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Arrays of prime ideals in commutative rings

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Abstract

If R is a commutative ring, A and B ideals of R, and S and T multiplicative submonoids of R, we note an elementary necessary and sufficient condition for there to exist prime ideals P and Q in R such that P contains A and is disjoint from S, Q contains B and is disjoint from T, and $P \subseteq Q$. We then study conditions for the existence of larger families of prime ideals satisfying similar systems of relations. When the inclusion relations specified in the given system define a "tree order," the necessary and sufficient conditions are quite tractable; otherwise, they are much less so. We apply these results to the case where R is a tensor product of two algebras over a field k, and end with some observations on the behavior of arrays of prime ideals in a k-algebra under base extension.

Keywords: Commutative ring; Prime ideal; Multiplicative monoid; Partially ordered set; Tensor product

1. Introduction and some basic results

Throughout this note, R will be a commutative ring.

We recall the following well-known result, useful for finding prime ideals P of R with specified properties.

Lemma 1 ([2, Theorem 9.2.2], [7, Proposition 7.3], [8, Theorem 1], [11, Theorem 1.2.1]). *Let A be an ideal of R, and S a multiplicative submonoid of R (a subset of R, containing* 1 *and closed under multiplication) disjoint from A. Then there exists a prime ideal P of R containing A and disjoint from S.*

Is there a result which could be used similarly to get pairs of primes $P \subseteq Q$? A first guess might be that given a pair of ideals $A \subseteq B$ and a pair of multiplicative monoids

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S0021-8693(02)00666-X/FLA AID:9425 Vol.●●●(●●●) Yjabr9425 P.2 (1-22)
ELSGMLTM(JABR):m1 2002/12/16 Prn:19/12/2002; 8:50 by:Rima p. 2

$S \supseteq T$ such that A is disjoint from S and B from T, there would exist prime ideals $P \subseteq Q$ containing A and B respectively, and disjoint from S and T, respectively. However, this is not true, as may be seen by taking $R = \mathbb{Z}$, $A = 6\mathbb{Z}$, $B = 3\mathbb{Z}$, $S = \{3^n\}$, $T = \{1\}$. Nevertheless, we shall see that there are elementary criteria for the existence of such a pair of primes, and of larger arrays of primes satisfying similar families of conditions. (For the "$P \subseteq Q$" case, we shall see that the relevant conditions do not even include the hypotheses $A \subseteq B$ and $S \supseteq T$ suggested above.)

Let us first take a closer look at the preceding lemma. We make

Definition 2. If A is an ideal of R and S a multiplicative submonoid of R, then a *realization* of the pair (A, S) will mean a prime ideal containing A and disjoint from S.

So Lemma 1 says that the set of realizations of (A, S) is nonempty if and only if $A \cap S = \emptyset$.

Let us next fix some notation for operations on sets of ring elements.

Definition 3. If X and Y are subsets of R, then $X + Y$ will denote $\{x + y \mid x \in X,\ y \in Y\}$, XY will denote $\{xy \mid x \in X,\ y \in Y\}$, and $X \div Y$ will denote $\{r \in R \mid (\exists y \in Y)\ ry \in X\}$.

In the absence of parentheses, the order of operations will be: multiplication, then $\div$, then addition; and after all of these, intersection. (Thus, $A \div ST + B \cap U$ will mean $((A \div (ST)) + B) \cap U$.)

We shall also write $R - X$ for $\{r \in R \mid r \notin X\}$.

Note that the sum of two ideals is an ideal, and the product of two multiplicative monoids is a multiplicative monoid, that for A an ideal and S a multiplicative monoid, $S + A$ is a multiplicative monoid and $A \div S$ an ideal, and that if P is a prime ideal, $R - P$ is a multiplicative monoid. The above symbols do not give us a way of writing the conventional "product" of two ideals, that is, the ideal of sums of products of elements; but that construction will not be needed in this note.

Observe that the assumption in Lemma 1,

$$A \cap S = \emptyset, \tag{1}$$

is equivalent to

$$A \div S \cap \{1\} = \emptyset, \tag{2}$$

and also to

$$\{0\} \cap A + S = \emptyset. \tag{3}$$

Each of these equations says that a certain ideal is disjoint from a certain multiplicative monoid, so in each case we may ask for a characterization of the prime ideals containing the indicated ideal and disjoint from the monoid; in the language of Definition 2, of the prime

S0021-8693(02)00666-X/FLA AID:9425 Vol.●●●(●●●) Yjabr9425 P.3 (1-22)
ELSGMLTM(JABR):m1 2002/12/16 Prn:19/12/2002; 8:50 by:Rima p. 3

ideals realizing the indicated pair. Of course, the condition that a prime ideal contain $\{0\}$, or be disjoint from $\{1\}$, is vacuous, so they can be dropped in the statements of our results:

Lemma 4. *Let A be an ideal of R and S a multiplicative submonoid of R. Then*

(i) *A prime ideal Q of R contains the ideal $A \div S$ if and only if it contains a prime ideal P which realizes the pair (A, S).*
(ii) *A prime ideal P of R is disjoint from the monoid $S + A$ if and only if it is contained in a prime ideal Q which realizes the pair (A, S).*

Proof. To prove (i), first note that a prime ideal P realizing (A, S) must contain $A \div S$; hence so must any prime Q containing P. Conversely, if a prime Q contains $A \div S$, then the monoid $R - Q$ is disjoint from $A \div S$, which means that $S(R - Q)$ is disjoint from A, hence by Lemma 1 we can find a prime ideal P realizing the pair $(A, S(R - Q))$. Hence P will realize (A, S) and be disjoint from $R - Q$, i.e., contained in Q, as required.

Likewise, to get (ii), observe that a prime Q containing A and disjoint from S must be disjoint from $S + A$, hence so must any prime P contained in Q; and conversely, if P is a prime disjoint from $S + A$, then $P + A$ is disjoint from S, hence there exists a prime ideal Q realizing $(P + A, S)$, which will realize (A, S) and contain P, as required. □

We can iterate the application of Lemma 4 starting with a single pair (A, S): Part (i) of that lemma says that prime ideals which contain primes realizing (A, S) are those realizing $(A \div S, \{1\})$. Applying part (ii) to this situation, we see that prime ideals contained in primes of the latter sort, i.e., in primes which contain primes which realize (A, S) are those realizing $(\{0\}, \{1\} + A \div S)$. Another iteration gives a characterization of primes containing primes contained in primes containing primes realizing (A, S); and so on. This yields an infinite family of successively weaker conditions, since rings can be found having pairs of prime ideals connected by an up-and-down chain of any given length, but by no shorter chain. For example, for any integer n, consider the ring

$$R = \bigl\{(f_1, \ldots, f_n) \in \mathbb{Q}[x]^n \bigm| f_i(i) = f_{i+1}(i)\ (i = 1, \ldots, n-1)\bigr\},$$

and in it, the prime ideals

$$\begin{aligned} \bigl\{(f_i) \mid f_1 = 0\bigr\} &\subset \bigl\{(f_i) \mid f_1(1) = 0 = f_2(1)\bigr\} \\ &\supset \bigl\{(f_i) \mid f_2 = 0\bigr\} \\ &\subset \bigl\{(f_i) \mid f_2(2) = 0 = f_3(2)\bigr\} \\ &\ \ \vdots \\ &\supset \bigl\{(f_i) \mid f_n = 0\bigr\} \\ &\subset \bigl\{(f_i) \mid f_n(n) = 0\bigr\}. \end{aligned}$$

S0021-8693(02)00666-X/FLA AID:9425 Vol.●●●(●●●) Yjabr9425 P.4 (1-22)
ELSGMLTM(JABR):m1 2002/12/16 Prn:19/12/2002; 8:50 by:Rima p. 4

If we let A be the first of these ideals, and $S = R - A$, then the first $2n$ of the conditions discussed above give distinct sets of primes.

