Comments, corrections, and related references welcomed, as always!

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SOME FRUSTRATING QUESTIONS ON DIMENSIONS OF PRODUCTS OF POSETS

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Abstract. Definitions, in particular that of the dimension of a poset, and examples, are recalled. For a subposet P of a direct product of $d > 0$ chains, and an integer $n > 0$, a condition is developed which implies that for any family of n chains $(T_j)_{j\in n}$, one has $\dim(P \times \prod_{j\in n} T_j) \leq d$. Applications are noted.

Open questions, old and new, on dimensions of product posets are noted, and some numerical invariants of posets that seem useful for studying these questions are developed.

In a final section (independent of the other results), we note that by a result of model theory, an infinite poset P will have finite dimension d if and only if d is the supremum of the dimensions of its finite subposets.

1. Definitions and examples

We assume the reader familiar with the definition of a *partially ordered set*, for which we will use the short term *poset*, and of the special case of a *totally ordered set*, also called a *chain*. We will formally consider a poset P to be an ordered pair $(|P|, \leq_P)$, where $|P|$ is the underlying set and $\leq$ the order relation; but when there no danger of ambiguity, we shall write $p \in P$ to mean $p \in |P|$, and $\leq$ for $\leq_P$.

Posets will always be understood to be nonempty.

We will write $x \geq y$, $x \geq_P y$, etc., for $y \leq x$, $y \leq_P x$, etc., and will use $<$, $<_P$, $>$, $>_P$, etc. for the conjunctions of the relations $\leq$ etc. with the relation $\neq$. We shall use $\not\leq$ for the negation of $\leq$, and likewise $\not\geq$ for the negation of $\geq$, etc..

Recall that a set-map $f : P \to Q$ between posets is called *isotone* if

(1.1) $\quad p \leq p' \implies f(p) \leq f(p')$ for all $p, p' \in P$,

and is called an *embedding* if it is isotone, and also satisfies

(1.2) $\quad p \not\leq p' \implies f(p) \not\leq f(p')$ for all $p, p' \in P$.

Clearly, an embedding of posets is one-to-one; but a one-to-one isotone map need not be an embedding.

A *linearization* of a partial ordering $\leq$ on a set X means a *total* ordering $\leq'$ on X which extends $\leq$, in the sense that for $x, y \in X$, if $x \leq y$ then $x \leq' y$; in other words, such that the identity map of X is an isotone map $(X, \leq) \to (X, \leq')$. It is easy to verify that every partial ordering on a set admits a linearization, and in fact that

(1.3) Every partial ordering $\leq$ on a set X is (as a set of ordered pairs) the intersection of its linearizations.

(Idea of proof: Given $x, y \in X$ such that $x \not\leq y$, show that there is a strengthening $\leq'$ of $\leq$ such that $x >' y$. Iterating this process infinitely many times if necessary, we get a linearization of $\leq$. Moreover, by our choice of the pair with $x \not\leq y$ that we start with, we can insure that any prechosen order-relation that does not hold under $\leq$ fails to hold in our linearization. Hence looking at *all* linearizations, we have (1.3).)

By the *product* $P\times Q$ of two posets P and Q one understands the poset whose underlying set is $|P|\times|Q|$, ordered so that $(p, q) \leq (p', q')$ if and only if $p \leq p'$ and $q \leq q'$; and analogously for products of larger families of posets. (These are in fact products in the category of posets and isotone maps, though we shall not use category-theoretic language in this note.) Thus (1.3) says that every poset P embeds in the product

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of the chains obtained from P using all linearizations of $\leq_P$. Occasionally, I shall refer to a product of posets as their 'direct product', when this seems desirable for clarity.

We now come to the concept we will be studying in this note.

Definition 1.1. *The dimension* $\dim(P)$ *of a poset* P *is the least cardinal* κ *such that*
(i) *the relation* $\leq_P$ *is an intersection of* κ *total orderings on* $|P|$,
equivalently, such that
(ii) P *is embeddable in a product of* κ *totally ordered sets.*

In the above definition, the implication (i) $\implies$ (ii) is clear. To get (ii) $\implies$ (i) (= [16, Theorem 10.4.2]) first consider any isotone map $f : P \to T$ where T is a totally ordered set. For each $t \in T$, regard the inverse image of t in P as a subposet, and choose a linearization $\leq_t$ of the partial ordering of that subposet. We can now strengthen $\leq_P$ to a linear ordering $\leq'$ of $|P|$ by making $p \leq' q$ if either $f(p) <_T f(q)$, or $f(p) = f(q)$ and $p \leq_{f(p)} q$. Thus, given an embedding of P in a product of κ posets T_α, if we construct as above from each of the projections $f(\alpha) : P \to T_\alpha$ a linearization $\leq'_\alpha$ of $\leq_P$, we see that the intersection of these linearizations will again be the partial ordering $\leq_P$.

We understand the product of the *empty* family of sets to be a singleton. Thus a poset has dimension 0 if and only if its underlying set is a singleton.

In the literature on dimensions of posets, condition (i) above is generally the preferred definition; but here we will more often use (ii).

This note will focus almost entirely on *finite-dimensional* posets, though we will allow underlying sets of these posets to be infinite. In indexing finite families of maps etc., we will follow the set-theorists' convention,

(1.4) For n a natural number, $n = \{0, \ldots, n-1\}$.

For positive integers n, two important examples of posets of dimension n are:

(1.5) The n-cube 2^n, i.e., the n-fold direct product $2 \times \cdots \times 2$, where 2 denotes the poset $\{0, 1\}$ with the ordering $0 < 1$,

and, for $n \geq 2$,

(1.6) The "standard example" S_n, whose underlying set consists of $2n$ elements $\{a_0, \ldots, a_{n-1}, b_0, \ldots, b_{n-1}\}$, with the ordering that makes each a_i less than every b_j *other* than b_i, and with no other order-relations among these $2n$ elements.

For $n \geq 3$, S_n can be identified with a subposet of 2^n, by identifying each a_i with the n-tuple that has value 1 in the i-th coordinate and 0 elsewhere, and each b_i with the n-tuple that has value 0 in the i-th coordinate and 1 elsewhere. (S_n is sometimes called the "crown" of dimension n. This nicely fits the appearance of the diagram when $n = 3$, but not so nicely for higher n.)

