{z};q)$*

Proof. The reverse of a σ-almost increasing sequence is $(w_0\sigma)$-almost decreasing. In particular, if the block minima of $\rho_{\mathbf{r}}$ are σ-almost increasing, then those of $\rho_{\mathbf{r}'}$ in Lemma 6.1.12 are $(w_0^k\sigma)$-almost decreasing. The corollary now follows from Lemmas 6.1.12 and 6.1.13. □

Lemma 6.1.16. *Suppose we are given weak compositions $\mathbf{r} = (r_1, \dots, r_k)$, $\mathbf{r}' = (r'_1, \dots, r'_k)$ such that $r_i \leq r'_i$ for all i, a permutation $\sigma \in S_k$, and a choice of $\rho_{\mathbf{r}}$, $\rho_{\mathbf{r}'}$ with associated block maxima and minima, in keeping with Convention 6.1.2. Assume that $\rho_{\mathbf{r}}$ and $\rho_{\mathbf{r}'}$ have the same block maxima M_i, and that the block minima m_i of $\rho_{\mathbf{r}'}$ and n_i of $\rho_{\mathbf{r}}$ are σ-almost decreasing and satisfy $m_i \leq n_i$ for all i (note that $r_i \leq r'_i$ already implies $m_i \leq n_i$ for $r'_i > 0$).*

Let $\mathbf{z}$ (resp. $\mathbf{z}'$) be a list of $|\mathbf{r}|$ (resp. $|\mathbf{r}'|$) variables, subdivided into blocks $Z_1, \dots, Z_k$ (resp. $Z'_1, \dots, Z'_k$) of lengths $r_1, \dots, r_k$ (resp. $r'_1, \dots, r'_k$). Let $\lambda_{(1)}, \dots, \lambda_{(k)}$ be partitions with $\ell(\lambda_{(i)}) \leq r'_i$, and define $\lambda + \rho_{\mathbf{r}'}$ as in Theorem 6.1.3, so that by Lemma 6.1.13, $(\mathbf{z}')^{-\rho_{\mathbf{r}'}} E^{\sigma}_{\mathbf{r}',\lambda+\rho_{\mathbf{r}'}}(\mathbf{z}';q)$ is a symmetric polynomial in each block of variables Z'_i, and if $\ell(\lambda_{(i)}) \leq r_i$ for all i, then $\mathbf{z}^{-\rho_{\mathbf{r}}} E^{\sigma}_{\mathbf{r},\lambda+\rho_{\mathbf{r}}}(\mathbf{z};q)$ is a symmetric polynomial in each block of variables Z_i.

Upon specializing r_i of the variables in each block Z'_i to Z_i and setting the other variables to zero, we then have

$$(163)\qquad (\mathbf{z}')^{-\rho_{\mathbf{r}'}} E^{\sigma}_{\mathbf{r}',\lambda+\rho_{\mathbf{r}'}}(\mathbf{z}';q)\big|_{Z'_i \mapsto Z_i} = \begin{cases} \mathbf{z}^{-\rho_{\mathbf{r}}} E^{\sigma}_{\mathbf{r},\lambda+\rho_{\mathbf{r}}}(\mathbf{z};q) & \text{if } \ell(\lambda_{(i)}) \leq r_i \text{ for all } i, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Observe that the specialization property (163) is 'transitive'—that is, given $\mathbf{r} \leq \mathbf{r}' \leq \mathbf{r}''$ coordinate-wise, if (163) holds for specialization from $\mathbf{z}''$ to $\mathbf{z}'$ and from $\mathbf{z}'$ to $\mathbf{z}$, then it holds for specialization from $\mathbf{z}''$ to $\mathbf{z}$. This is true even if $\mathbf{r}$, $\mathbf{r}'$, $\mathbf{r}''$ are weak compositions, with some blocks of variables empty. Using this and Lemma 6.1.5, we can reduce to the case that the block minima of $\rho_{\mathbf{r}}$ and $\rho_{\mathbf{r}'}$ differ by a unit coordinate vector ε_j.

If $r'_j = 0$, then $\mathbf{r} = \mathbf{r}'$ and the specialization property is trivial, so we assume that $r'_j > 0$. Then $r_j = r'_j - 1$, and $r_i = r'_i$ for all $i \neq j$. Thus, we are specializing one variable in Z'_j to zero and leaving the other blocks Z'_i unchanged, apart from renaming the variables. We can also assume for simplicity that $r_i > 0$ for $i \neq j$, since deleting any parts $r_i = r'_i = 0$ preserves the hypotheses and does not change the conclusion.

Let $m = (m_1, \dots, m_k)$ be the sequence of block minima of $\rho_{\mathbf{r}'}$. That of $\rho_{\mathbf{r}}$ is then $m + \varepsilon_j$. Let $\widehat{m}$ be the concatenation of constant blocks $(m_i^{r_i})$, except for the j-th block, which we take to be $((m_j + 1)^{r_j}, m_j)$, so its length is $r'_j = r_j + 1$. The assumption that m and $m + \varepsilon_j$ are σ-almost decreasing implies that $\widehat{m}$ and $\widehat{m} + \varepsilon_{\widehat{\jmath}}$ are $\widehat{\sigma}$-almost decreasing, where $\widehat{\jmath} = r_1 + \cdots + r_j + 1$ is the last index in the j-th block.

Now consider the right hand side of (163). By Lemma 6.1.11, since $\lambda + \rho_{\mathbf{r}'} \geq \widehat{m}$, the coefficient $\langle (z'_{\widehat{\jmath}})^{m_j} \rangle\, E^{\widehat{\sigma}}_{\lambda+\rho_{\mathbf{r}'}}(\mathbf{z}';q)$ becomes $E^{\widehat{\sigma}}_{\lambda+\rho_{\mathbf{r}}}(\mathbf{z};q)$ after renaming the variables if $\ell(\lambda_{(j)}) \leq r_j$, or zero if $\ell(\lambda_{(j)}) = r'_j$. Note that $\widehat{\sigma}$ in $E^{\widehat{\sigma}}_{\lambda+\rho_{\mathbf{r}}}$ is defined with respect to $\mathbf{r}$, and that this is the τ in Lemma 6.1.11 when we take σ there to be $\widehat{\sigma}$ defined with respect to $\mathbf{r}'$, and j to be $\widehat{\jmath}$. Hence, the right hand side of (163) is given by

$$(164)\qquad \mathbf{z}^{-\rho_{\mathbf{r}}} \boldsymbol{\delta}_{\mathbf{r}} \Big(\big(\langle (z'_{\widehat{\jmath}})^{m_j} \rangle\, E^{\widehat{\sigma}}_{\lambda+\rho_{\mathbf{r}'}}(\mathbf{z}';q) \big)\big|_{\hat{\mathbf{z}}' \to \mathbf{z}} \Big),$$

where $\hat{\mathbf{z}}'$ stands for $\mathbf{z}'$ with $z'_{\widehat{\jmath}}$ omitted.

To complete the proof we need to show that (164) is equal to the left hand side of (163), which by definition is

$$(\mathbf{z}')^{-\rho_{\mathbf{r}'}} \boldsymbol{\delta}_{\mathbf{r}'} \, E^{\widehat{\sigma}}_{\lambda+\rho_{\mathbf{r}'}}(\mathbf{z}';q)\big|_{Z'_i \mapsto Z_i}. \tag{165}$$

For this it suffices to show that

$$(\mathbf{z}')^{-\rho_{\mathbf{r}'}} \boldsymbol{\delta}_{\mathbf{r}'} \, (\mathbf{z}')^{\nu}\big|_{Z'_i \mapsto Z_i} = \mathbf{z}^{-\rho_{\mathbf{r}}} \boldsymbol{\delta}_{\mathbf{r}} \Big(\big(\langle (z'_{\widehat{\jmath}})^{m_j} \rangle \, (\mathbf{z}')^{\nu} \big)\big|_{\hat{\mathbf{z}}' \to \mathbf{z}} \Big) \tag{166}$$

for all terms $(\mathbf{z}')^{\nu}$ occurring in $E^{\widehat{\sigma}}_{\lambda+\rho_{\mathbf{r}'}}(\mathbf{z}';q)$. Thus, by Lemma 6.1.9, we can assume $\nu \geq \widehat{m}$.