However, if R is an integral domain, then any two primes have the prime $\{0\}$ as a common lower bound, so we get only a small number of distinct conditions; and the same is true if R is a local ring, where any two primes have the maximal ideal as a common upper bound.

A straightforward but useful observation is

Lemma 5. *If A, A' are ideals and S, S' are multiplicative submonoids of R, then a prime P realizes both (A, S) and (A', S') if and only if it realizes $(A + A', SS')$.*

Let us now turn to conditions involving more than one prime ideal.

2. Pairs of primes

Lemma 6. *Let A and B be ideals of R, and S and T multiplicative submonoids of R. Then the following conditions are equivalent*:

(i) *There exist prime ideals P, Q such that P realizes (A, S), Q realizes (B, T), and $P \subseteq Q$.*
(ii) *The ideal A is disjoint from the multiplicative monoid $S(T + B)$.*
(iii) *The ideal $B + A \div S$ is disjoint from the multiplicative monoid T.*

In fact, a prime ideal P realizes (A, S) and is contained in a prime Q realizing (B, T) if and only if P realizes $(A, S(T + B))$; and a prime ideal Q realizes (B, T) and contains a prime P realizing (A, S) if and only if Q realizes $(B + A \div S, T)$.

Proof. It will suffice to prove the assertions of the final paragraph. By Lemma 4(ii), P is contained in a prime realizing (B, T) if and only if it realizes $(\{0\}, T + B)$. By Lemma 5, the conjunction of this condition and the condition that P realize (A, S) is equivalent to the condition of realizing $(A + \{0\}, S(T + B)) = (A, S(T + B))$. The last assertion is gotten similarly, using Lemma 4(i). □

Can we see directly the equivalence of conditions (ii) and (iii) of the above lemma? Yes; each of them says that

There do *not* exist elements $a \in A,\ s \in S,\ b \in B,\ t \in T,\ x \in R$ satisfying

$$t + b + x = 0, \qquad sx = a. \tag{4}$$

Just as the condition of Lemma 1, namely (1), has the equivalent formulations (2) and (3), so the equivalent conditions of Lemma 6(ii) and (iii), in symbols,

$$A \cap S(T + B) = \emptyset, \tag{5}$$

S0021-8693(02)00666-X/FLA AID:9425 Vol.●●●(●●●) Yjabr9425 P.5 (1-22)
ELSGMLTM(JABR):m1 2002/12/16 Prn:19/12/2002; 8:50 by:Rima p. 5

$$B + A \div S \cap T = \emptyset, \tag{6}$$

which can be thought of as stating the nonexistence condition (4) in terms of the element a and the element t, respectively, can also be formulated in terms of the elements s, b, and x respectively as

$$A \div (T + B) \cap S = \emptyset, \tag{7}$$

$$B \cap T + A \div S = \emptyset, \tag{8}$$

$$A \div S \cap T + B = \emptyset. \tag{9}$$

Again, since each of these equations says that an ideal is disjoint from a multiplicative monoid, one can ask for characterizations of the primes realizing these ideal–monoid pairs. Let us work this out for (7). A prime P realizes $(A \div (T + B), S)$ if and only if it contains $A \div (T + B)$ and is disjoint from S. By Lemma 4(i) the former condition is equivalent to containing an ideal P' that realizes the pair $(A, T + B)$, i.e., that contains A and is disjoint from $T + B$, and by Lemma 4(ii) the latter condition is equivalent to saying that P' is contained in an ideal Q that realizes (B, T). Note that in the above situation P, which is disjoint from S, contains P', which contains A; hence both of these prime ideals contain A and are disjoint from S, i.e., realize (A, S). So the condition that P realize $(A \div (T + B), S)$ can be described as saying that it realizes (A, S) and contains a prime P' also realizing (A, S) which is contained in a prime Q realizing (B, T). Note, incidentally, that the *existence* of such a P is equivalent to the existence of a prime that realizes (A, S) contained in a prime that realizes (B, T) (for if such a pair exists, we can take both P and P' to be the former prime).

Similar reasoning shows that a prime realizes the pair indicated in (8) if and only if it realizes (B, T), and is contained in a prime which also realizes (B, T) and contains a prime realizing (A, S).

The condition corresponding to (9) is the most natural: An application of the two parts of Lemma 4 shows that a prime realizes $(A \div S, T + B)$ if and only if it contains a prime realizing (A, S) and is contained in a prime realizing (B, T).

It is easy to see that the existence of a prime satisfying the above reformulation of any of (7)–(9) is equivalent to the existence of a pair of primes as in Lemma 4(i), confirming our observation that each of (7)–(9) is, like (5) and (6), a translation of (4). We could, of course, go on and apply to each of (5)–(9) the fact that every condition (1) has equivalent formulations (2) and (3), and get still more conditions equivalent to those listed; e.g., $A \div S(T + B) \cap \{1\} = \emptyset$, $\{0\} \cap S(T + B) + A = \emptyset$, etc.; and, using Lemma 4, characterize the prime ideals realizing such pairs.

Here, as at the end of the first section, we have "played around" with equivalent formulations of the conditions that we have characterized, getting results tangential to the main point of the section, in order to develop some familiarity with our techniques, and see where those tangents led. In subsequent sections, however, we shall limit ourselves more closely to our main line of investigation.

S0021-8693(02)00666-X/FLA AID:9425 Vol.●●●(●●●) Yjabr9425 P.6 (1-22)
ELSGMLTM(JABR):m1 2002/12/16 Prn:19/12/2002; 8:50 by:Rima p. 6

3. Realizing arrays of pairs

Generalizing the situation of the preceding section, suppose we are given a family of ideal-and-monoid pairs (A_i, S_i), and wish to know whether we can find a family of prime ideals P_i realizing these pairs, and satisfying specified inclusion relations. Let us set up the language and notation to say this precisely.

Definition 7. By a template over R we shall mean a pair $(I, (A_i, S_i)_{i\in I})$, where I is a partially ordered set, and for each $i \in I$, A_i is an ideal of R and S_i a multiplicative submonoid of R. Given such a template, we shall denote by $\mathrm{Spec}_R(I, (A_i, S_i)_{i\in I})$ the set of all I-tuples $(P_i)_{i\in I}$ such that for each $i \in I$, P_i is a prime ideal realizing the pair (A_i, S_i), and for all $i, j \in I$ with $i \leqslant j$, we have $P_i \subseteq P_j$. A member of this set will be called a realization of the given template.

In writing a template $(I, (A_i, S_i)_{i\in I})$, we will generally suppress the subscript on the second component, simply writing $(I, (A_i, S_i))$. In particular, if J is a subset of the indexing partially ordered set I, the subtemplate $(J, (A_i, S_i)_{i\in J})$ will be written $(J, (A_i, S_i))$. When I is a singleton, if the unique member of our I-tuple of pairs is (A, S), then we may abbreviate $\mathrm{Spec}_R(I, (A, S))$ to $\mathrm{Spec}_R(A, S)$.