To see that (1.5) and (1.6) both have dimension n, note first the inequalities

(1.7) $\dim(S_n) \leq \dim(2^n) \leq n$,

which are clear in view of the representation of S_n as a subposet of 2^n, and of 2^n as a direct product of n chains. So it will suffice to show that the ordering on S_n is not an intersection of fewer than n total orderings.

To see this, consider any family of total orderings of $|S_n|$ whose intersection is the ordering of S_n. Note that for each $i \in n$, since $a_i \not\leq b_i$ in S_n, that family of total orders must have at least one member $\leq_i$ such that $a_i >_i b_i$. I claim these orderings $\leq_i$ must be distinct. Indeed, if for some $i \neq j$, $\leq_i$ is the same as $\leq_j$, let us call their common value $\leq_{i,j}$. Interchanging the roles of i and j if necessary, we may assume $a_j >_{i,j} a_i$. Then $a_j >_{i,j} a_i >_{i,j} b_i$, hence under the intersection of our family of orderings, $a_j \not\leq b_i$, contradicting the definition of S_n. So the orderings $\leq_i$ $(i \in n)$ in our family are indeed distinct, so $\dim(S_n) \geq n$, so by (1.7), both (1.5) and (1.6) indeed describe posets of dimension n.

For another class of examples of dimensions of posets, recall that a poset P in which every pair of distinct elements is incomparable is called an *antichain*. We claim that

(1.8) Every antichain P with more than one element has dimension 2.

Indeed, if one chooses any total ordering of $|P|$, then the intersection of that ordering and the opposite ordering is the antichain ordering, so $\dim(P) \leq 2$; and since $|P|$ has more than one element, no single linear ordering makes it an antichain.

We remark that under an isotone bijection of posets which is *not* an isomorphism, the dimension may increase, decrease, or remain unchanged. For instance, starting with an antichain of 8 points, a bijection onto the poset 2^3 is an isotone map that increases the dimension from 2 to 3, while a linearization decreases the latter dimension from 3 to 1. On the other hand, a bijection from a 4-element antichain to the poset 2^2 is isotone and leaves the dimension, 2, unchanged.

Let us note one more elementary poset construction (a generalization of one we applied to chains in proving the equivalence of the two conditions in Definition 1.1). Given a poset P, and for each $p \in P$ a poset Q_p, we may define the set

$$|\textstyle\sum_P Q_p| \;=\; \{(p,q) \mid p \in P,\ q \in Q_p\}, \tag{1.9}$$

and partially order this lexicographically; that is, by defining

$$(p,q) \;\leq\; (p',q') \text{ if and only if either } p <_P p', \text{ or } p = p' \text{ and } q \leq_{Q_p} q'. \tag{1.10}$$

It is not hard to verify that for this construction,

$$\dim(\textstyle\sum_P Q_p) \text{ is the supremum of } \{\dim(P)\} \cup \{\dim(Q_p) \mid p \in P\}. \tag{1.11}$$

In the verification, one takes a representation for P as in Definition 1.1(i), and representations of the Q_p as in Definition 1.1(ii) (which may, but do not need to satisfy the stronger condition (i)). By repeating mappings if necessary, one can assume that the number of chains used in each of these representations is the supremum indicated above. One can then easily combine these to get an embedding of $\sum_P Q_p$ in a product of that number of chains.

2. Our main result, and some consequences

To lead up to our main result, let us first note that for any two posets P and Q (which, we recall, are required to be nonempty), we must have

$$\max(\dim(P), \dim(Q)) \;\leq\; \dim(P \times Q) \;\leq\; \dim(P) + \dim(Q). \tag{2.1}$$

The properties of vector-space dimension suggest that the second inequality should be equality; but the two concepts of dimension are not alike in that respect. To see an interesting way that equality can fail, consider a poset P of dimension d, represented as a subposet of a product of d chains, $P \subseteq \prod_{i\in d} T_i$, and consider a nontrivial chain C. Suppose that for each i we let $T_i^* = T_i \times C$, ordered lexicographically – intuitively, the chain gotten from T_i by replacing each element t by a miniature copy of C. It is easy to see that the map $f : P \times C \to \prod_{i\in d} T_i^*$ taking $((p_0, \dots, p_{d-1}), c)$ to $((p_0, c), \dots, (p_{d-1}, c))$ is one-to-one and isotone. Can it fail to be an embedding?

In general, yes. If $p < p'$ in P and $c > c'$ in C, then in the product poset $P \times C$, the elements (p, c) and (p', c') are by definition incomparable; but if it happens that for all i, we have $p_i < p'_i$, then under the ordering described above, $f(p, c) < f(p', c')$; so their images are not incomparable.

However, *if* the subposet $P \subseteq \prod_{i\in d} T_i$ has the property that whenever two elements p, p' satisfy $p < p'$, the elements $p,\, p' \in \prod_{i\in d} T_i$ agree in *at least one coordinate*, then looking at that coordinate, we see that we do get incomparability between the indicated elements of $f(P \times C)$; so f is an embedding, so we indeed have $\dim(P \times C) = d = \dim(P)$.

The condition that every pair of comparable elements of $P \subseteq \prod_{i\in d} T_i$ agree in at least one coordinate may seem unnatural, but for $d \geq 3$ it is easy to see that it holds in the poset S_d, regarded as a subposet of 2^d (sentence following (1.6)).

In fact, more is true there: Every such pair of elements $a_i < b_j$ agree in one coordinate where their common value is 0 (the j-th), and one where their common value is 1 (the i-th); and the above construction can be adapted to show, as a consequence, that taking the product of S_d with *two* chains does not increase its dimension.

The next result gives this argument in detail for a still more general situation, that applies to products of P with possibly more than two chains. Note, however, that in that theorem, d is not assumed to be the dimension of P (as it is in the above example), but simply an integer such that P is a subposet of a product of d chains with certain properties.