If ν is not $GL_{\mathbf{r}'}$ regular, then $(\mathbf{z}')^{-\rho_{\mathbf{r}'}} \boldsymbol{\delta}_{\mathbf{r}'} \, (\mathbf{z}')^{\nu} = 0$. If ν is $GL_{\mathbf{r}'}$ regular, then $(\mathbf{z}')^{-\rho_{\mathbf{r}'}} \boldsymbol{\delta}_{\mathbf{r}'} \, (\mathbf{z}')^{\nu} = \pm \chi_{\mu}(GL_{\mathbf{r}'})$, where $\nu_+ = \mu + \rho_{\mathbf{r}'}$ is the dominant $GL_{\mathbf{r}'}$ weight in the $S_{\mathbf{r}'}$ orbit of ν, that is, ν_+ is the weight obtained by sorting each block of ν into decreasing order. As in the proof of Lemma 6.1.13, only polynomial characters $\chi_{\mu}(GL_{\mathbf{r}'})$ arise, so we can regard the index μ as a tuple of partitions $(\mu_{(1)}, \ldots, \mu_{(k)})$ with $\ell(\mu_{(i)}) \leq r'_i$.

If $\nu_{\widehat{\jmath}} > m_j$, the right hand side of (166) vanishes. Since $\nu \geq \widehat{m}$, all entries of ν in the j-th block (of length $r'_j = r_j + 1$) are greater than m_j, and the same holds for ν_+. If ν is not $GL_{\mathbf{r}'}$ regular, the left side of (166) vanishes immediately. If ν is $GL_{\mathbf{r}'}$ regular, then the corresponding character $\chi_{\mu}(GL_{\mathbf{r}'})$ has $\ell(\mu_{(j)}) = r'_j$. In that case $\chi_{\mu}(GL_{\mathbf{r}'})$ vanishes upon specializing one of the variables Z'_j to zero. Hence the left hand side of (166) vanishes in either case.

If $\nu_{\widehat{\jmath}} = m_j$, then $\big(\langle (z'_{\widehat{\jmath}})^{m_j} \rangle \, (\mathbf{z}')^{\nu} \big)\big|_{\hat{\mathbf{z}}' \to \mathbf{z}} = \mathbf{z}^{\kappa}$, where κ is just ν with the entry in position $\widehat{\jmath}$ deleted. In this case $\nu_{\widehat{\jmath}}$ is strictly less than all other entries in the j-th block. Hence, ν is $GL_{\mathbf{r}'}$ regular if and only if κ is $GL_{\mathbf{r}}$ regular, and when these hold, the permutation $v \in S_{r_j} \subseteq S_{\mathbf{r}}$ such that $\kappa = v(\kappa_+)$ has the same length as the permutation $w \in S_{r'_j} \subseteq S_{\mathbf{r}'}$ such that $\nu = w(\nu_+)$. Indeed, the two permutations agree on the interval $[\widehat{\jmath} - r_j, \widehat{\jmath} - 1]$, and w fixes $\widehat{\jmath}$.

Hence, we have $(\mathbf{z}')^{-\rho_{\mathbf{r}'}} \boldsymbol{\delta}_{\mathbf{r}'} \, (\mathbf{z}')^{\nu} = \pm \chi_{\mu}(GL_{\mathbf{r}'})$ and $\mathbf{z}^{-\rho_{\mathbf{r}}} \boldsymbol{\delta}_{\mathbf{r}} \, \mathbf{z}^{\kappa} = \pm \chi_{\mu}(GL_{\mathbf{r}})$ for the same tuple of partitions μ such that $\ell(\mu_{(i)}) \leq r_i$, and with the same sign. Now (166) follows because $\chi_{\mu}(GL_{\mathbf{r}'})$ specializes to $\chi_{\mu}(GL_{\mathbf{r}})$ upon setting one variable in the j-th block Z'_j to zero. ☐

Corollary 6.1.17. *If the block minima of $\rho_{\mathbf{r}}$ and $\rho_{\mathbf{r}'}$ are σ-almost increasing instead of σ-almost decreasing, then Lemma 6.1.16 holds with $(\mathbf{z}')^{-\rho_{\mathbf{r}'}} F^{\sigma}_{\mathbf{r}',\lambda+\rho_{\mathbf{r}'}}(\mathbf{z}';q)$ and $\mathbf{z}^{-\rho_{\mathbf{r}}} F^{\sigma}_{\mathbf{r},\lambda+\rho_{\mathbf{r}}}(\mathbf{z};q)$ in place of $(\mathbf{z}')^{-\rho_{\mathbf{r}'}} E^{\sigma}_{\mathbf{r}',\lambda+\rho_{\mathbf{r}'}}(\mathbf{z}';q)$ and $\mathbf{z}^{-\rho_{\mathbf{r}}} E^{\sigma}_{\mathbf{r},\lambda+\rho_{\mathbf{r}}}(\mathbf{z};q)$.*

Proof. Follows from Lemma 6.1.16 using Lemma 6.1.12 in the same way that Corollary 6.1.15 follows from Lemma 6.1.13. ☐

Proof of Theorem 6.1.3. First we consider the case when $\mathbf{r} = \mathbf{s}$. Inverting the variables y_i and q, we are to prove

$$\frac{\prod_{i<j} \Omega[-q^{-1}\, t\, X_i \overline{Y_j}]}{\prod_{i\leq j} \Omega[-t\, X_i \overline{Y_j}]} = \sum_{\lambda} t^{|\lambda|}\, \mathbf{x}^{-\rho_{\mathbf{r}}} E^{\sigma}_{\mathbf{r},\lambda+\rho_{\mathbf{r}}}(\mathbf{x};q)\, \overline{\mathbf{y}^{-\rho_{\mathbf{r}}} F^{\sigma}_{\mathbf{r},\lambda+\rho_{\mathbf{r}}}(\mathbf{y};q)}, \tag{167}$$

where the index λ ranges over all k-tuples of partitions with $\ell(\lambda_{(i)}) \leq r_i$. By Corollary 6.1.15, the functions $\overline{\mathbf{y}^{-\rho_{\mathbf{r}}} F^{\sigma}_{\mathbf{r},\lambda+\rho_{\mathbf{r}}}(\mathbf{y};q)}$ appearing in the sum form a basis of $\Bbbk[y_1^{-1}, \ldots, y_r^{-1}]^{S_{\mathbf{r}}}$. We need to show that the coefficient of each such basis element in the product on the left hand side is $t^{|\lambda|}\,\mathbf{x}^{-\rho_{\mathbf{r}}} E^{\sigma}_{\mathbf{r},\lambda+\rho_{\mathbf{r}}}(\mathbf{x};q)$. By Proposition 5.3.1, the coefficient in question is given by

$$\Big\langle\, \mathbf{y}^{-\rho_{\mathbf{r}}} E^{\sigma}_{\mathbf{r},\lambda+\rho_{\mathbf{r}}}(\mathbf{y};q),\ \frac{\prod_{i<j}\Omega[-q^{-1}\,t\,X_i\overline{Y_j}]}{\prod_{i\leq j}\Omega[-t\,X_i\overline{Y_j}]}\,\Big\rangle^{\mathbf{r}}_{q}, \tag{168}$$

where the inner product is in the $\mathbf{y}$ variables. Note that it is permissible to perform such operations term by term in the power series in t on each side of (167).