A template may be shown diagrammatically by drawing a picture of the partially ordered set I, and writing in place of each $i \in I$ the pair (A_i, S_i).

Note that $\mathrm{Spec}_R(\{0\}, \{1\})$ can be identified with the underlying set of the usual prime spectrum of R. More generally, given a pair (A, S), $\mathrm{Spec}_R(A, S)$ may be identified with the spectrum of the localization of R/A gotten by inverting the images of all elements of S; however we shall not use this observation.

Lemmas 1 and 6 give necessary and sufficient conditions for templates of the respective forms

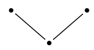
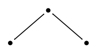
$$(A, S) \quad \text{and} \quad \begin{array}{c} (B, T) \\ | \\ (A, S) \end{array}$$

to have nonempty spectra, and they describe the sets of primes occurring as each coordinate of members of these spectra. The reader will not find it hard to obtain from those results similar results for templates of the forms

$$\begin{array}{c} (C, U) \\ | \\ (B, T) \\ | \\ (A, S) \end{array}, \qquad \begin{array}{ccc} (B, T) & & (C, U) \\ & \diagdown \quad \diagup & \\ & (A, S) & \end{array}, \quad \text{and} \quad \begin{array}{ccc} & (C, U) & \\ & \diagup \quad \diagdown & \\ (A, S) & & (B, T) \end{array},$$

S0021-8693(02)00666-X/FLA AID:9425 Vol.●●●(●●●) Yjabr9425 P.7 (1-22)
ELSGMLTM(JABR):m1 2002/12/16 Prn:19/12/2002; 8:50 by:Rima p. 7

i.e., templates indexed by the 3-element partially ordered sets

 ,  , and  .

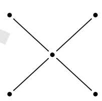
What is the most general partially ordered set for which we can get such results? To answer this we need

Definition 8 (cf. [10, third paragraph of Introduction]). A finite partially ordered set I will be called a *tree order* if its Hasse diagram (the graph showing the elements of I as vertices and the minimal order relations as edges), regarded as an *unoriented* graph, is a tree. A finite partially ordered set such that each connected component of its unoriented Hasse diagram is a tree may similarly be called a *forest order*.

The above definition is indirect, since it uses the order structure of I only via a conventional way of diagramming it. In fact, the characterization of tree orders that we will use below will not be the definition but the following easily verified recursive description: the unique one-element partially ordered set is a tree order, and a connected partially ordered set I of $n+1$ elements is a tree order if and only if it can be obtained from an n-element tree order I_0 by adjoining one element i, and an order relation between this new element and a single element of I_0. Such an element i, i.e., a terminal vertex of the associated unoriented graph, is called a "leaf;" we shall also use the fact, easily seen by induction, that every finite tree order of more than one element has at least two leaves.

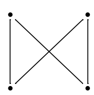
One can see that the spectrum of a general template $(I, (A_i, S_i))$ over R is the direct product of the spectra of the subtemplates indexed by the connected components of the partially ordered set I; so in studying such spectra we may restrict our attention to the case where I is connected. Thus, we shall not speak further of forest orders; results on these will be implied by our results on tree orders.

Note that for I a finite connected partially ordered set and J a subset connected under the induced ordering, there is no implication between the conditions "I is a tree order" and "J is a tree order." That a connected partially ordered set which is not a tree order can contain a subset which is a tree order is clear; the reverse situation is illustrated by the tree order



(10)

and its subset



(11)

(Actually, McKenzie (unpublished) has shown that this is "essentially the only way" a connected subset of a tree order can fail to be one. Namely, he has shown that a finite connected partially ordered set I is a tree order if and only if every minimal cycle in I is of the form (11), and is contained in a subset of I of the form (10).)

We can now prove

Theorem 9. *Suppose $(I,(A_i,S_i))$ is a finite template over R such that I is a tree order. Then*

(i) *The condition that* $\mathrm{Spec}_R(I,(A_i,S_i))$ *be nonempty is equivalent to the condition that there be no solution in R to a certain system of $2|I|-1$ equations, each having one of the forms $x+y+z=0$, $xy-z$, or $x=y$, in $4|I|-2$ variables, namely $|I|$ variables a_i $(i\in I)$, subject to the restrictions $a_i\in A_i$, $|I|$ variables s_i $(i\in I)$, subject to the restriction $s_i\in S_i$, and $2|I|-2$ unrestricted variables.*

(ii) *For each $j\in I$ there exist an ideal $A^{(j)}$ and a multiplicative monoid $S^{(j)}$ such that the prime ideals which occur as the jth components, P_j, of realizations of the template $(I,(A_i,S_i))$ are precisely the realizations of the pair $(A^{(j)},S^{(j)})$. These $A^{(j)}$ and $S^{(j)}$ are expressible in terms of the given ideals and monoids A_i and S_i using the four operations of adding ideals A and B to get an ideal $A+B$, multiplying monoids S and T to get a monoid ST, adding a monoid S and an ideal A to get a monoid $S+A$, and enlarging an ideal A with the help of a monoid S to get an ideal $A\div S$.*

The explicit construction of the equations of (i) *and the ideals and monoids of* (ii) *are described in the proof below.*

Proof. We shall use induction on $|I|$.

If $|I|=1$, let us write $I=\{0\}$. Then (i) holds using the single equation $a_0=s_0$, and (ii) holds with $A^{(0)}=A_0$, $S^{(0)}=S_0$. For the inductive step, let me first outline the form of the argument, then fill in the details for the respective assertions (i) and (ii).

Given $|I|>1$, we shall choose a leaf in the Hasse diagram of I, denote this leaf 1, and denote the unique element of $I-\{1\}$ to which 1 is connected in that diagram 0. Assuming inductively that the desired result holds for templates indexed by the partially ordered set $I-\{1\}$, we will then consider the two cases $1>0$ and $1<0$.

If $1>0$, we will apply the inductive assumption on templates indexed by $I-\{1\}$ to the template gotten from $(I-\{1\},(A_i,S_i))$ by the single change of replacing the monoid S_0 with the monoid $S_0(S_1+A_1)$. By the second paragraph of Lemma 6, a prime P_0 realizing the pair $(A_0,S_0(S_1+A_1))$ is equivalent to a prime P_0 realizing (A_0,S_0) and contained in a prime P_1 realizing (A_1,S_1); hence a family of primes will realize this modified template if and only if it realizes the template $(I-\{1\},(A_i,S_i))$ and can be extended to a realization of $(I,(A_i,S_i))$.

If $1<0$, we will use, in the same way, the template gotten from $(I-\{1\},(A_i,S_i))$ by replacing the ideal A_0 with the ideal $A_0+A_1\div S_1$.

Now for the details of the proof of (i). Let $0,1\in I$ be as above, and assume we have a system of equations of the desired sort for templates indexed by $I-\{1\}$. If $1>0$, we introduce four new variables $a_1\in A_1$, $s_1\in S_1$, $x_{01},y_{01}\in R$, and two equations, $x_{01}+s_1+a_1=0$ and $y_{01}=s_0x_{01}$, and then replace all occurrences of s_0 in the *equations* of the original system with y_{01}, but leave unchanged the membership relation $s_0\in S_0$ of that

S0021-8693(02)00666-X/FLA AID:9425 Vol.●●●(●●●) Yjabr9425 P.9 (1-22)
ELSGMLTM(JABR):m1 2002/12/16 Prn:19/12/2002; 8:50 by:Rima p. 9

system. The newly added equations and relations and the old relation $s_0 \in S_0$ together say that $y_{01} \in S_0(S_1 + A_1)$; thus the nonexistence of a solution to these equations is equivalent to the realizability of the template obtained from $(I - \{1\}, (A_i, S_i))$ by replacing S_0 with $S_0(S_1 + A_1)$, hence, as discussed above, to the realizability of our given template.