Theorem 2.1. *Let d and n be nonnegative integers, $(T_i)_{i\in d}$ a d-tuple of chains, which for notational convenience we will assume pairwise disjoint, P a subposet of $\prod_{i\in d} T_i$, and $(M_j)_{j\in n}$ an n-tuple of pairwise disjoint subsets of $\bigcup_{i\in d} T_i$. such that*

(2.2) *For every comparable pair of elements $p \leq p'$ in P, and every $j \in n$, there exists $i \in d$ such that the i-th coordinates of p and p' are equal, and their common value is a member of M_j.*

Then for any n-tuple of chains $(C_j)_{j\in n}$ we have

$$\dim(P \times \textstyle\prod_{j\in n} C_j) \;\leq\; d. \tag{2.3}$$

Proof. For notational simplicity, we may assume all the C_j are equal to a common chain C, and that $\bigcup_{j\in n} M_j = \bigcup_{i\in d} T_i$. Indeed, given structures as in the statement of the theorem, we may embed all the C_j in a common chain C, and if we prove (2.3) with the C_j all replaced by C, this will imply the same inequality for the original choices of C_j. Likewise, if we enlarge some of the M_j (still keeping them disjoint) so that their union becomes the whole set $\bigcup_{i\in d} T_i$, then the hypothesis about comparable pairs of elements $p \leq p'$ that was assumed for the original choices of M_j remains true.

To obtain (2.3), we need an embedding of the poset on the left-hand side, which we can now write $P \times C^n$, in a product of d chains T'_i $(i \in d)$. For this purpose we define

(2.4) $T'_i \ = \ T_i \ltimes C$, the set-theoretic product of T_i and C ordered lexicographically, i.e., so that

(2.5) $(t, c) \ \leq \ (t', c')$ if and only if $t < t'$, or $t = t'$ and $c \leq c'$.

Also, since the M_j $(j \in n)$ partition $\bigcup_{i\in d} T_i$, we can set the notation

(2.6) For each $t \in \bigcup_{i\in d} T_i$, $m(t) \in n$ will denote the unique value such that $t \in M_{m(t)}$.

We now define a map $f : P \times C^n \to \prod_{i\in d} T'_i$ as follows.

(2.7) For $p = (p_i)_{i\in d} \in P \subseteq \prod_{i\in d} T_i$, and $c = (c_j)_{j\in n} \in C^n$, let $f(p, c)$ be the element of $\prod_{i\in d} T'_i$ whose i-th coordinate is $(p_i, c_{m(p_i)})$ for each $i \in d$.

We wish to show that f is an embedding of posets.

First, f is isotone, i.e.,

(2.8) If $(p, c) \leq (p', c')$ in $P \times C^n$, then $f(p, c) \leq f(p', c')$ in $\prod_{i\in d} T'_i$.

This follows easily from (2.5) and (2.7).

To complete the proof that f is an embedding, we need to show that

(2.9) If $(p, c) \not\leq (p', c')$, then $f(p, c) \not\leq f(p', c')$.

This breaks down into two cases. Suppose first that

(2.10) $p \ \not\leq \ p'$.

In that case, for some i we have $p_i \not\leq p'_i$, i.e., $p_i > p'_i$, and looking at the i-th coordinates of $f(p, c)$ and $f(p', c')$, namely $(p_i, c_{m(p_i)})$ and $(p'_i, c'_{m(p_i)})$, we see from the lexicographic ordering of T'_i that the former is $>$ the latter, giving the conclusion of (2.9). If, on the other hand,

(2.11) $p \ \leq \ p'$, but $c \ \not\leq \ c'$,

then in view of the second inequality above, we may choose a $j \in n$ such that

(2.12) $c_j \ > \ c'_j$.

Now by the *first* inequality of (2.11) and the hypothesis (2.2), there is some $i \in d$ such that

(2.13) $p_i \ = \ p'_i \in M_{m(j)}$.

Note that by (2.7),

(2.14) $f(p, c)$ and $f(p', c')$ have i-th terms (p_i, c_j) and (p'_i, c'_j) respectively.

Since the T_i-coordinates of the above two i-th terms are the same by (2.13), the order-relation between them is that of the C-coordinates, which satisfy (2.12). This completes the proof of (2.9), as required. $\square$

Remark: I came up with the above result after pondering [4], which showed by an explicit construction that $\dim(S_3 \times 2 \times 2) = 3$, i.e., is equal to $\dim(S_3)$. The next corollary includes that case.

Corollary 2.2. *Suppose $d \geq 2$ and $(T_i)_{i\in d}$ is a family of chains, each having a least element 0_i and a greatest element 1_i, and P is a subposet of $\prod_{i\in d} T_i$ consisting of elements each of which has at least one coordinate of the form 0_i and at least one of the form $1_{i'}$. Then for any two chains C_0 and C_1, we have*

(2.15) $\dim(P \times C_0 \times C_1) \ \leq \ d$.

In particular, if $d \geq 3$ and P is any subposet of $2^d \setminus \{0, 1\}$, containing the subposet S_d (e.g., if $P = S_d$), then for any two chains C_0 and C_1,

(2.16) $\dim(P \times C_0 \times C_1) \ = \ d \ = \ \dim(P)$.

Proof. In the context of the first assertion, given $p \leq p' \in P$, since p has at least one coordinate of the form 1_i, $p' \geq p$ must also have 1_i as its i-th coordinate; and similarly, since p' has a coordinate $0_{i'}$, p must agree with p' in that coordinate. Applying Theorem 2.1 with $n = 2$, $M_0 = \{0_i \mid i \in d\}$ and $M_1 = \{1_i \mid i \in d\}$, we get (2.15).

The second statement follows because when each T_i is the 2-element set $\{0_i, 1_i\}$, the exclusion of 0 and 1 from P forces every element to have at least one coordinate of the form 1_i and one of the form $0_{i'}$; and the assumption that P contains S_d, which we saw following (1.6) has dimension d, turns the inequality (2.15) into equality. □

If, instead of assuming that every element has at least one 0-coordinate and at least one 1-coordinate, we only assume one of these conditions, it is easy to check that the analogous reasoning gives a conclusion half as strong:

Corollary 2.3. *Suppose $d \geq 1$, and $(T_i)_{i \in d}$ is a family of chains each having a least element, 0_i, and P is a subposet of $\prod_{i \in d} T_i$ consisting of elements each of which has at least one coordinate of the form 0_i. Then for any chain C we have*

(2.17) $\quad \dim(P \times C) \ \leq\ d.$

In particular, if P is any subposet of $2^d \setminus \{1\}$ containing the subposet S_d (e.g., if $P = S_d \cup \{0\}$) then for any chain C we have

(2.18) $\quad \dim(P \times C) \ =\ d \ =\ \dim(P).$

The analogous statements hold with least elements 0_i and 0 everywhere replaced by greatest elements 1_i and 1. □

To get examples of Theorem 2.1 with $n > 2$, we shall use a different way of choosing n subsets M_j of $\bigcup_{i \in d} T_i$, based on the subscript i rather than the distinction between greatest and least elements of T_i.