By Lemma 6.1.13, $\mathbf{y}^{-\rho_{\mathbf{r}}} E^{\sigma}_{\mathbf{r},\lambda+\rho_{\mathbf{r}}}(\mathbf{y};q)$ is in fact a polynomial in $\mathbf{y}$ and not just a Laurent polynomial. Moreover, it is homogeneous of degree $|\lambda|$. Hence, the result follows if we show that

$$f(t\,\mathbf{x}) = \Big\langle\, f(\mathbf{y}),\ \frac{\prod_{i<j}\Omega[-q^{-1}\,t\,X_i\overline{Y_j}]}{\prod_{i\leq j}\Omega[-t\,X_i\overline{Y_j}]}\,\Big\rangle^{\mathbf{r}}_{q} \tag{169}$$

for every $S_{\mathbf{r}}$ invariant polynomial $f\in \Bbbk[\mathbf{y}]^{S_{\mathbf{r}}}$. Let X_i and Y_i outside the plethystic brackets stand for the list of variables in each block, so that $f(\mathbf{y}) = f(Y_1,\ldots,Y_k)$, for instance. Using the definition (110) of the inner product, the right hand side of (169) can then be written

$$\langle 1_{GL_{\mathbf{r}}}\rangle\, f(Y_1,\ldots,Y_k)\,\frac{\prod_{i<j}\Omega[-q^{-1}\,t\,X_i\overline{Y_j}]}{\prod_{i\leq j}\Omega[-t\,X_i\overline{Y_j}]}\prod_{i<j}\frac{\Omega[-Y_i\overline{Y_j}]}{\Omega[-q^{-1}Y_i\overline{Y_j}]}, \tag{170}$$

where the coefficient $\langle 1_{GL_{\mathbf{r}}}\rangle$ is taken in the $\mathbf{y}$ variables. Note that this is the same as taking the coefficient of $1_{GL_{r_i}}(Y_i)$ in each block of variables separately.

The only part of (170) that involves $\overline{Y_1}$ is the factor $\Omega[-tX_1\overline{Y_1}]^{-1} = \Omega[tX_1\overline{Y_1}]$; everything else involves only symmetric polynomials in Y_1. The classical Cauchy identity implies

$$\langle 1_{GL_{r_1}}\rangle g(Y_1)\,\Omega[tX_1\overline{Y_1}] = g(tX_1) \tag{171}$$

for every symmetric polynomial $g(Y_1)$, by reducing to the case that g is a Schur function. Taking the coefficient $\langle 1_{GL_{\mathbf{r}}}\rangle$ by starting with GL_{r_1} and using (171) reduces (170) to

$$\langle 1_{GL_{(r_2,\ldots,r_k)}}\rangle\, f(t\,X_1,Y_2,\ldots,Y_k)\,\frac{\prod_{1<i<j}\Omega[-q^{-1}\,t\,X_i\overline{Y_j}]}{\prod_{1<i\leq j}\Omega[-t\,X_i\overline{Y_j}]}\prod_{1<i<j}\frac{\Omega[-Y_i\overline{Y_j}]}{\Omega[-q^{-1}Y_i\overline{Y_j}]}, \tag{172}$$

once we observe that after removing the factor $\Omega[t\,X_1\overline{Y_1}]$ and setting $Y_1 = t\,X_1$ in the rest, all factors with index $i=1$ cancel. We can assume by induction on k that (172) reduces to $f(t\,\mathbf{x})$.

For the general case, choose an integer N less than or equal to all M_i, m_i and n_i. Define $\mathbf{r}' = (r'_1,\ldots,r'_k)$ by $r'_i = M_i - N + 1$ and choose $\rho_{\mathbf{r}'}$ to have block maxima M_i and (hence) constant block minima $m'_i = N$. The constant sequence $(N,\ldots,N)$ is both σ-almost decreasing and σ-almost increasing, so the case of the theorem with equal compositions holds for $\mathbf{r}'$ and $\rho_{\mathbf{r}'}$, by what was shown above (note that since we chose $N\leq M_i$ for all i, $\rho_{\mathbf{r}'}$ has no artificial zero-length blocks). Denote the blocks of variables in this case by X'_i, Y'_i. By the choice of N, we have $r_i, s_i \leq r'_i$, so there are at least as many variables in each block X'_i, Y'_i as in X_i, Y_i.

Specializing r_i of the variables in each X'_i to X_i and s_i of the variables in Y'_i to Y_i, and setting the other variables to zero, the left hand side of (147) with both compositions equal to $\mathbf{r}'$ reduces to the left hand side for compositions $\mathbf{r}$ and $\mathbf{s}$. By Lemma 6.1.16 and Corollary 6.1.17, the same thing happens on the right hand side. Thus, the general case follows from the case already proven. □

6.2. **Winding permutations.** We use the following notions from [6, Definition 5.2.1].

Definition 6.2.1. A permutation $\sigma \in S_k$ is a *winding permutation* if $\sigma(1), \ldots, \sigma(k)$ are in the same relative order as $c_1, \ldots, c_k$, where $c_i = \{y + x\, i\}$ are the fractional parts of an arithmetic progression, for any real x, y with x assumed irrational, so the c_i are distinct.

The *descent indicator* of σ is the $\{0,1\}$-valued vector $(\eta_1, \ldots, \eta_{k-1})$ defined by

$$\eta_i = \chi(\sigma(i) > \sigma(i+1)). \tag{173}$$

The *head* and *tail* of σ are the permutations $\tau, \theta \in S_{k-1}$ such that $\tau(1), \ldots, \tau(k-1)$ are in the same relative order as $\sigma(1), \ldots, \sigma(k-1)$ and $\theta(1), \ldots, \theta(k-1)$ are in the same relative order as $\sigma(2), \ldots, \sigma(k)$.

Proposition 6.2.4, below, is the counterpart of [6, Proposition 5.2.2] for semi-symmetric Hall-Littlewood polynomials. We start with a more general identity.

Lemma 6.2.2. *Let $\tau, \theta \in S_k$ and $\eta \in \mathbb{Z}^k$ be such that $|\eta_i - \eta_j| \leq 1$ for all i, j, and*

(a) *$\eta_i = \eta_j$ implies $\tau(i) < \tau(j) \Leftrightarrow \theta(i) < \theta(j)$, i.e., τ and θ are in the same relative order in positions i, j;*

(b) *$\eta_i - \eta_j = 1$ implies $\tau(i) > \tau(j)$ and $\theta(i) < \theta(j)$, i.e., $\theta(\eta)$ is dominant and $\tau(\eta)$ is antidominant.*

Given a composition $\mathbf{r} = (r_1, \ldots, r_k)$, let $\widehat{\eta} = (\eta_1^{r_1}, \ldots, \eta_k^{r_k})$ be the concatenation of constant blocks $(\eta_i^{r_i})$. Then for every $\mu \in X^{++}(GL_{\mathbf{r}})$ we have the identities

$$E^{\theta^{-1}}_{\mathbf{r},\mu}(\mathbf{z}; q) = \mathbf{z}^{\widehat{\eta}}\, E^{\tau^{-1}}_{\mathbf{r},\mu-\widehat{\eta}}(\mathbf{z}; q), \tag{174}$$

$$F^{\theta^{-1}}_{\mathbf{r},\mu}(\mathbf{z}; q) = \mathbf{z}^{\widehat{\eta}}\, F^{\tau^{-1}}_{\mathbf{r},\mu-\widehat{\eta}}(\mathbf{z}; q). \tag{175}$$