If $1 < 0$, we again introduce variables $a_1 \in A_1$, $s_1 \in S_1$, $x_{01}, y_{01} \in R$, and this time equations $x_{01}s_1 = a_1$, $y_{01} + a_0 + x_{01} = 0$, and replace all occurrence of a_0 in our earlier equations (but not in the condition $a_0 \in A_0$) with y_{01}. Thus, x_{01} now represents an element of $A_1 \div S_1$, and y_{01} an element of $A_0 + A_1 \div S_1$, and our modified system of conditions again has the desired property.

Turning to assertion (ii), note that an element $j \in I$ is singled out in that statement; hence in proving the inductive step, let us use the fact that every finite tree order of more than one element has at least *two* leaves, to choose a leaf $1 \neq j$. (This is for convenience; we could alternatively choose an arbitrary leaf 1, and use different arguments when $1 = j$ and $1 \neq j$.) By induction we can get expressions $A^{(j)}, S^{(j)}$ in the ideals and monoids A_i, S_i $(i \in I - \{1\})$ and the operations $+$, $\div$, and multiplication, such that the realizations of the pair $(A^{(j)}, S^{(j)}$ are precisely the jth coordinates of realizations of the template $(I - \{1\}, (A_i, S_i))$. Now if $1 > 0$ (where 0 again denotes the vertex to which 1 is attached, which may or may not be j), we modify the formulas for $A^{(j)}$ and $S^{(j)}$ by replacing all occurrences of S_0 with $S_0(S_1 + A_1)$, while if $1 < 0$ we instead replace occurrences of A_0 with $A_0 + A_1 \div S_1$. In each case, the resulting pair will, by our earlier discussion, have the desired property. □

In Section 6 below we shall show that for templates based on finite partially ordered sets that are not tree orders, such neat results cannot hold. On the other hand, we shall see in the next section (the results of which will not be used in subsequent sections) that from the results obtained above, we *can* get similar results for infinite templates.

4. Infinite arrays of primes

Infinite templates may be studied in terms of their finite subtemplates using

Proposition 10. *Let $(I, (A_i, S_i)_{i \in I})$ be a template over R, and let F be a family of subsets of I which is directed under inclusion (i.e., such that given $I', I'' \in F$, there exists $I''' \in F$ containing $I' \cup I''$), and has I as its union. Then*

(i) $\mathrm{Spec}_R(I, (A_i, S_i))$ *can be identified with the inverse limit over $I' \in F$ of the sets* $\mathrm{Spec}_R(I', (A_i, S_i))$.
(ii) $\mathrm{Spec}_R(I, (A_i, S_i))$ *is nonempty if and only if for all $I' \in F$,* $\mathrm{Spec}_R(I', (A_i, S_i))$ *is nonempty.*
(iii) *For each $j \in I$, the set of primes P_j occurring as jth coordinates in realizations of* $\mathrm{Spec}_R(I, (A_i, S_i))$ *is the intersection, over all $I' \in F$ which contain j, of the sets of primes occurring as jth coordinates in realizations of* $\mathrm{Spec}_R(I', (A_i, S_i))$.

S0021-8693(02)00666-X/FLA AID:9425 Vol.●●●(●●●) Yjabr9425 P.10 (1-22)
ELSGMLTM(JABR):m1 2002/12/16 Prn:19/12/2002; 8:50 by:Rima p. 10

Proof. Note that given arbitrary subsets $I' \supseteq I''$ of I, there is a natural map $\mathrm{Spec}_R(I', (A_i, S_i)) \to \mathrm{Spec}_R(I'', (A_i, S_i))$, sending each I'-tuple $(P_i)_{i\in I'}$ to its restriction $(P_i)_{i\in I''}$. Regarding our sets as connected by this family of mappings, assertion (i) is immediate from the definition of $\mathrm{Spec}_R(-,-)$.

We claim next that once we have proved statement (ii), statement (iii) will follow. Indeed, given any prime P_j realizing the pair (A_j, S_j), let us form a new template, agreeing with $(I, (A_i, S_i))$, except in the jth position, where (A_j, S_j) is replaced by $(P_j, R - P_j)$. Then realizations of this new template correspond to realizations of our original template having P_j as jth coordinate. Now (ii) applied to this modified template gives (iii).

The "only if" direction of (ii) is clear from (i). The "if" direction will be an application of elementary model theory.

Note first that to specify a realization $(P_i)_{i\in I}$ of our given template is equivalent to assigning a truth value to each member of the set of propositions "$r \in P_i$," where r ranges over R and i over I, in a way consistent with a certain family of implications. (These are:

(a) the conditions saying that each P_i is an ideal, namely $0 \in P_i$, $[(r \in P_i) \wedge (r' \in P_i) \Rightarrow (r + r' \in P_i)]$, and $[(r \in P_i) \Rightarrow (rr' \in P_i)]$, for all $r, r' \in R$;
(b) the condition saying that this ideal is prime, namely $[(rr' \in P_i) \Rightarrow (r \in P_i) \vee (r' \in P_i)]$,
(c) the conditions saying that each P_i contains A_i and is disjoint from S_i, and
(d) the implications saying that for all i, i' with $i < i'$, one has $P_i \subseteq P_{i'}$.)

By the Compactness Theorem of model theory [12], there will exist a set of truth values satisfying all of these conditions if and only if for every finite subset X of these conditions, there is a set of truth values satisfying the members of X. Now any such finite X involves the relation of membership in P_i for only finitely many $i \in I$, and these finitely many i will all be contained in some $I' \in F$. By assumption, $\mathrm{Spec}_R(I', (A_i, S_i))$ is nonempty; let $(P_i)_{i\in I'} \in \mathrm{Spec}_R(I', (A_i, S_i))$. This I'-tuple determines an assignment of truth values to all the propositions "$r \in P_i$" with $i \in I'$, which satisfies the finitely many conditions in X. If we extend this assignment in an arbitrary way to the remaining propositions, it will continue to satisfy these conditions; hence by the Compactness Theorem, our full set of conditions can be satisfied simultaneously. □

Remarks on the above proof:

What logicians call compactness results can, in fact, generally be obtained by topological compactness arguments; let us note how this may be done in the above case. We recall that the prime spectrum of a commutative ring R, in addition to the Zariski topology, with its basis of open sets consisting of the sets $U_r = \{P \mid r \notin P\}$ $(r \in R)$, admits another topology, which Hochster [5] names the "patch" topology, in which the sets U_r *and their complements* form a subbasis of open sets; and that this topology is compact and Hausdorff. It is straightforward to verify that each $\mathrm{Spec}_R(I', (A_i, S_i))$ is closed in the I'-fold direct product of copies of $\mathrm{Spec}\,R$ under the product of these patch topologies, hence is compact and Hausdorff in the subspace topology, and that the natural maps among these compact spaces are continuous. Statement (ii) is now a consequence of the fact that the inverse limit of a system of nonempty compact Hausdorff spaces and continuous maps is nonempty.