We will need a bit more notation. For every positive integer d and every pair of integers a, b with $0 \leq a < b \leq d$,

(2.19) Let $P_d^{a,\,b}$ denote the subset of 2^d consisting of those elements in which the number of coordinates of the form 1_i is either a or b.

(Thus, for $d \geq 3$, $S_d = P_d^{1,d-1}$.)

Then we have

Corollary 2.4. *For every positive integer d, every integer a with $0 \leq a \leq d-1$, and every family of chains $(C_j)_{j \in n}$ with*

(2.20) $\quad n \ \leq\ d/2,$

we have, in the notation of (2.19),

(2.21) $\quad \dim(P_d^{a,\,a+1} \times \prod_{j \in n} C_j) \ \leq\ d.$

Proof. Let us define subsets $M_0, \dots, M_{n-1}$ of $\bigcup_{i \in d} \{0_i, 1_i\}$ by

(2.22) $\quad M_j \ =\ \{0_{2j}, 1_{2j}, 0_{2j+1}, 1_{2j+1}\}.$

Condition (2.20) guarantees that all of the M_j are contained in $\bigcup_{i \in d} \{0_i, 1_i\}$.

Now if $p \leq p'$ are elements of $P_d^{a\,,a+1}$, then since the cardinalities of the subsets of d on which they assume values of the form 1_i differ by at most 1, they must agree in all but at most one coordinate. Since each M_j contains both 0_i and 1_i for *two* values of i, one of those values of i must have the property that p and p' agree on the i-th coordinate. Hence these subsets M_j satisfy the hypotheses of Theorem 2.1, completing the proof. □

If the posets $P_d^{a\,,a+1} \subseteq 2^d$ had, like S_d, dimension d, then the above result would show that for every n, there exist finite posets that don't change their dimension on taking a direct product with n chains. However, such subposets of 2^d generally have dimension less than d. There are many results in the literature obtaining bounds on the dimensions of subposets of 2^d; cf. [5], [8], [9, Theorems III and V], [10, Theorem 7.1], [12], [13, Theorems 7, 10, 12, 13], and [18, Theorem 2]. Let us obtain one such result here, as another consequence of Theorem 2.1. (In the statement and proof, we shall not need to look at the disjoint union of the factors of 2^d, hence we shall not, as in the preceding corollaries, treat these as disjoint sets $\{0_i, 1_i\}$, but as the same set $2 = \{0, 1\}$.)

Corollary 2.5. *Let* d *be a positive integer, define*

(2.23) $\quad P_d^{[2,\,d-2]} \ = \ \{p \in 2^n \mid 2 \ \leq \ \mathrm{card}(\{i \mid p_i = 1\}) \ \leq \ d{-}2\,\}.$

and let

(2.24) $\quad P \ = \ P_d^{[2,\,d-2]} \setminus \{(0,\dots,0,1,1),\ (1,\dots,1,0,0)\}$

(*where by* $(0,\dots,0,1,1)$ *we mean the* d*-tuple whose first* $d{-}2$ *coordinates are* 0 *and whose last* 2 *coordinates are* 1, *and by* $(1,\dots,1,0,0)$, *the same with the roles of* 0 *and* 1 *reversed*). *Then*

(2.25) $\quad \dim(P) \ \leq \ d-2.$

Hence the same upper bound holds for the dimension of any subposet of P; *in particular, for any poset of the form* $P_d^{a,\,b}$ (*see* (2.19)) *with* $3 \leq a < b \leq d-3$.

Proof. I claim that as a subset of 2^d,

(2.26) $\quad P \ \subseteq \ P_{d-2}^{[1,\,d-3]} \times 2 \times 2;$

in other words, that the first $d-2$ coordinates of every element $p \in P$ contain at least one 1 and at least one 0. To see the former condition, note that by (2.23), p itself must have at least two coordinates 1. If none of these were among the first $d-2$ coordinates, this would force $p = (0,\dots,0,1,1)$, but that element is excluded in (2.24). The second condition follows in the same way.

Now by Corollary 2.2, with $d-2$ in place of d, and each T_i taken to be 2, we see that the direct product on the right-hand side of (2.26) has dimension $\leq d-2$, hence the same is true of its subposet P, proving (2.25)

The final sentence of the corollary follows immediately. $\square$

3. An old result, and questions old and new

Having seen that dimension of posets is *not* in general additive on direct products, it is striking that in an important class of cases, it is. The theorem below was proved, for products over index sets of arbitrary cardinality, in K. Baker's unpublished undergraduate thesis [1]. That proof is summarized in [10] for pairwise products (from which the case of arbitrary *finite* products follows), and assuming the dimensions finite. I give below a version of the proof (also for pairwise products of finite-dimensional posets) which I find easier to follow.

Theorem 3.1. ([1, p.9, Property 3], [10, p.179, last 11 lines]) *Let* P *and* Q *be finite-dimensional posets, each having a least element* 0 *and a greatest element* 1. *Then*

(3.1) $\quad \dim(P \times Q) \ = \ \dim(P) + \dim(Q).$

Proof. Clearly $\leq$ holds in (3.1), so it suffices to show $\geq$. For this we shall use version (i) of Definition 1.1, and show that if we have an expression for the partial order of $P \times Q$ as an intersection of

(3.2) $\quad n$ total orderings, $\ \leq_0,\ \dots,\ \leq_{n-1},\ $ on $|P \times Q|$,

then we can split that family into two disjoint subsets, such that a certain natural map of P into the product of chains determined by one of those subsets is an embedding, as is a map of Q into the product determined by the other. Thus the former subset must consist of $\geq \dim(P)$ orderings, and the latter of $\geq \dim(Q)$ orderings, so $n \geq \dim(P) + \dim(Q)$, as claimed.

The trick to finding this partition is to look at the relative order, under each of the orderings (3.2), of the elements $(0_P, 1_Q)$ and $(1_P, 0_Q)$ of $|P \times Q|$. Reindexing those n orderings if necessary, we may assume that for some $m \leq n$,

(3.3) $\quad (0_P, 1_Q) <_i (1_P, 0_Q)$ for $0 \leq i < m$, while $(1_P, 0_Q) <_i (0_P, 1_Q)$ for $m \leq i < n$.