Proof. First we show that (174) implies (175). Note that $w_0\tau$, $w_0\theta$ and $-\eta$ satisfy the same hypothesis as τ, θ and η. Hence, assuming the validity of (174), we have

$$\begin{aligned} F^{\theta^{-1}}_{\mathbf{r},\mu}(\mathbf{z}; q) &= \mathbf{z}^{\rho_{\mathbf{r}} - w_0^{\mathbf{r}}(\rho_{\mathbf{r}})}\, \overline{E^{(w_0\theta)^{-1}}_{\mathbf{r},-w_0^{\mathbf{r}}(\mu)}(\mathbf{z}; q)} \\ &= \mathbf{z}^{\rho_{\mathbf{r}} - w_0^{\mathbf{r}}(\rho_{\mathbf{r}})}\, \overline{\mathbf{z}^{-\widehat{\eta}} E^{(w_0\tau)^{-1}}_{\mathbf{r},-w_0^{\mathbf{r}}(\mu)+\widehat{\eta}}(\mathbf{z}; q)} = \mathbf{z}^{\widehat{\eta}}\, F^{\tau^{-1}}_{\mathbf{r},\mu-\widehat{\eta}}(\mathbf{z}; q), \end{aligned} \tag{176}$$

using (94) and (107). It remains to prove (174).

Let $\sim$ denote equality up to a non-zero scalar factor. Since $\widehat{\eta}$ is $S_{\mathbf{r}}$ invariant, (100) implies that both sides of (174) have leading term $\mathbf{z}^{\rho_{\mathbf{r}}} \chi_{\mu-\rho_{\mathbf{r}}}$ with coefficient 1. Hence, if (174) holds up to $\sim$ equivalence, then it holds with equality.

Let $l = |\mathbf{r}|$, and let λ_+ denote the dominant weight in the S_l orbit of any GL_l weight $\lambda \in \mathbb{Z}^l$. The entries of η and $\widehat{\eta}$ take at most two values c and $c+1$. As in the proof of [6, Proposition 5.2.2], this implies that there is a $w \in S_l$ such that $\mu = w(\mu_+)$ and $\mu - \widehat{\eta} = w((\mu - \widehat{\eta})_+)$. Note that $(\mu - \widehat{\eta})_+ = \mu_+ - w^{-1}(\widehat{\eta})$. Hence, up to $\sim$ equivalence, (174) can be written

$$\boldsymbol{\delta}_{\mathbf{r}}\, T^{-1}_{\widetilde{\theta}}\, T_{\widetilde{\theta} w}(\mathbf{z}^{\mu_+}) \sim \mathbf{z}^{\widehat{\eta}}\, \boldsymbol{\delta}_{\mathbf{r}}\, T^{-1}_{\widetilde{\tau}}\, T_{\widetilde{\tau} w}(\mathbf{z}^{-w^{-1}(\widehat{\eta})}\, \mathbf{z}^{\mu_+}), \tag{177}$$

where $\widetilde{\theta} = (\widehat{\theta^{-1}})^{-1}$, $\widetilde{\tau} = (\widehat{\tau^{-1}})^{-1}$. Since multiplication by the $S_{\mathbf{r}}$ invariant monomial $\mathbf{z}^{\widehat{\eta}}$ commutes with $\boldsymbol{\delta}_{\mathbf{r}}$, (177) follows if we prove the operator identity $T_{\widetilde{\theta}}^{-1}\, T_{\widetilde{\theta}w} \sim \mathbf{z}^{\widehat{\eta}}\, T_{\widetilde{\tau}}^{-1}\, T_{\widetilde{\tau}w}\, \mathbf{z}^{-w^{-1}(\widehat{\eta})}$, or equivalently

$$(178)\qquad T_{\widetilde{\theta}w}^{-1}\, T_{\widetilde{\theta}}\, \mathbf{z}^{\widehat{\eta}}\, T_{\widetilde{\tau}}^{-1}\, T_{\widetilde{\tau}w} \sim \mathbf{z}^{w^{-1}(\widehat{\eta})}.$$

We now prove (178) for all $w \in S_l$ by induction on $\ell(w)$. For this we use the well-known operator identities (the same as [6, (115)])

$$(179)\qquad T_i\, \mathbf{z}^{\mu}\, T_i^{-1} = T_i^{-1}\, \mathbf{z}^{\mu}\, T_i = \mathbf{z}^{\mu} = \mathbf{z}^{s_i\mu} \quad \text{if } \langle \alpha_i^{\vee}, \mu\rangle \overset{\text{def}}{=} \mu_i - \mu_{i+1} = 0,$$

$$(180)\qquad T_i\, \mathbf{z}^{\mu}\, T_i = q\, \mathbf{z}^{s_i\mu} \quad \text{if } \langle \alpha_i^{\vee}, \mu\rangle = -1,$$

$$(181)\qquad T_i^{-1}\, \mathbf{z}^{\mu}\, T_i^{-1} = q^{-1}\, \mathbf{z}^{s_i\mu} \quad \text{if } \langle \alpha_i^{\vee}, \mu\rangle = 1,$$

which follow directly from the definition of T_i.

The base case $w = 1$ of the induction is trivial. Otherwise, let $w = vs_i > v$, and assume by induction that

$$(182)\qquad T_{\widetilde{\theta}v}^{-1}\, T_{\widetilde{\theta}}\, \mathbf{z}^{\widehat{\eta}}\, T_{\widetilde{\tau}}^{-1}\, T_{\widetilde{\tau}v} \sim \mathbf{z}^{v^{-1}(\widehat{\eta})}.$$

We have $T_{\widetilde{\theta}w}^{-1} = T_i^{e_1} T_{\widetilde{\theta}v}^{-1}$ and $T_{\widetilde{\tau}w} = T_{\widetilde{\tau}v} T_i^{e_2}$, where

$$(183)\qquad e_1 = \begin{cases} -1 & \widetilde{\theta}vs_i > \widetilde{\theta}v \\ 1 & \widetilde{\theta}vs_i < \widetilde{\theta}v \end{cases}, \qquad e_2 = \begin{cases} 1 & \widetilde{\tau}vs_i > \widetilde{\tau}v \\ -1 & \widetilde{\tau}vs_i < \widetilde{\tau}v \end{cases}.$$

Then (182) implies (178), provided we show that

$$(184)\qquad T_i^{e_1} \mathbf{z}^{v^{-1}(\widehat{\eta})} T_i^{e_2} \sim \mathbf{z}^{s_i v^{-1}(\widehat{\eta})} = \mathbf{z}^{w^{-1}(\widehat{\eta})}.$$

Set $a = v(i)$, $b = v(i+1)$, and let $a', b' \in [k]$ be the indices of the blocks containing a and b in the partition of $[l]$ into intervals of lengths $r_1, \ldots, r_k$. Note that $vs_i > v$ implies $a < b$, and that, in the same way, we have $\widehat{\tau}vs_i > \widehat{\tau}v \Leftrightarrow \widehat{\tau}(a) < \widehat{\tau}(b)$ and $\widehat{\theta}vs_i > \widehat{\theta}v \Leftrightarrow \widehat{\theta}(a) < \widehat{\theta}(b)$.

Case I: $\widehat{\eta}_a = \widehat{\eta}_b$. One way this can happen is if $a' = b'$, so a and b are in the same block. Since $\widetilde{\tau}$ and $\widetilde{\theta}$ are increasing on each block, we then have $\widetilde{\theta}vs_i > \widetilde{\theta}v$, $\widetilde{\tau}vs_i > \widetilde{\tau}v$, $e_1 = -1$, $e_2 = 1$. Otherwise, if $a' \neq b'$, we have $\eta_{a'} = \eta_{b'}$. By hypothesis, we then have $\tau(a') < \tau(b') \Leftrightarrow \theta(a') < \theta(b')$. By construction, this implies $\widetilde{\tau}(a) < \widetilde{\tau}(b) \Leftrightarrow \widetilde{\theta}(a) < \widetilde{\theta}(b)$, or equivalently $\widetilde{\tau}vs_i > \widetilde{\tau}v \Leftrightarrow \widetilde{\theta}vs_i > \widetilde{\theta}v$, hence $e_2 = -e_1$. Thus, we have $e_2 = -e_1$ either way, and since $\langle \alpha_i^{\vee}, v^{-1}(\widehat{\eta})\rangle = \widehat{\eta}_a - \widehat{\eta}_b = 0$, (184) reduces to (179).