S0021-8693(02)00666-X/FLA AID:9425 Vol.●●●(●●●) Yjabr9425 P.11 (1-22)
ELSGMLTM(JABR):m1 2002/12/16 Prn:19/12/2002; 8:50 by:Rima p. 11

For the reader who likes ultraproducts, here is a sketch of yet another version of the above argument. Since F is directed, the subsets of F of the form $C_{I'} = \{I'' \in F \mid I'' \supseteq I'\}$ $(I' \in F)$ generate a proper filter on F. Choose an ultrafilter U on F containing this filter. For each $I' \in F$ choose a realization $(P_{I',i})_{i \in I'}$ of $(I', (A_i, S_i))$, and extend each of these realizations to an I-tuple of subsets of R by letting $P_{I',i}$ be arbitrary for $i \notin I'$. For each i, the F-tuple $(P_{I',i})_{I' \in F}$ of subsets of R will induce a subset Q_i of the ultrapower $R^+ = R^F/U$. Because of the way we chose U, each of the conditions required for an I-tuple (P_i) of subsets of R to be a realization of $(I, (A_i, S_i))$ is satisfied by $(P_{I',i})_{i \in I}$ for "almost all" (relative to U) $I' \in F$. One can deduce that the Q_i will be prime ideals of R^+, and that letting $P_i = Q_i \cap R$, we get a realization of $(I, (A_i, S_i))$.

We now want to use the above lemma to extend Theorem 9 to appropriate cases where I may be infinite. But what should the infinite analog of a tree order be? An infinite partially ordered set does not, in general, have a "Hasse diagram," since it may have few or no minimal order relations; so we cannot use the definition we gave in the finite case. It would also not be appropriate to define a general partially ordered set to be a tree order if and only if the induced partial orderings on all connected finite subsets are tree orders, because as noted, even finite tree orders can have connected subsets that are not tree orders. The result of McKenzie noted parenthetically following (11) above suggests that one might define a not-necessarily-finite tree order to mean a partially ordered set in which every minimal cycle is of the form (11), and is contained in a subset of the form (10); but it is not clear that a partially ordered set I with this property must be a directed union of finite subsets with the same property, as would be needed to apply Proposition 10. So I will not try to define "infinite tree order;" rather, let us simply assume the condition needed to apply that proposition.

Corollary 11. *Let $(I, (A_i, S_i)_{i \in I})$ be a template over R, and suppose that for every finite subset $J \subseteq I$ there exists a finite subset $I' \subseteq I$ which contains J and which is a tree order under the induced ordering. Then*

(i) *$(I, (A_i, S_i)_{i \in I})$ is realizable if and only if for every finite $I' \subseteq I$ which is a tree order, the condition for realizability of the finite template $(I', (A_i, S_i)_{i \in I'})$ referred to in Theorem* 9(i) *holds.*

(ii) *For each $j \in I$, there exists an ideal $A^{(j)} \subseteq R$ and a monoid $S^{(j)} \subseteq R$ such that the prime ideals of R occurring as jth coordinates of realizations of $(I, (A_i, S_i)_{i \in I})$ are the realizations of the pair $(A^{(j)}, S^{(j)})$. Here $A^{(j)}$ is the union of a directed system of ideals each obtained from finitely many of the A_i and S_i as described in Theorem* 9(ii), *and $S^{(j)}$ is the union of a similarly constructed directed system of multiplicative monoids.*

5. How to construct counterexamples

Consider a template $(I, (A_i, S_i))$, where I is the three-element chain $0 < 1 < 2$. The method of Theorem 9(ii) shows that the prime ideals occurring as $i = 0$ coordinates of realizations of this template comprise the set

$$\mathrm{Spec}_R\big(A_0, S_0\big(S_1(S_2 + A_2) + A_1\big)\big).$$

One might wonder whether this description can be simplified to

$$\mathrm{Spec}_R\big(A_0, S_0(S_1 + A_1)(S_2 + A_2)\big).$$

Now in fact, the latter set can be seen to be the set of $i = 0$ coordinates of realizations of the template based on the same family of ideals and monoids, but with the order relations on I reduced to $0 < 1$ and $0 < 2$, with 1 and 2 incomparable. So our question is whether the sets of $i = 0$ coordinates of realizations of these two templates are always the same.

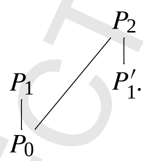
To see that they are not, take any ring R with three prime ideals P_0, P_1, P_2 such that $P_0 \subseteq P_1$ and $P_0 \subseteq P_2$, but $P_1 \not\subseteq P_2$. For each i, let

$$A_i = P_i, \qquad S_i = R - P_i.$$

Then regardless of what ordering we put on I, the only element that could possibly belong to $\mathrm{Spec}_R(I, (A_i, S_i))$ is $(P_i)_{i\in I}$. Under the ordering noted above with 1 and 2 incomparable, this unique element indeed belongs to $\mathrm{Spec}_R(I, (A_i, S_i))$ but under the original ordering it clearly does not.

Here is a similar question. For I again the set $\{0, 1, 2\}$ with $0 < 1 < 2$, and $(I, (A_i, S_i))$ a template indexed by this I, is the condition for readability of this template just the conjunction of the realizability conditions for the three subtemplates $(\{0, 1\}, (A_i, S_i))$, $(\{1, 2\}, (A_i, S_i))$, and $(\{0, 2\}, (A_i, S_i))$, where each 2-element subset is given the induced ordering?

Again the answer is "no," and we can prove it in a similar way. Let R be a ring in which four prime ideals P_0, P_1, P'_1, P_2 satisfy $P_0 \subseteq P_1$, $P'_1 \subseteq P_2$, and $P_0 \subseteq P_2$, but no other inclusion relations:



Define A_i and S_i as in the previous example for $i = 0, 2$, and define $A_1 = P_1 \cap P'_1$, $S_1 = R - (P_1 \cup P'_1)$. Then for $i = 0, 2$, P_i is again the only prime realizing (A_i, S_i), while it is easy to check that the set of primes realizing the pair (A_i, S_i) is precisely $\{P_1, P'_1\}$. From these facts we can see that the three subtemplates referred to above are all realizable, but the original template is not.

In these examples, we have taken for granted that we could find rings with families of prime ideals satisfying specified inclusion and non-inclusion relations; and indeed, such rings are not hard to find in the cases considered above. But in later sections we will need examples of more complicated situations; hence let us record

Lemma 12. *Let I be a partially ordered set. Then there exists a ring R having a family of prime ideals $(P_i)_{i\in I}$ such that for $i, j \in I$,*

$$P_i \subseteq P_j \Leftrightarrow i \leqslant j.$$

S0021-8693(02)00666-X/FLA AID:9425 Vol.●●●(●●●) Yjabr9425 P.13 (1-22)
ELSGMLTM(JABR):m1 2002/12/16 Prn:19/12/2002; 8:50 by:Rima p. 13

In fact, R can be taken to be a polynomial ring in an I-tuple of indeterminates over any integral domain k, and the P_i to be ideals generated by subsets of the set of indeterminates.

Proof. Recall that any partially ordered set I is isomorphic to a family of subsets of some set J under the inclusion ordering; in particular, we can take $J = I$, mapping each $i \in I$ to $L_i = \{j \in I \mid j \leqslant i\}$. We next note that if we form the polynomial algebra over any integral domain k in a J-tuple of indeterminates X_j, then the ideal generated by any subset of the indeterminates is prime, and the order structure on this set of primes is that of the power set of J. The desired conclusion follows immediately. □

We also used in the second of our above examples the observation that for primes P_1 and P_1' which are incomparable under inclusion, the realizations of the pair $(P_1 \cap P_1', (R - (P_1 \cup P_1')))$ are precisely P_1 and P_1'. Let us record a few general observations of this sort (where R is once again an arbitrary commutative ring).