Let us now show that the map

(3.4) $\quad P \to (|P \times Q|, \leq_0) \times \cdots \times (|P \times Q|, \leq_{m-1})$ given by $p \mapsto ((p, 0_Q), \dots, (p, 0_Q))$

is an embedding. Since (3.4) is clearly isotone, it remains to show that if $p,\, p' \in P$ satisfy

(3.5) $\quad p \ \not\leq \ p',$

then the corresponding condition holds on the images of p and p' under (3.4).

To get this, note that under the product ordering on $P \times Q$, the relation (3.5) implies that $(p, 0_Q) \not\leq (p', 1_Q)$. Hence we can find some i such that

$$(3.6) \qquad (p, 0_Q) \not\leq_i (p', 1_Q).$$

Now if $i \geq m$, we would have $(p, 0_Q) \leq_i (1_P, 0_Q) <_i (0_P, 1_Q) \leq_i (p', 1_Q)$, contradicting (3.6); so $i < m$. Since (3.6) implies $(p, 0_Q) \not\leq_i (p', 0_Q)$, this completes the proof that (3.4) respects $\not\leq$, hence is an embedding. The analogous argument shows that the map $Q \to (|P \times Q|, \leq_m) \times \cdots \times (|P \times Q|, \leq_{n-1})$ given by $q \mapsto ((0_P, q), \ldots, (0_P, q))$ likewise gives an embedding of Q. As noted in the first paragraph, this proves (3.1). $\square$

Though we have seen that (3.1) does not hold without the assumptions that P and Q have upper and lower bounds, the deviations from equality in all known examples are ≤ 2, leading to the longstanding open question:

Question 3.2. (i) *If P and Q are finite-dimensional posets, must* $\dim(P \times Q) \geq \dim(P) + \dim(Q) - 2$ *?*

In particular,

(ii) *If P is a finite-dimensional poset, and n a positive integer such that* $\dim(P \times 2^n) = \dim(P)$, *must* $n \leq 2$ *?*

In [10, last line of p.179 and top line of p.180] it was conjectured that $\dim(P) + \dim(Q)$ can exceed $\dim(P \times Q)$ by at most the number of members of the set $\{P, Q\}$ that do *not* have both a greatest and least element. Thus, the case where that number is 0 is Theorem 3.1, while the case where there is no restriction on P or Q corresponds to Question 3.2(i) above. The case of that conjecture where that number is 1, however, turned out to be false: from Corollary 2.2 we see that for any $n \geq 3$, $\dim(S_n \times 2^2) = \dim(S_n)$, so $\dim(S_n) + \dim(2^2)$ exceeds $\dim(S_n \times 2^2) = \dim(S_n)$ by 2, though 2^2 has both greatest and least elements.

However, one might ask about different intermediate cases between those of Theorem 3.1 and Question 3.2(i), suggested by Corollary 2.3, and noted in Question 3.3 below. C. Lin [13] poses part (iii) of that question, though Theorems 10 and 11 of that paper suggest the stronger implications of parts (i) and (ii).

Question 3.3. (i) *If P and Q are finite-dimensional posets such that P has a least element and Q has a greatest element, must* $\dim(P \times Q) \geq \dim(P) + \dim(Q) - 1$ *?*

(ii) *If P and Q are finite-dimensional posets each of which has a least element, or each of which has a greatest element, must* $\dim(P \times Q) \geq \dim(P) + \dim(Q) - 1$ *?*

And finally, a possible implication weaker than either of the above two:

(iii) [13, p.80] *If P and Q are finite-dimensional posets such that P has a least or a greatest element, and Q has both, must* $\dim(P \times Q) \geq \dim(P) + \dim(Q) - 1$ *?*

Turning back to Question 3.2, I wondered whether the result of Theorem 3.1, where no dimensionality was lost in forming a product of posets, could be related to the hypothesis that the sets of minimal and of maximal elements of each factor were singletons, and hence had dimension 0. If so, could one use the fact that for arbitrary finite P and Q the sets of minimal and maximal elements are antichains, hence have dimension ≤ 2, to get a positive answer to Question 3.2 for finite posets? But I see no way to adapt the proof of Theorem 3.1.

Moving on to other questions, recall that in Theorem 2.1, the result did not depend on the lengths of the chains C_j. This suggests

Question 3.4. (*Cf.* [15], [17, Conjecture 1]) *If P is a finite-dimensional poset, and C any chain of more than one element, must* $\dim(P \times C) = \dim(P \times 2)$ *?*

The next question at first seemed "obviously" to have an affirmative answer; but I see no argument proving it.

Question 3.5. *If P is a finite-dimensional poset, and* $\dim(P \times 2) = \dim(P) + 1$, *must* $\dim(P \times 2 \times 2) = \dim(P) + 2$ *?*

Much more generally (but much more vaguely), we may ask

Question 3.6. *Given finite-dimensional posets P_0, P_1, P_2, if we know their dimensions, and those of $P_0 \times P_1$, $P_0 \times P_2$ and $P_1 \times P_2$, what can we say about* $\dim(P_0 \times P_1 \times P_2)$ *?* (*Anything more than that it is greater than or equal to the dimensions of each of the pairwise products, and less than or equal to the values of* $\dim(P_i \times P_j) + \dim(P_k)$ *for* $\{i, j, k\} = \{0, 1, 2\}$ *?*)

One example of "misbehavior" on this front: Suppose P is an antichain of more than one element, and $P' = P \times 2$. Now P, P^2, and P^3 are all antichains, hence all have dimension 2; but P', P'^2, and P'^3 have the forms $P \times 2$, $P^2 \times 2^2$, and $P^3 \times 2^3$, and from (1.11) we see that they have dimensions 2, 2, and 3 respectively. So the dimensions of three posets, and of their pairwise direct products, do not determine the dimension of the product of all three.

4. Absorbency

The following concept might be helpful in studying questions of the sort we have been considering.

Definition 4.1. *If P is a finite-dimensional poset, let us define its* absorbency *to be*

(4.1) $\mathrm{abs}(P)$ = *the largest natural number n such that* $\dim(P \times \prod_{i\in n} T_i) = \dim(P)$ *for every n-tuple of chains* $(T_i)_{i\in n}$.

As an example,

(4.2) For all $d \geq 3$, $\mathrm{abs}(S_d) = 2$.