Case II: $\widehat{\eta}_a - \widehat{\eta}_b = \eta_{a'} - \eta_{b'} = 1$. Then $\tau(a') > \tau(b')$ and $\theta(a') < \theta(b')$ by hypothesis, which implies $\widetilde{\tau}(a) > \widetilde{\tau}(b)$ and $\widetilde{\theta}(a) < \widetilde{\theta}(b)$ by construction. In other words, $\widetilde{\tau}vs_i > \widetilde{\tau}v$ and $\widetilde{\theta}vs_i < \widetilde{\theta}v$, so $e_1 = e_2 = -1$. In this case, $\langle \alpha_i^{\vee}, v^{-1}(\widehat{\eta})\rangle = 1$, so (184) reduces to (181).

Case III: $\widehat{\eta}_a - \widehat{\eta}_b = \eta_{a'} - \eta_{b'} = -1$. The reasoning in Case II with a and b exchanged gives $e_1 = e_2 = 1$. Since $\langle \alpha_i^{\vee}, v^{-1}(\widehat{\eta})\rangle = -1$, (184) reduces to (180). □

Remark 6.2.3. In the proof of Lemma 6.2.2, we implicitly assumed that $\mathbf{r}$ was a strict composition. However, with the conventions in Remark 5.2.2 (ii), the weak composition case follows from the strict composition case.

Proposition 6.2.4. *If $\sigma \in S_{k+1}$ is a winding permutation, then its descent indicator $\eta = (\eta_1, \dots, \eta_k)$ and head and tail $\tau, \theta \in S_k$ satisfy the hypotheses of Lemma 6.2.2. Hence, given a (strict or weak) composition $\mathbf{r} = (r_1, \dots, r_k)$, identities (174–175) hold for every $\mu \in X^{++}(GL_{\mathbf{r}})$.*

Proof. By definition, η is $\{0,1\}$-valued, so $|\eta_i - \eta_j| < 1$ for all i, j. Let $\mathbf{c} = (c_1, \dots, c_{k+1})$ be the sequence $c_i = \{y + xi\}$ in the definition of the winding permutation σ. Since adding an integer to x does not change $\mathbf{c}$, we can assume $0 < x < 1$. Then, since σ and $\mathbf{c}$ are in the same relative order, we have

$$\begin{aligned} \eta_i = 1 \;\Leftrightarrow\; \sigma(i) > \sigma(i+1) \;\Leftrightarrow\; c_i > c_{i+1} \;\Leftrightarrow\; c_{i+1} = c_i + x - 1, \\ \eta_i = 0 \;\Leftrightarrow\; \sigma(i) < \sigma(i+1) \;\Leftrightarrow\; c_i < c_{i+1} \;\Leftrightarrow\; c_{i+1} = c_i + x. \end{aligned} \tag{185}$$

If $\eta_i = \eta_j$, then $c_{i+1} - c_i = c_{j+1} - c_j$, so $c_{i+1} - c_{j+1} = c_i - c_j$ and $c_i < c_j \Leftrightarrow c_{i+1} < c_{j+1}$. Then $\sigma(i) < \sigma(j) \Leftrightarrow \sigma(i+1) < \sigma(j+1)$, or equivalently, $\tau(i) < \tau(j) \Leftrightarrow \theta(i) < \theta(j)$. This shows that hypothesis (a) in Lemma 6.2.2 is satisfied.

If $\eta_i - \eta_j = 1$, that is, if $\eta_i = 1$ and $\eta_j = 0$, then $c_{i+1} = c_i + x - 1$ and $c_{j+1} = c_j + x$ imply $c_{i+1} - c_{j+1} = c_i - c_j - 1$. Since $|c_i - c_j| < 1$ and $|c_{i+1} - c_{j+1}| < 1$, we must have $c_i - c_j > 0$ and $c_{i+1} - c_{j+1} < 0$. Then $\sigma(i) > \sigma(j)$ and $\sigma(i+1) < \sigma(j+1)$, or equivalently, $\tau(i) > \tau(j)$ and $\theta(i) < \theta(j)$. This shows that hypothesis (b) in Lemma 6.2.2 is satisfied.

By Remark 6.2.3, we can conclude that (174–175) hold even if $\mathbf{r}$ is a weak composition. □

7. Proof of the main results

7.1. **Stable form of the main theorem.** In §7.2, we prove the combinatorial Theorem 3.5.1 by restricting to the polynomial part of a stronger infinite series identity, given by the following theorem, which expresses the full Catalanimal in (39) in terms of LLT series.

Theorem 7.1.1. *Given a positive integer h and real numbers s, p with p irrational, let*

$$b_i = \lfloor s - p\,(i-1) \rfloor - \lfloor s - p\,i \rfloor, \qquad c_i = \{s - p\,(i-1)\} \tag{186}$$

for $i = 1, \dots, h+1$, where $\{a\} = a - \lfloor a \rfloor$ denotes the fractional part of a. Let $\sigma \in S_{h+1}$ be the permutation such that $\sigma(1), \dots, \sigma(h+1)$ are in the same relative order as $c_1, \dots, c_{h+1}$, and let $\tau, \theta \in S_h$ be its head and tail. Let $(u_1, \dots, u_h)$, $(v_1, \dots, v_h)$ be integer sequences which are respectively θ^{-1}-almost decreasing and τ^{-1}-almost increasing, that is,

$$u_i \geq u_j - \chi(\theta(i) > \theta(j)), \quad v_i \leq v_j + \chi(\tau(i) < \tau(j)) \qquad \text{for all } i < j, \tag{187}$$

and let $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_h) \in \mathbb{Z}_+^h$ be a sequence of positive integers with first differences

$$\gamma_{i+1} - \gamma_i = u_i - v_{i+1}. \tag{188}$$

Fix $\rho_{w_0(\boldsymbol{\gamma})}, \rho'_{w_0(\boldsymbol{\gamma})}$ satisfying (82) for $GL_{w_0(\boldsymbol{\gamma})}$, with block minima u_{h+1-i} for $\rho_{w_0(\boldsymbol{\gamma})}$ and v_{h+1-i} for $\rho'_{w_0(\boldsymbol{\gamma})}$. Then

$$\begin{aligned} &H(R_q, R_t, R_{qt}, ((u_1 - v_1 + b_1)^{\gamma_1}, \dots, (u_h - v_h + b_h)^{\gamma_h})) \\ &\qquad = \sum_{\lambda} t^{|\lambda|} \mathcal{L}^{\tau w_0}_{w_0(\boldsymbol{\gamma}),\, ((0;\lambda) + (b_h^{\gamma_h}, \dots, b_1^{\gamma_1}) + \rho_{w_0(\boldsymbol{\gamma})}) / ((\lambda;0) + \rho'_{w_0(\boldsymbol{\gamma})})}(\mathbf{z}; q), \end{aligned} \tag{189}$$

where λ ranges over tuples of partitions $(\lambda_{(h-1)},\ldots,\lambda_{(1)})$ such that $\ell(\lambda_{(i)})\leq \min(\gamma_i,\gamma_{i+1})$, we form $(0;\lambda)$ and $(\lambda;0)$ by prepending or appending an empty partition to λ, and we interpret these as weights in $X^+(GL_{w_0(\boldsymbol{\gamma})})$ by padding the i-th component to length γ_{h+1-i} and concatenating, as in Theorem 6.1.3.