Lemma 13. *Let X be a set of prime ideals of R. Then the following conditions are equivalent*:

(i) $X = \mathrm{Spec}_R(A, S)$ *for some ideal A and multiplicative monoid S in R.*
(ii) *X contains all prime ideals Q such that $\bigcap_{P\in X} P \subseteq Q \subseteq \bigcup_{P\in X} P$.*

If we call a set of primes satisfying these equivalent conditions convex, then for any set Y of primes, the least convex set of primes containing Y is

$$\mathrm{Spec}_R\left(\bigcap_{P\in Y} P,\ R - \left(\bigcup_{P\in Y} P\right)\right).$$

If Y is finite, this can be described as the set of all primes Q such that $P_0 \subseteq Q \subseteq P_1$ for some $P_0, P_1 \in Y$. Hence if Y is finite and no two distinct primes in Y are comparable, Y is itself convex.

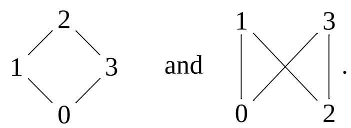
Proof. Assuming (i), $\bigcap_{P\in X} P$ will contain A, and $\bigcup_{P\in X} P$ will be contained in the complement of S, from which we can see (ii). Conversely, assuming (ii), the choices $A = \bigcap_{P\in X} P$ and $S = R - \bigcup_{P\in X} P$ give (i). The first sentence of the final paragraph is clear from these observations.

To see the characterization of the convex closure of a finite set Y of primes, it suffices to know that for such a Y the only primes containing $\bigcap_{P\in Y} P$ are the primes that contain some $P_0 \in Y$, and the only primes contained in $\bigcup_{P\in Y} P$ are those contained in some $P_1 \in Y$. The former fact is well-known, the latter less so; for both, see [1, §II.1.1, Propositions 1–2]. (In each statement, only the ideal(s) on the larger side of the inclusion must be assumed prime.)

The final assertion clearly follows. □

6. What if I is not a tree order?

We saw in Section 3 that if I is a tree order, the conditions for a template $(I, (A_i, S_i))$ to be realizable, and the set of primes occurring as ith coordinates in its realizations, have convenient descriptions. What if I is not a tree order? The simplest non-tree orders are the "diamond" and the "two-peaked crown,"

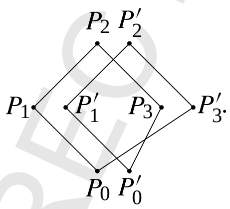


(12)

We shall investigate the "diamond" below.

Let $(I, (A_i, S_i))$ be a template over R such that I is the above diamond, and suppose we want to characterize prime ideals P_0 that can occur as $i = 0$ coordinates in realizations of this template. Using the methods of Section 3, we can write down the conditions for a prime P_0 realizing (A_0, S_0) to be contained in a prime realizing (A_1, S_1) which is contained in a prime realizing (A_2, S_2); or the stronger condition for a prime realizing (A_0, S_0) to be contained in a prime realizing (A_1, S_1) which is contained in a prime realizing (A_2, S_2) which contains a prime realizing (A_3, S_3) which contains a prime realizing (A_0, S_0); and so forth. We may ask whether if we go sufficiently far along in this family of conditions, or perhaps take the infinite conjunction of this family, the resulting condition, clearly *necessary* for P_0 to occur as the $i = 0$ coordinate of a realization of our template, is also sufficient.

The answer is no. To see this, we note that by Lemma 12 there exists a ring R containing 8 primes whose inclusion relations are precisely those shown below:



(13)

Let us now define the ideals and monoids of our template by $A_i = P_i \cap P_i'$, $S_i = R - (P_i \cup P_i')$ $(i = 0, \ldots, 3)$. Then by Lemma 13, the only primes realizing each pair (A_i, S_i) are P_i and P_i'. We see that P_0 satisfies all the conditions just referred to (being contained in a prime which is contained in a prime which contains a prime which contains a prime, etc.), but is not the $i = 0$ coordinate of a realization of our template; indeed, the template has no realizations, since (13) clearly contains no isotone image of the "diamond" with vertices in the required subsets.

So let us take a slightly different approach. Given as before a template $(I, (A_i, S_i))$ with I the "diamond" of (12), if we specify a prime P_0 realizing (A_0, S_0), can we determine whether there exist P_1, P_2, P_3 such that the 4-tuple (P_i) is a realization of our template? In such a realization, P_1 must be a realization of (A_1, S_1) which contains P_0; in other words, it

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S0021-8693(02)00666-X/FLA AID:9425 Vol.●●●(●●●) Yjabr9425 P.15 (1-22)
ELSGMLTM(JABR):m1 2002/12/16 Prn:19/12/2002; 8:50 by:Rima p. 15

must be a realization of $(A_1 + P_0, S_1)$; similarly, P_3 must be a realization of $(A_3 + P_0, S_3)$. In fact, we see that the necessary and sufficient condition for the desired P_1, P_2, P_3 to exist is that the template

$$\begin{array}{ccc} & (A_2, S_2) & \\ \swarrow & & \searrow \\ (A_1 + P_0, S_1) & & (A_3 + P_0, S_3) \end{array}$$

be realizable. The proof of Theorem 9 shows that this condition is equivalent to

$$\big(A_2 + (A_1 + P_0) \div S_1 + (A_3 + P_0) \div S_3\big) \cap S_2 = \emptyset. \tag{14}$$

Note that P_0 is the only prime or monoid occurring more than once in (14); thus a failure of (14) means the existence of *two* elements $x, x' \in P_0$ that together satisfy a certain family of equations involving elements, *one* each, of $A_1, A_2, A_3, S_1, S_2, S_3$, and a certain number of unrestricted elements of R. Let us now drop the assumption that P_0 has been pre-chosen, and let X denote the collection of all pairs (x, x') of elements of R for which there exist elements of $A_1, \ldots, S_3$ and R which satisfy, with x and x', the family of equations just referred to. Then we see that a prime P_0 occurs as the $i = 0$ component of a realization of our template if and only if P_0 is a realization of (A_0, S_0) such that for every $(x, x') \in X$, P_0 contains *at most one* of x, x'. This characterization of such primes is, in its way, as "concrete" as the conditions of Theorem 9, but it is certainly not as simple. I do not know whether this set of primes will in general be convex in the sense of Lemma 13.

If we look for conditions for a prime P_2 to occur as the $i = 2$ component of a realization of our template, the analysis begins in much the same way. The condition we get is that

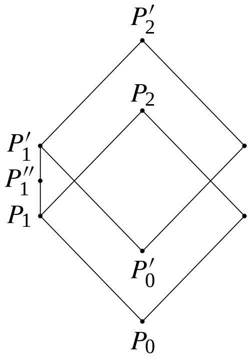
$$A_0 \cap S_0\big(S_1(R - P_2) + A_1\big)\big(S_3(R - P_2) + A_3\big) = \emptyset,$$

which says that for each member of a certain set of pairs (y, y'), at most one of y, y' should belong to $R - P_2$. But note that this says that *at least one* of y, y' should belong to P_2, and since P_2 is to be a prime ideal, this is equivalent to the condition that the product yy' belong to P_2. Hence if we write A_2^+ for the ideal of R generated by A_2 and the set of such products yy', the primes occurring as the $i = 2$ components of realizations of our template are precisely the realizations of the pair (A_2^+, S_2).