Here $\geq$ follows from the $P = S_d$ case of (2.16). To get the reverse inequality, we will need to call on a result from the literature. Note that if $\mathrm{abs}(S_d)$ were ≥ 3, then since $S_d \subseteq 2^d$, the value of $\dim(S_d \times S_d)$ would be $\leq \dim(S_d \times 2^d) \leq \dim(S_d \times 2^3) + \dim(2^{d-3}) = d + (d-3) = 2d-3$. However, Trotter [18, Theorem 2] shows that for all $d \geq 3$, $\dim(S_d \times S_d) = 2d-2$.

An immediate property of absorbency is

(4.3) $\mathrm{abs}(P) \leq \dim(P)$,

since for any $n > \dim(P)$, the product of P with, say, 2^n will contain a copy of 2^n, and hence have dimension greater than that of P.

Here are some other easy properties.

Lemma 4.2. *Let P and Q be finite-dimensional posets.*

(4.4) *If* $\mathrm{abs}(P) \geq \dim(Q)$, *then* $\dim(P \times Q) = \dim(P)$, *and* $\mathrm{abs}(P \times Q) \geq \mathrm{abs}(P) - \dim(Q) + \mathrm{abs}(Q)$.

The case where neither the hypothesis of (4.4) *nor the corresponding inequality with the roles of P and Q reversed holds is covered by*

(4.5) *If* $\min(\dim(P), \dim(Q)) \geq \max(\mathrm{abs}(P), \mathrm{abs}(Q))$, *then*
$\max(\dim(P), \dim(Q)) \leq \dim(P \times Q) \leq \dim(P) + \dim(Q) - \max(\mathrm{abs}(P), \mathrm{abs}(Q))$.

Proof. In (4.4), the first conclusion holds because Q embeds in a product of $\dim(Q) \leq \mathrm{abs}(P)$ chains, which P can "absorb" without increasing its dimension.

To get the final inequality of (4.4), we must show that the dimension of $P \times Q$ is not increased on multiplying it by a product of $\mathrm{abs}(P) - \dim(Q) + \mathrm{abs}(Q)$ chains. Let us write such a product as $X \times Y$, where X is a product of $\mathrm{abs}(P) - \dim(Q)$ chains, and Y a product of $\mathrm{abs}(Q)$ chains. Then if we write $P \times Q \times (X \times Y)$ as $P \times (X \times (Q \times Y))$, we see that $Q \times Y$ has dimension $\dim(Q)$ (by our choice of Y), hence $X \times (Q \times Y)$ has (by our choice of X) dimension at most $(\mathrm{abs}(P) - \dim(Q)) + \dim(Q) = \mathrm{abs}(P)$, so its product with P has dimension $\dim(P)$, which we have noted equals $\dim(P \times Q)$, so we have indeed shown that the dimension of $P \times Q$ has not been increased, as required.

In the conclusion of (4.5), the left-hand inequality is immediate. The right-hand inequality is equivalent to saying that $\dim(P \times Q)$ is bounded above by $\dim(P) + \dim(Q)$ minus either of $\mathrm{abs}(P)$ and $\mathrm{abs}(Q)$, so by symmetry it suffices to prove the bound involving $\mathrm{abs}(P)$. If we embed Q in a product of $\dim(Q)$ nontrivial chains, and break this into a product X of $\mathrm{abs}(P)$ chains and a product Y of $\dim(Q) - \mathrm{abs}(P)$ chains, then we have $\dim(P \times Q) \leq \dim(P \times X \times Y) \leq \dim(P \times X) + \dim(Y) = \dim(P) + (\dim(Q) - \mathrm{abs}(P))$, as desired. ☐

We may ask

Question 4.3. *Under the hypothesis of* (4.4), *does equality always hold in the final inequality of that implication?*

If we ask the same question about the final inequality of (4.5), the peculiarities noted in the paragraph following Question 3.6 make trouble. But those involved posets constructed using antichains, and I know of no cases not arising in that way. To express this precisely, let us set up some terminology.

Definition 4.4. *We shall call a poset P* disconnected *if it satisfies the following equivalent conditions.*
(i) *P can be written as a union of subposets P_0 and P_1 such that all elements of P_0 are incomparable with all elements of P_1.*
(ii) *P can be written as a union of a family of more than one subposets P_i, such that for $i \neq i'$, all elements of P_i are incomparable with all elements of $P_{i'}$.*
(iii) *P is isomorphic to a poset constructed as in* (1.9) *and* (1.10) *with the index-poset* (*there called P*) *an antichain of cardinality* > 1.

On the other hand, a poset P will be called connected *if it is not disconnected; equivalently, if the equivalence relation on $|P|$ generated by the relation $\leq_P$ is the indiscrete equivalence relation.*

We can now ask

Question 4.5. *If the posets P and Q in* (4.5) *are connected, must equality hold in the final inequality of the conclusion thereof?*

Since (1.11) gives us a way of computing the dimension of a disconnected poset from the dimensions of its connected components, a positive answer to Question 4.5 (together with the first assertion of (4.4)) would allow us to compute exactly the dimension of a product of two arbitrary finite-dimensional posets, given the dimensions and absorbencies of their connected components. It would, in particular, imply positive answers to Questions 3.4 and 3.5.

But as long as we do not know a positive answer to Question 4.5, here are some further points worth considering:

If Question 3.4 has a negative answer, one could consider variants of the absorbency concept, depending on the lengths of the chains involved; though this is likely to get messy.

Assuming Question 3.4 has a positive answer but Question 3.5 does not, one might define the "eventual absorbency" of a finite-dimensional poset P as the supremum, as $n \to \infty$, of $\dim(P) + n - \dim(P \times 2^n)$.

Given a poset P and a nonnegative integer d about which we know that $\dim(P) \leq d$, one might define the absorbency of P *relative* to d to be the largest n such that the product of P with every n-tuple of chains has dimension $\leq d$. So, for instance, given finite-dimensional posets P and Q, the absorbency of $P \times Q$ relative to $\dim(P) + \dim(Q)$ will be at least the sum of the absorbencies of P and of Q.

5. Another function related to dimension, and a few from the literature

Let us call a poset *bounded* if it has a greatest element and a least element.

(I wish I could think of a better term for this condition, since it has very different properties from the familiar sense of "bounded". E.g., a subposet of a bounded poset is not, in general, bounded. However, "bounded" is used in this way in places in the literature; e.g., [10, p.179, 12th line from bottom].)