The root sets are defined by $\alpha_{ij}\in R_q=R_t$ if $i<j$ are in distinct blocks of the partition of $\{1,\ldots,|\boldsymbol{\gamma}|\}$ into intervals of length γ_i, and $\alpha_{ij}\in R_{qt}$ if $i<j$ are in distinct, non-adjacent blocks; in other words, $R_q=R_t=R_+\setminus R_+(GL_{\boldsymbol{\gamma}})$ and $R_{qt}=[R_q,R_t]$.

Example 7.1.2. Before turning to the proof, we give an example to clarify the notation in Theorem 7.1.1. Let $h=4$, $p\approx .67$, and $s=6.5$. Then $(b_1,\ldots,b_5)=(1,0,1,1,0)$ and $(c_1,\ldots,c_5)\approx(.5,.83,.16,.49,.82)$. The permutation σ with the same relative order as the numbers c_i is $\sigma=(3,5,1,2,4)$ in one-line notation. Its head and tail are $\tau=(3,4,1,2)$ and $\theta=(4,1,2,3)$.

The sequences $(u_1,\ldots,u_4)=(1,2,0,0)$ and $(v_1,\ldots,v_4)=(1,0,3,2)$ satisfy the required almost-decreasing/increasing conditions. The sequence $\boldsymbol{\gamma}=(3,4,3,1)$ has first differences $(u_1,u_2,u_3)-(v_2,v_3,v_4)=(1,-1,-2)$.

On the left hand side of (189) we have the Catalanimal

$$H(R_q,R_t,R_{qt},(1,1,1,2,2,2,2,-2,-2,-2,-1))$$

in $|\boldsymbol{\gamma}|=11$ variables $\mathbf{z}=z_1,\ldots,z_{11}$. A root α_{ij} $(i<j)$ belongs to $R_q=R_t$ if i and j are in distinct blocks of the partition $\{\{1,2,3\},\{4,5,6,7\},\{8,9,10\},\{11\}\}$, and to R_{qt} if i and j are in non-adjacent blocks.

The terms on the right hand side of (189) are indexed by triples of partitions $\lambda_{(3)}=(\lambda_{3,1})$, $\lambda_{(2)}=(\lambda_{2,1},\lambda_{2,2},\lambda_{2,3})$, $\lambda_{(1)}=(\lambda_{1,1},\lambda_{1,2},\lambda_{1,3})$ of lengths at most 1, 3 and 3, respectively. More explicitly, the term indexed by a given triple is

$$\mathcal{L}^{(2,1,4,3)}_{(1,3,4,3),\beta/\alpha}(\mathbf{z};q),\quad\text{where}\quad \begin{aligned}\beta &= (\;1,\;1,\;1,\;1,\;0,\;0,\;0,\;0,\;1,\;1,\;1\;)\\ &\quad+(\;0,\;2,\;1,\;0,\;5,\;4,\;3,\;2,\;3,\;2,\;1\;)\\ &\quad+(\;0,\;\lambda_{3,1},\;0,\;0,\;\lambda_{2,1},\lambda_{2,2},\lambda_{2,3},0,\lambda_{1,1},\lambda_{1,2},\lambda_{1,3}\;)\\ \alpha &= (\;2,\;5,\;4,\;3,\;3,\;2,\;1,\;0,\;3,\;2,\;1\;)\\ &\quad+(\;\lambda_{3,1},\lambda_{2,1},\lambda_{2,2},\lambda_{2,3},\lambda_{1,1},\lambda_{1,2},\lambda_{1,3},0,\;0,\;0,\;0\;).\end{aligned}$$

Note that for $\lambda=\varnothing$, $\beta-\alpha$ reduces to the reverse of the weight vector in the Catalanimal.

Proof of Theorem 7.1.1. We will prove a stronger identity

$$\begin{aligned}(190)\quad & \mathbf{z}^{((u_1-v_1+b_1)^{\gamma_1},\ldots,(u_h-v_h+b_h)^{\gamma_h})}\frac{\prod_{\alpha\in R_{qt}}(1-q\,t\,\mathbf{z}^{\alpha})}{\prod_{\alpha\in R_t}(1-t\,\mathbf{z}^{\alpha})}\\ &=\sum_{\lambda}t^{|\lambda|}w_0\Big(F^{(\tau w_0)^{-1}}_{w_0(\boldsymbol{\gamma}),((0;\lambda)+(b_h^{\gamma_h},\ldots,b_1^{\gamma_1})+\rho_{w_0(\boldsymbol{\gamma})})}(\mathbf{z};q)\,\overline{E^{(\tau w_0)^{-1}}_{w_0(\boldsymbol{\gamma}),((\lambda;0)+\rho'_{w_0(\boldsymbol{\gamma})})}(\mathbf{z};q)}\Big).\end{aligned}$$

Then (189) follows from Proposition 5.4.3 after applying the operator $\mathbf{H}_q^{\boldsymbol{\gamma}}$ in (124) to both sides of (190).

By construction, we have $\gamma_i+u_i=\gamma_{i+1}+v_{i+1}$. Define

$$(191)\qquad M_1=\gamma_1+v_1-1,\ M_2=\gamma_2+v_2-1=\gamma_1+u_1-1,\ \ldots,\ M_{h+1}=\gamma_h+u_h-1.$$

We now apply our Cauchy identity, Theorem 6.1.3, with $k = h+1$, taking the σ there to be $(\sigma w_0)^{-1}$, setting

$$\mathbf{r} = (0, \gamma_h, \dots, \gamma_1), \quad \mathbf{s} = (\gamma_h, \dots, \gamma_1, 0), \tag{192}$$

and taking $\rho_{\mathbf{r}}$ and $\rho_{\mathbf{s}}$ to have block maxima $M_{h+1}, \dots, M_1$. We ascribe arbitrarily large artificial minima denoted by ∞ to the empty first block of $\rho_{\mathbf{r}}$ and last block of $\rho_{\mathbf{s}}$. The block minima are then

$$\rho_{\mathbf{r}} \colon (\infty, v_h, \dots, v_1), \quad \rho_{\mathbf{s}} \colon (u_h, \dots, u_1, \infty). \tag{193}$$

By (187), these sequences are $(\sigma w_0)^{-1}$-almost decreasing and $(\sigma w_0)^{-1}$-almost increasing, respectively, so the hypotheses of Theorem 6.1.3 are satisfied. Note that (193) also implies that $\rho_{\mathbf{r}} = \rho'_{w_0(\boldsymbol{\gamma})}$ and $\rho_{\mathbf{s}} = \rho_{w_0(\boldsymbol{\gamma})}$.