So in this case, the set of such primes *is* convex. This is more like the criterion of Theorem 9; except that the ideal A_2^+ does not have as simple a description as the ideals $A^{(i)}$ of that theorem. The nature of the constructions $A \div S$, $S + A$, etc., has the consequence that in the situation of that theorem, the predicate of membership in each of the sets $A^{(i)}$ and $S^{(i)}$ is expressible by a first-order sentence in the ring operations and the predicates of membership in the ideals and monoids of the given template; but here the condition $b \in A_2^+$ is equivalent to the existence of an equation $b = a + \sum_i r_i y_i y_i'$ with an unspecified number of terms in the summation. It would be interesting to know whether this difference has any significant consequences for the behavior of these sets.

S0021-8693(02)00666-X/FLA AID:9425 Vol.●●●(●●●) Yjabr9425 P.16 (1-22)
ELSGMLTM(JABR):m1 2002/12/16 Prn:19/12/2002; 8:50 by:Rima p. 16

Finally, if we turn to the set of primes occurring as the $i = 1$ coordinates of realizations of our template, this is *not* in general convex. To show this, let R be a ring containing a family of 9 primes with precisely the order relations shown below:



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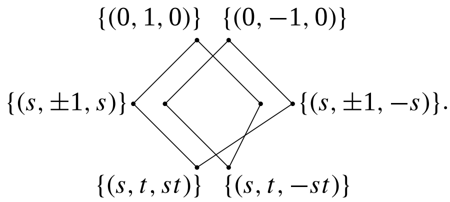
and let us construct a template $(I, (A_i, S_i))$ by defining, for $i = 0, 2, 3$, $A_i = P_i \cap P_i'$, and $S_i = R - (P_i \cup P_i')$ as before, while for (A_1, S_1) we take any pair whose realizations include both P_1 and P_1'. (For instance, the pair $(P_1, R - P_1')$, or the pair $(\{0\}, \{1\})$.) For $i = 0, 2, 3$, the fact that P_i and P_i' are incomparable means that P_i and P_i' are the only primes that can occur in the ith coordinate of a realization of our template. From this fact and the order relations among our primes, we see that every such realization must have in these coordinates either precisely P_0, P_2 and P_3, or precisely P_0', P_2' and P_3'. Turning to the $i = 1$ coordinate, we see from the two obvious realizations (P_0, P_1, P_2, P_3) and (P_0', P_1', P_2', P_3') of our template that P_1 and P_1' can each occur in this position; however P_1'' cannot, since it neither contains P_0' nor is contained in P_2. Thus the set of primes occurring as $i = 1$ coordinates of realizations of this template includes P_1 and P_1', but not the prime P_1'' lying between them; so it is not convex.

Let us end this section by returning to the "double covering of the diamond," (13), and recording for later use a simpler example of a family of prime ideals having that order structure than the one produced by the construction of Lemma 12. Let k be a field of characteristic $\neq 2$ and $R = k[x, y, z]$. It is immediate that the desired order relations are satisfied by the prime ideals

$$P_0, P_0' = (xy \pm z), \qquad P_1, P_1' = (x - z, y \pm 1), \qquad P_2, P_2' = (x, z, y \pm 1),$$
$$P_3, P_3' = (x + z, y \pm 1), \tag{15}$$

where in each case, the minus sign goes with the unprimed symbol and the plus sign with the primed symbol. We illustrate this below by showing the corresponding subvarieties of affine 3-space, each expressed as the set of points of a given form. (E.g., $\{(s, -1, s)\}$ denotes the set of points whose first and third coordinates are equal, and whose middle coordinate is -1. The two varieties at the top and the two at the bottom are labeled explicitly, while the pairs at the middle level are combined using the $\pm$ sign, for reasons of spacing. Which sign corresponds to which vertex at that level can easily be seen by comparing with the precisely labeled vertex above or below.)



(16)

We note a further property of this example, that the varieties determined by corresponding "primed" and "unprimed" ideals are interchanged by the map $(s,t,u) \leftrightarrow (-s,-t,-u)$; equivalently, the ideals are interchanged by the k-algebra automorphism of R which acts by $x \mapsto -x$, $y \mapsto -y$, $z \mapsto -z$.

7. Prime ideals in tensor products

I will confess at this point that the origin of this note was the desire to prove for myself the known fact that the Krull dimension of a tensor product algebra $R^{(0)} \otimes_k R^{(1)}$ over a field k is at least the sum of the Krull dimensions of $R^{(0)}$ and $R^{(1)}$. (Recall that the *Krull dimension* of a commutative ring is the supremum of the lengths n of chains $P_0 \subseteq P_1 \subseteq \cdots \subseteq P_n$ of prime ideals of R.) Using the standard result Lemma 1, it is easy to show that given prime ideals $P^{(0)} \subseteq R^{(0)}$, $P^{(1)} \subseteq R^{(1)}$, there exists a prime $P \subseteq R^{(0)} \otimes_k R^{(1)}$ which intersects the given rings in $P^{(0)}$ and $P^{(1)}$, respectively. But it was not clear whether *inclusions* of ideals could similarly be lifted to the tensor product, as would be needed to estimate its Krull dimension. This led me to look for an analog of Lemma 1 for inclusions of primes, which led to the results of the preceding sections, which I then tried to apply to the original question about tensor product rings.

I have realized subsequently that a better approach to the lifting of general arrays of prime ideals to tensor product rings is probably via the fact that when k is *algebraically closed*, a tensor product over k of integral domains is an integral domain ([6, Lemma 1.54, p. 97]; cf. [4, Exercises 1.3.15, p. 22, II.3.15, p. 93]); hence that in this situation, if $P^{(0)}$, $P^{(1)}$ are prime ideals of $R^{(0)}$ and $R^{(1)}$, the ideal $P^{(0)} \otimes_k R^{(1)} + R^{(0)} \otimes_k P^{(1)}$ of $R^{(0)} \otimes_k R^{(1)}$, i.e., the kernel of the map

$$R^{(0)} \otimes_k R^{(1)} \to \bigl(R^{(0)}/P^{(0)}\bigr) \otimes_k \bigl(R^{(1)}/P^{(1)}\bigr),$$

will be prime, giving us a choice-free order-preserving way of lifting primes. For non-algebraically-closed k, the corresponding problem should probably be approached by first studying the lifting of arrays of primes in the given algebras under algebraic extension of the base field, which is where the complications come in, and then using the above result on tensor products over algebraically closed fields.

However, it was fairly easy to obtain from the preceding results of this paper a result which includes the abovementioned estimate of the Krull dimension of a tensor product

S0021-8693(02)00666-X/FLA AID:9425 Vol.•••(•••) Yjabr9425 P.18 (1-22)
ELSGMLTM(JABR):m1 2002/12/16 Prn:19/12/2002; 8:50 by:Rima p. 18

algebra, and I will give this below. In the final section, I will give an example showing that complications indeed arise in lifting arrays of primes under base field extension.