This allows us to define another function which is useful in studying dimensions.

Definition 5.1. *For P a finite-dimensional poset, the* bounded dimension *of P, denoted* bd-dim(P), *will be defined to be*

(i) *the greatest integer n such that P contains a bounded subposet P' satisfying* $\dim(P') = n$, *equivalently,*

(ii) *the maximum, over all pairs of elements $p \leq p'$ in P, of* $\dim(\{x \mid p \leq x \leq p'\})$.

Since every bounded subset P' of P is contained in an interval $\{x \mid p \leq x \leq p'\}$, and every such interval is bounded, we see that the two versions of the above definition are equivalent. Because of Theorem 3.1, this function behaves very nicely under direct products:

Lemma 5.2. *For finite-dimensional posets P and Q,*

$$\text{bd-dim}(P \times Q) \ = \ \text{bd-dim}(P) \ + \ \text{bd-dim}(Q). \tag{5.1}$$

Proof. For elements $p \leq p'$ of P, let us write

$$[p, p'] \ = \ \{p'' \in P \mid p \ \leq \ p'' \ \leq \ p'\}, \text{ regarded as a subposet of } P. \tag{5.2}$$

Thus, by version (ii) of the definition of bounded dimension, bd-dim$(P \times Q)$ is the greatest of the values $\dim([(p,q),\,(p',q')])$ for $(p,q) \leq (p',q')$ in $P \times Q$.

But by the order-structure of a direct product, $[(p,q),\,(p',q')]$ is isomorphic to $[\,p,\,p'] \times [\,q,\,q']$, and by Theorem 3.1 the dimension of this product is $\dim([\,p,p']) + \dim([\,q,q'])$. Taking the maximum over all pairs $p \leq p'$ and $q \leq q'$, we get bd-dim(P) + bd-dim(Q). ☐

Clearly, for any poset P,

(5.3) $\quad$ $\text{bd-dim}(P) \ \leq \ \dim(P).$

However, the two sides of (5.3) can be far from equal, as can be seen by taking $P = S_d$ for large d. Then the left-hand side of (5.3) is easily seen to be 1, while the right-hand side is d.

The concept of bounded dimension allows us to get an upper bound on absorbency:

(5.4) $\quad$ $\text{abs}(P) \ \leq \ \dim(P) \ - \ \text{bd-dim}(P).$

Indeed, if we take the product of P with more nontrivial chains than the number on the right, then since P contains a bounded set of dimension $\text{bd-dim}(P)$, and each of the chains contains a nontrivial bounded chain, the whole product will have dimension $> \text{bd-dim}(P) + (\dim(P) - \text{bd-dim}(P)) = \dim(P)$. Thus, to be "absorbed", a family of nontrivial chains can have at most the number of members shown on the right in (5.4). (Incidentally, this argument is easily adapted to give the same upper bound for the "eventual absorbency" function defined in the next-to-last paragraph of the preceding section.)

The two sides of (5.4) can also be far from equal. For instance, for $P = S_d$, where $d \geq 3$, we have seen in (4.2) that the left-hand side of (5.4) is 2. On the other hand, the right-hand side is $d - 1$.

From (5.4) we get a strengthening of (4.3); namely, we can add to that inequality the observation,

(5.5) $\quad$ The only finite-dimensional posets P for which $\text{abs}(P) = \dim(P)$ are the antichains,

since a poset that is not an antichain contains a copy of 2, a bounded poset of dimension 1.

Let us note some examples regarding the inequalities in the conclusion of (4.5) if we do not make the connectedness assumption of Question 4.5. (These translate the "misbehavior" noted in the last paragraph of §3.) Let D be a 2-element antichain, and 2, as usual, the 2-element chain. Then D has dimension and absorbency 2, while the poset 2 has dimension 1 and, by (5.4), absorbency 0. Hence by (4.4), $D \times 2$ has dimension 2 and absorbency at least 1, so by (5.4) its absorbency must be exactly 1. The analogous considerations show that $(D \times 2)^2$, i.e., $D^2 \times 2^2$, again has dimension 2, but has absorbency 0. Hence taking $P = Q = D \times 2$, the inequalities in the conclusion of (4.5) become $2 \leq 2 \leq 2 + 2 - 1$, so the first is equality and the second a strict inequality. On the other hand, if we take $P = Q = D \times 2^2$, then the corresponding considerations show that this poset again has dimension 2, but has absorbency 0, while $P \times Q = D^2 \times 2^4$ has dimension 4; so the inequalities become $2 \leq 4 \leq 2 + 2 - 0$, with the *first* strict and the *second* an equality. (If, for a further comparison, we take $P = Q = D$, we find that the inequalities in question are $2 \leq 2 \leq 2 + 2 - 2$, hence are both equalities.)

Trotter, in [18, Conjecture 2] suggested that for all $n \geq 2$ there exist posets P of dimension n such that $P \times P$ also has dimension n. Reuter [17, Theorem 13] showed that no such P exists for $n = 3$; but it is conceivable that there exist such P for higher n. Such an example would imply a negative answer to Question 4.5, in view of (5.4).

Returning to the bounded-dimension function, it might be of interest to look at the variants of $\text{bd-dim}(P)$ in which *bounded* is replaced by *bounded above*, respectively, *bounded below*.

I will end this section by sketching some other functions related to dimension that appear in the literature. A striking example, studied in [2], [14], [20] is based on the concept of a *Boolean representation* of a poset P. Here one considers families of $d > 0$ total orderings of $|P|$, which are *not* assumed to be strengthenings of the ordering of P, but merely to have the property that the order-relation or incomparability between elements $x \neq y$ in P can be expressed in terms of which of the d given orderings have $x < y$ and which have $x > y$. For example, the order relation of S_n $(n \geq 3)$ can be so described in terms of the four total orderings

(5.6) $\quad$ $a_0 <_0 \cdots <_0 a_i <_0 \cdots <_0 a_{n-1} <_0 b_0 <_0 \cdots <_0 b_i <_0 \cdots <_0 b_{n-1},$

(5.7) $\quad$ $b_0 <_1 \cdots <_1 b_i <_1 \cdots <_1 b_{n-1} <_1 a_0 <_1 \cdots <_1 a_i <_1 \cdots <_1 a_{n-1},$

(5.8) $\quad$ $a_0 <_2 b_0 <_2 \cdots <_2 a_i <_2 b_i <_2 \cdots <_2 a_{n-1} <_2 b_{n-1},$

(5.9) $\quad$ $b_0 <_3 a_0 <_3 \cdots <_3 b_i <_3 a_i <_3 \cdots <_3 b_{n-1} <_3 a_{n-1}.$

Namely, $x < y$ in S_n if and only if on the one hand, x precedes y in (5.6) but follows it in (5.7) (which together say that x has the form a_i and y the form b_j), and, further, the relative order of x and y is the same in (5.8) as in (5.9) (which says that they are *not* a pair of the form $\{a_i, b_i\}$). This is called a *Boolean representation* of S_n in terms of the four orderings (5.6)-(5.9). The *Boolean dimension* of a poset P, $\text{bdim}(P)$, is the least d such that P has a Boolean representation in terms of d total orderings.