We set the variables $\mathbf{x}$ and $\mathbf{y}$ in (147) to $\mathbf{x} = w_0(\overline{\mathbf{z}})$, $\mathbf{y} = w_0(\mathbf{z})$, that is, $x_i = z_{l+1-i}^{-1}$, $y_i = z_{l+1-i}$, where $l = |\boldsymbol{\gamma}|$. Grouping the variables $\mathbf{x}$ into blocks X_i of size r_i, the variables $\mathbf{y}$ into blocks Y_i of size s_i, and the variables $\mathbf{z}$ into blocks Z_i of size γ_i, we then have $(X_1, \dots, X_{h+1}) = (0, \overline{Z_h}, \dots, \overline{Z_1})$ and $(Y_1, \dots, Y_{h+1}) = (Z_h, \dots, Z_1, 0)$. Theorem 6.1.3 now yields

$$\begin{aligned} \frac{\prod_{\alpha \in R_{qt}} (1 - q\, t\, \mathbf{z}^\alpha)}{\prod_{\alpha \in R_t} (1 - t\, \mathbf{z}^\alpha)} &= \frac{\prod_{i+1<j} \Omega[-q\, t\, Z_i \overline{Z_j}]}{\prod_{i<j} \Omega[-t\, Z_i \overline{Z_j}]} \\ &= \mathbf{z}^{((v_1-u_1)^{\gamma_1}, \dots, (v_h-u_h)^{\gamma_h})} \sum_\lambda t^{|\lambda|} w_0 \Big(F^{(\sigma w_0)^{-1}}_{\mathbf{s},(0;\lambda)+\rho_{w_0(\boldsymbol{\gamma})}}(\mathbf{z}; q) \overline{E^{(\sigma w_0)^{-1}}_{\mathbf{r},(\lambda;0)+\rho'_{w_0(\boldsymbol{\gamma})}}(\mathbf{z}; q)} \Big), \end{aligned} \tag{194}$$

where we used $\rho_{\mathbf{r}} = \rho'_{w_0(\boldsymbol{\gamma})}$, $\rho_{\mathbf{s}} = \rho_{w_0(\boldsymbol{\gamma})}$, and

$$\mathbf{y}^{-\rho_{\mathbf{s}}} \mathbf{x}^{-\rho_{\mathbf{r}}} = w_0(\mathbf{z}^{\rho'_{w_0(\boldsymbol{\gamma})} - \rho_{w_0(\boldsymbol{\gamma})}}) = \mathbf{z}^{((v_1-u_1)^{\gamma_1}, \dots, (v_h-u_h)^{\gamma_h})}. \tag{195}$$

Because of the zero-length first block in $\mathbf{r}$ and last block in $\mathbf{s}$, the sum in (147) is over tuples of partitions of the form $(0, \lambda_{(h-1)}, \dots, \lambda_{(1)}, 0)$, where $\ell(\lambda_{(i)}) \le \min(\gamma_i, \gamma_{i+1})$. These become $(\lambda; 0)$ and $(0; \lambda)$ when interpreted as weights for $GL_{\mathbf{r}}$ and $GL_{\mathbf{s}}$, respectively.

Dropping zero-length blocks as in Remark 5.2.2 (ii) gives

$$E^{(\sigma w_0)^{-1}}_{\mathbf{r},(\lambda;0)+\rho'_{w_0(\boldsymbol{\gamma})}}(\mathbf{z}; q) = E^{(\tau w_0)^{-1}}_{w_0(\boldsymbol{\gamma}),(\lambda;0)+\rho'_{w_0(\boldsymbol{\gamma})}}(\mathbf{z}; q), \tag{196}$$

$$F^{(\sigma w_0)^{-1}}_{\mathbf{s},(0;\lambda)+\rho_{w_0(\boldsymbol{\gamma})}}(\mathbf{z}; q) = F^{(\theta w_0)^{-1}}_{w_0(\boldsymbol{\gamma}),(0;\lambda)+\rho_{w_0(\boldsymbol{\gamma})}}(\mathbf{z}; q). \tag{197}$$

To complete the proof, we observe that

$$b_i = p + c_{i+1} - c_i = \begin{cases} \lfloor p \rfloor, & c_i > c_{i+1}, \\ \lceil p \rceil, & c_i < c_{i+1}. \end{cases} \tag{198}$$

Hence $(b_h, \dots, b_1) = \eta + \lfloor p \rfloor (1^h)$, where η is the descent indicator of the permutation $\sigma w_0 \in S_{h+1}$. Since σw_0 is a winding permutation with head θw_0^h and tail τw_0^h, (107) and (175) imply

$$F^{(\theta w_0)^{-1}}_{w_0(\boldsymbol{\gamma}),(0;\lambda)+\rho_{w_0(\boldsymbol{\gamma})}}(\mathbf{z}; q) = \mathbf{z}^{-(b_h^{\gamma_h}, \dots, b_1^{\gamma_1})} F^{(\tau w_0)^{-1}}_{w_0(\boldsymbol{\gamma}),(0;\lambda)+(b_h^{\gamma_h}, \dots, b_1^{\gamma_1})+\rho_{w_0(\boldsymbol{\gamma})}}(\mathbf{z}; q). \tag{199}$$

Combining (194), (196–197), and (199) gives (190). □

7.2. **Proof of the main combinatorial theorem.** We now prove Theorem 3.5.1, using Theorem 7.1.1. Before giving the full proof, we outline the argument. Theorem 7.1.1 gives an expansion

$$H = \sum_{\lambda} t^{|\lambda|} \mathcal{L}^{\tau w_0}_{w_0(\boldsymbol{\gamma}),\, \beta/\alpha}(\mathbf{z}; q)$$

for the Catalanimal H in Theorem 3.5.1, in which β and α depend on λ. Applying Theorem 5.5.4, we find that the surviving terms in H_{pol} belong to those λ for which we have $\beta \geq \alpha$ coordinate-wise, and these correspond one-to-one with nests in the given den by Lemma 3.2.1. Theorem 5.5.4 also gives us an expression $t^{|\lambda|}\mathcal{L}^{\tau w_0}_{w_0(\boldsymbol{\gamma}),\, \beta/\alpha}(\mathbf{z}; q)_{\mathrm{pol}} = \sum t^{|\lambda|} q^{h_{\tau w_0}(\boldsymbol{\beta}/\boldsymbol{\alpha})} \mathcal{G}_{\tau w_0(\boldsymbol{\beta}/\boldsymbol{\alpha})}(\mathbf{z}; q^{-1})$ for these terms. We then complete the proof by using the results of §§3.3–3.4 to verify that for the corresponding nest π, we have $a(\pi) = |\lambda|$, $\mathrm{dinv}_p(\pi) = h_{\tau w_0}(\boldsymbol{\beta}/\boldsymbol{\alpha})$, and $\tau w_0(\boldsymbol{\beta}/\boldsymbol{\alpha}) = \boldsymbol{\nu}(\pi)$.

Proof of Theorem 3.5.1. Let $(h, p, \mathbf{d}, \mathbf{e})$ be the given den, with $\mathbf{g}$ as in (24). We fix s such that the line $y + p\,x = s$ passes above the heads and feet, as in §3.4, and define $f_i = \lfloor s - p\,i \rfloor$ to be the y-coordinate of the highest lattice point below the line $y + p\,x = s$ at $x = i$. For $i = 1, \ldots, h$, we set

$$u_i = f_i - e_i, \quad v_i = f_{i-1} - d_{i-1}. \tag{200}$$

We will apply Theorem 7.1.1 with these values of s, p, u_i and v_i. The numbers in (186) are then given by $b_i = f_{i-1} - f_i$ and $c_{i+1} = s - p\,i - f_i$.

We start by verifying that the hypotheses in (187) are satisfied. The hypothesis on the u_i can be restated as $u_i \geq u_j - \chi(c_{i+1} > c_{j+1})$ for $i < j$. Since the c_i are distinct and $c_i \in [0, 1)$ for all i, this is equivalent to $u_i + c_{i+1} > u_j + c_{j+1} - 1$, or to $u_i - p\,i - f_i > u_j - p\,j - f_j - 1$. The latter reduces to condition (22) in the definition of a den. Similarly, the hypothesis on the v_i is equivalent to $v_i + c_i < v_j + c_j + 1$ for $i < j$, which reduces to (21).

In (188), we have $u_i - v_{i+1} = d_i - e_i$, so we can take $\gamma_i = g_i$. We also have $u_i - v_i + b_i = d_{i-1} - e_i$. Hence, the Catalanimal in (189) coincides with the Catalanimal H in (39).