This number is always $\leq \dim(P)$, but often strictly less; e.g., the above example shows that for all n, $\mathrm{bdim}(S_n) \leq 4$. In [20] it is shown that $\mathrm{bdim}(P)$ agrees with $\dim(P)$ when the latter is 3.

(In the formal definition of a Boolean representation of a poset P in terms of total orderings $<_0, \dots, <_{d-1}$ of $|P|$, one begins by mapping $|P| \times |P|$ to 2^d by sending (x, y) to the d-tuple which has 1 in the i-th position if and only if $x <_i y$. The ordering of P is then determined by a function $\tau : 2^d \to \{0, 1\}$, such that $x <_P y$ if and only if τ takes the d-tuple determined by (x, y) to 1. For instance, in the above example describing S_n, τ is the function on 2^4 taking the value 1 only at the two 4-tuples $(1, 0, 0, 0)$ and $(1, 0, 1, 1)$. Here the condition that for a 4-tuple to be taken to 1, its first entry must be 1 and its second entry 0 translate the conditions that x must precede y in (5.6) but not in (5.7); and the condition that the last two entries of the 4-tuple be equal translates the condition that x and y must occur in the same order in (5.8) and in (5.9). The term "Boolean dimension" refers to the fact that a map $\tau : 2^d \to 2$ can be expressed by a Boolean word in d variables. Incidentally, I think that in the definition of a Boolean representation, it would be desirable to allow $d = 0$, in which case the Boolean dimension of an antichain would be 0, determined by the function $\tau : 2^0 \to \{0, 1\}$ sending the unique element of 2^0 to 0, meaning that no pairs (x, y) satisfy $x < y$.)

Another variant of dimension described in [2] is the *local dimension,* $\mathrm{ldim}(P)$, the least d such that there exists a family of linearizations of *subposets* of P such that each member of P appears in $\leq d$ of these linearized subposets, and such that for every pair (x, y) of distinct elements of P, there are enough linearized subposets containing both so that the relationship between x and y in P (an order-relation, or incomparability) is the relation between their images in the product of these linearizations. As with the Boolean dimension, this is $\leq \dim(P)$, and considerably less for the S_n (in this case always ≤ 3).

Still another variant of dimension: If instead of looking at lists of linearizations of $\leq_P$, such that every relation $x \not< y$ in P is realized in at least one member of our list, and letting the dimension of P be the least cardinality of such a list, one can look at lists of linearizations with a "weight", a positive real number, attached to each, such that $x \not< y$ in P if and only if the sum of the weights of the linearizations for which $x > y$ is at least 1, and define the *fractional dimension* of P to be the infimum, over all weighted lists which determine the ordering of P in this way, of the sum of the weights of the listed linearizations. (The term "weight" is not used in the literature; it is my way of giving an intuitive description of the definition.) See [11, p.5, and references given there]. ([11] also introduces several dimension-like functions specific to the type of posets there named "convex geometries".)

Less exotic invariants of posets considered in [2] and elsewhere are the *height,* i.e., the supremum of the cardinalities of chains contained in P, and the *width,* the supremum of the cardinalities of antichains in P. The function associating to a poset P the largest n such that P contains a subposet isomorphic to S_n is studied, under the name $\mathrm{se}(P)$, in [3], [13, §3], [19, §5.2.1].

The function $\dim(P)$, though usually (as in this note) simply called the dimension of P, is sometimes, as in the title of [2], called the *Dushnik-Miller* dimension, to distinguish it amid this plethora of concepts.

6. When an infinite poset has finite dimension

I have left this topic to the end, because it assumes familiarity with a very different technique from those used in the other sections. Namely, by a straightforward application of the *Compactness Theorem* of first-order logic [6, Theorem VI.2.1(b)], [7, Theorem 6.1.1] (applied to a language with a constant for each member of $|P|$, which, we are not assuming countable), one can verify

Proposition 6.1. *Let P be a poset, and n a positive integer. Then P has dimension n if and only if n is the supremum of the dimensions of all finite subposets of P.* $\square$

Remark: I originally had two rather complicated proofs of the above result – one using an ultraproduct of the finite subposets of P, over any ultrafilter $\mathcal{U}$ on the set of such subposets extending the filter $\mathcal{F}$ generated by the nonempty up-sets under inclusion; the other based on indexing $|P|$ in any way by a well-ordered set I, and recursively constructing embeddings of the subposets determined by initial subsets of I, into products of n chains, in a way chosen at each step to guarantee further extensibility. But Theodore Slaman pointed out to me that the Compactness Theorem yields the same result much more easily.

Immediate consequences are

Corollary 6.2. *For any finite-dimensional poset P,* $\mathrm{bd\text{-}dim}(P)$ *is equal to the supremum of the values of* $\mathrm{bd\text{-}dim}(P')$ *on all finite subposets $P' \subseteq P$.* $\square$

Corollary 6.3. *For any finite-dimensional poset P,* $\mathrm{abs}(P)$ *is equal to the* infimum *of the values of* $\mathrm{abs}(P')$ *over the finite subposets P' of P satisfying* $\dim(P') = \dim(P)$. $\square$

I wonder about

Question 6.4. *Do there exist* infinite *cardinals* κ *and* λ *such that for every poset* P, *if all subposets* $P' \subseteq P$ *of cardinality* $< \kappa$ *have dimension* $\leq \lambda$, *then* P *itself has dimension* $\leq \lambda$ *?*

Thus, Proposition 6.1 is the corresponding statement with $\aleph_0$ in the role of κ, and the natural number n in place of the infinite cardinal λ.

7. Acknowledgements

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