Turning to the right hand side of (189), we use Theorem 5.5.4 to evaluate the polynomial part of $\mathcal{L}^{\tau w_0}_{w_0(\boldsymbol{\gamma}),\, ((0;\lambda)+(b_h^{\gamma_h},\ldots,b_1^{\gamma_1})+\rho_{w_0(\boldsymbol{\gamma})})/((\lambda;0)+\rho'_{w_0(\boldsymbol{\gamma})})}(\mathbf{z}; q) = \mathcal{L}^{\tau w_0}_{w_0(\boldsymbol{\gamma}),\, \beta/\alpha}(\mathbf{z}; q)$, where

$$\beta = (0; \lambda) + (b_h^{\gamma_h}, \ldots, b_1^{\gamma_1}) + \rho_{w_0(\boldsymbol{\gamma})} \tag{201}$$

$$\alpha = (\lambda; 0) + \rho'_{w_0(\boldsymbol{\gamma})}. \tag{202}$$

The weight β is the concatenation of blocks

$$\lambda_{(k)} + ((f_{k-1} - e_k)^{g_k}) + (g_k - 1, g_k - 2, \cdots, 0) \tag{203}$$

in the order $k = h, h-1, \ldots, 1$, if we set $\lambda_{(h)} = \varnothing$. Similarly, α is the concatenation of blocks

$$\lambda_{(k-1)} + ((f_{k-1} - d_{k-1})^{g_k}) + (g_k - 1, g_k - 2, \cdots, 0) \tag{204}$$

in the same order, with $\lambda_{(0)} = \varnothing$. The $\lambda_{(k)}$ for $1 \leq k \leq h-1$ vary over partitions of length $\ell(\lambda_{(k)}) \leq \min(g_k, g_{k+1})$. By Theorem 5.5.4, $\mathcal{L}^{\tau w_0}_{w_0(\boldsymbol{\gamma}),\, \beta/\alpha}(\mathbf{z}; q)_{\mathrm{pol}} = 0$ unless $\alpha \leq \beta$ coordinate-wise. From (203) and (204), we see that $\alpha \leq \beta$ if and only if $(\lambda_{(k-1)})_i - d_{k-1} \leq (\lambda_{(k)})_i - e_k$ for $k = 1, \ldots, h$ and $i \leq g_k$. Since $d_{k-1} = e_{k-1} + g_k - g_{k-1}$, this is equivalent to the condition

$e_k - g_k - (\lambda_{(k)})_i \leq e_{k-1} - g_{k-1} - (\lambda_{(k-1)})_i$ in Lemma 3.2.1. By that lemma, the indices λ for which $\mathcal{L}^{\tau w_0}_{w_0(\boldsymbol{\gamma}),\,\beta/\alpha}(\mathbf{z};q)_{\mathrm{pol}} \neq 0$ correspond to nests π in the given den.

For these indices, Theorem 5.5.4 gives

$$\mathcal{L}^{\tau w_0}_{w_0(\boldsymbol{\gamma}),\,\beta/\alpha}(\mathbf{z};q)_{\mathrm{pol}} = q^{h_{\tau w_0}(\boldsymbol{\beta}/\boldsymbol{\alpha})}\mathcal{G}_{\tau w_0(\boldsymbol{\beta}/\boldsymbol{\alpha})}(z_1,\ldots,z_l;q^{-1}), \tag{205}$$

where $l = |\mathbf{g}|$, and $\boldsymbol{\beta}/\boldsymbol{\alpha}$ is related to α and β by the recipe in (133) for blocks of lengths $g_h,\ldots,g_1$. Writing $\boldsymbol{\beta}/\boldsymbol{\alpha} = (\beta_{(h)}/\alpha_{(h)},\ldots,\beta_{(1)}/\alpha_{(1)})$ with decreasing indices and using (203) and (204), this recipe gives $\beta_{(k)} = ((f_{k-1}-e_k+g_k)^{g_k})+\lambda_{(k)}$ and $\alpha_{(k)} = ((f_{k-1}-d_{k-1}+g_k)^{g_k})+\lambda_{(k-1)} = ((f_{k-1}-e_{k-1}+g_{k-1})^{g_k})+\lambda_{(k-1)}$. The permutation σ in Definition 3.4.1 for the given den and choice of s is the same as τ in (205). Hence, by Remark 3.4.3, $\beta_{(k)}/\alpha_{(k)} = \boldsymbol{\nu}(\pi)_{\tau(k)}$, so $\tau w_0(\boldsymbol{\beta}/\boldsymbol{\alpha}) = \tau(\beta_{(1)}/\alpha_{(1)},\ldots,\beta_{(h)}/\alpha_{(h)}) = \boldsymbol{\nu}(\pi)$.

As noted in Definition 3.3.1, we have $a(\pi) = |\lambda|$ for the nest π corresponding to λ.

We now show that $\mathrm{dinv}_p(\pi) = h_{\tau w_0}(\boldsymbol{\beta}/\boldsymbol{\alpha})$. Because the components $\beta_{(i)}/\alpha_{(i)}$ of $\boldsymbol{\beta}/\boldsymbol{\alpha}$ are indexed in decreasing order, a (τw_0)-triple (a,b,c) in $\boldsymbol{\beta}/\boldsymbol{\alpha}$ has a, c in $\beta_{(j)}/\alpha_{(j)}$ and b in $\beta_{(i)}/\alpha_{(i)}$ for $i > j$, with content $c(b)$ equal to $c(a)$ if $\tau(i) > \tau(j)$, or to $c(a)+1$ if $\tau(i) < \tau(j)$. As in §3.4, the box $b \in \beta_{(i)}/\alpha_{(i)}$ corresponds to an element $(S,k) \in \mathbf{S}(\pi)$ with S on the line $x = i-1$, with $c(b)+c_i$ equal to the vertical distance between the line $y + p\,x = s$ and the south endpoint of S. Similarly, even though boxes a and c need not actually be in $\beta_{(j)}/\alpha_{(j)}$, the boundary between them corresponds to a non-sink lattice point P at $x = j-1$ on some path $\pi_{k'}$ in π, with $c(a)+c_j$ equal to the vertical distance between $y + p\,x = s$ and P.

The tuple (P,k',S,k) is counted by $\mathrm{dinv}_p(\pi)$ if and only if $0 < c(b)-c(a)+c_i-c_j < 1$. Since $c_i, c_j \in [0,1)$, we have $|c_i - c_j| < 1$, and by the definition of τ, we have $\tau(i) < \tau(j)$ if and only if $c_i < c_j$. If $\tau(i) > \tau(j)$, it follows that $0 < c(b)-c(a)+c_i-c_j < 1$ if and only if $c(b) = c(a)$, while if $\tau(i) < \tau(j)$, it follows that $0 < c(b)-c(a)+c_i-c_j < 1$ if and only if $c(b) = c(a)+1$. Hence, tuples (P,k',S,k) counted by $\mathrm{dinv}_p(\pi)$ are in bijective correspondence with (τw_0)-triples in $\boldsymbol{\beta}/\boldsymbol{\alpha}$, giving $\mathrm{dinv}_p(\pi) = h_{\tau w_0}(\boldsymbol{\beta}/\boldsymbol{\alpha})$.

From (205) and the expressions for $a(\pi)$ and $\mathrm{dinv}_p(\pi)$ we see that the polynomial part of the series on the right hand side of (189) is equal to

$$\sum_{\pi} t^{a(\pi)} q^{\mathrm{dinv}_p(\pi)} \mathcal{G}_{\boldsymbol{\nu}(\pi)}(z_1,\ldots,z_l,q^{-1}), \tag{206}$$

where the sum is over nests π in the given den. Since the Catalanimal on the left hand side of (189) is equal to H, this proves Theorem 3.5.1. $\square$

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