Definition 14. For the remainder of this section k will be an arbitrary field, and $R^{(0)}$, $R^{(1)}$ nonzero commutative k-algebras. We shall write $R^{(0)} \otimes R^{(1)}$ for $R^{(0)} \otimes_k R^{(1)}$, and identify $R^{(0)}$ and $R^{(1)}$ with their natural images in this ring. Thus, for subsets $X^{(0)} \subseteq R^{(0)}$, $X^{(1)} \subseteq R^{(1)}$, we may write $X^{(0)}X^{(1)}$ for the set of products $x^{(0)}x^{(1)} \in R^{(0)} \otimes R^{(1)}$ $(x^{(0)} \in X^{(0)},\ x^{(1)} \in X^{(1)})$. On the other hand, if $A^{(0)}$, $A^{(1)}$ are ideals of these respective rings, we shall write $A^{(0)} \otimes R^{(1)}$ and $R^{(0)} \otimes A^{(1)}$ for the ideals of $R^{(0)} \otimes R^{(1)}$ generated by the images of $A^{(0)}$ and $A^{(1)}$ therein (these ideals being clearly isomorphic to the corresponding external tensor products).

Lemma 15. *Let $P^{(0)} \subseteq R^{(0)}$, $P^{(1)} \subseteq R^{(1)}$ be prime ideals, let A denote the ideal $P^{(0)} \otimes R^{(1)} + R^{(0)} \otimes P^{(1)} \subseteq R^{(0)} \otimes R^{(1)}$, and let S denote the multiplicative monoid $(R^{(0)} - P^{(0)})(R^{(1)} - P^{(1)})$ of that ring. Then*

(i) $A \cap S = \emptyset$.
(ii) $A \div S = A$.
(iii) *A prime P in $R^{(0)} \otimes R^{(1)}$ is a realization of the pair (A, S) if and only if $P \cap R^{(0)} = P^{(0)}$ and $P \cap R^{(1)} = P^{(1)}$.*
(iv) *Every prime ideal Q of $R^{(0)} \otimes R^{(1)}$ containing A contains a prime ideal P which realizes the pair (A, S); that is, every prime whose intersections with $R^{(0)}$ and $R^{(1)}$ contain $P^{(0)}$ and $P^{(1)}$, respectively, contains a prime P whose intersections with these subrings are precisely those primes.*

Proof. As noted, A is the kernel of a homomorphism from $R^{(0)} \otimes R^{(1)}$ to a nontrivial ring; hence it is a proper ideal, so (i) will follow from (ii). To prove (ii), note that (ii) is equivalent to saying that no nonzero element of the $R^{(0)} \otimes R^{(1)}$-module $(R^{(0)} \otimes R^{(1)})/A$ is annihilated by any element of S. Now $(R^{(0)} \otimes R^{(1)})/A$ can be identified with $(R^{(0)}/P^{(0)}) \otimes_k (R^{(1)}/P^{(1)})$; hence it is free both as a module over $R^{(0)}/P^{(0)}$ and as a module over $R^{(1)}/P^{(1)}$. Since each of these rings is a domain, no element of that module is annihilated by a nonzero element of $R^{(0)}/P^{(0)}$ or of $R^{(1)}/P^{(1)}$; i.e., looking at it as an $R^{(0)} \otimes R^{(1)}$-module, none of its nonzero elements is annihilated by a member of $R^{(0)} - P^{(0)}$ or $R^{(1)} - P^{(1)}$; hence no nonzero element is annihilated by a member of the product S of these monoids, as required.

Statement (iii) holds because by Lemma 5 a prime realizes $(A, S) = (P^{(0)} \otimes R^{(1)} + R^{(0)} \otimes P^{(1)}, (R^{(0)} - P^{(0)})(R^{(1)} - P^{(1)}))$ if and only if it realizes both $(P^{(0)} \otimes R^{(1)}, R^{(0)} - P^{(0)})$ and $(R^{(0)} \otimes P^{(1)}, R^{(1)} - P^{(1)})$, i.e., meets $R^{(0)}$ in $P^{(0)}$, and $R^{(1)}$ in $P^{(1)}$. Finally, (iv) follows from (ii) in view of Lemma 4(i). □

From part (iv) of the above lemma, we see

Corollary 16. *Let Q be a prime ideal of $R^{(0)} \otimes R^{(1)}$, and let $Q^{(\alpha)} = Q \cap R^{(\alpha)}$* $(\alpha = 0, 1)$. *Then given any prime ideals $P^{(\alpha)} \subseteq Q^{(\alpha)}$ in $R^{(a)}$* $(\alpha = 0, 1)$, *there exists a prime ideal $P \subseteq Q$ of $R^{(0)} \otimes R^{(1)}$ such that $P \cap R^{(\alpha)} = P^{(\alpha)}$* $(a = 0, 1)$.

S0021-8693(02)00666-X/FLA AID:9425 Vol.●●●(●●●) Yjabr9425 P.19 (1-22)
ELSGMLTM(JABR):m1 2002/12/16 Prn:19/12/2002; 8:50 by:Rima p. 19

To formulate our application of this result, let us make

Definition 17. A *finite descending tree* will mean a member of the class of finite partially ordered sets defined recursively by the conditions that

(i) all one-element partially ordered sets are contained in the class, and
(ii) an $(n+1)$-element partially ordered set I is contained in the class if and only if it can be obtained by adjoining to an n-element partially ordered set I_0 in the class a single element j and a single order relation making j *less than* some element $i \in I_0$.

(*Ascending trees* may be defined analogously, replacing "less than" with "greater than.")

By starting at the top of such a tree and working downwards inductively, using Lemma 15(i) and (iii) at the first step, and Corollary 16 at each subsequent step, we can clearly get

Corollary 18. *Let I be a finite descending tree* (*as defined above*), *and let $(P_i^{(0)})_I$, $(P_i^{(1)})_I$ be families of prime ideals of $R^{(0)}$ and $R^{(1)}$, respectively, such that whenever $i \leqslant j$ in I, one has $P^{(\alpha)} \subseteq P_j^{(\alpha)}$ in $R^{(\alpha)}$ $(a = 0, 1)$.*

Then there exists a family of prime ideals $P_i \subseteq R^{(0)} \otimes R^{(1)}$ such that $P_i \cap R^{(\alpha)} = P_i^{(\alpha)}$ $(i \in I,\ \alpha = 0, 1)$ and $i \leqslant j \Rightarrow P_i \subseteq P_j$ $(i, j \in I)$.

In particular, if $R^{(0)}$ and $R^{(1)}$ have Krull dimensions at least m and n, respectively, then we can take for I a chain of length $m+n$, and map it into the partially ordered sets of prime ideals of $R^{(0)}$ and $R^{(1)}$ so that each link of the chain goes to a nontrivial interval in one or the other of those partially ordered sets. Then the above corollary gives a map into the prime ideals of $R^{(0)} \otimes R^{(1)}$ under which no link collapses, hence the Krull dimension of $R^{(0)} \otimes R^{(1)}$ is at least $m+n$.

Wadsworth [14] shows that the question of whether the Krull dimension of $R^{(0)} \otimes R^{(1)}$ is strictly larger than that of $m+n$, and if so, by how much, is quite subtle.

Finite descending trees can also be characterized as the finite connected partially ordered sets I such that no two incomparable elements of I have a common lower bound. Using this characterization, one can define not-necessarily-finite descending trees, and use Proposition 10 to extend Corollary 18 to that case.

Returning to Corollary 16, we remark that result does not remain true if we reverse the direction of our inequalities. For example, in $k[x,y] \cong k[x] \otimes k[y]$, "most" nonmaximal prime ideals P intersect $k[x]$ and $k[y]$ in the zero ideal, but such a P cannot in general be enlarged to a prime ideal Q which restricts to a specified pair of nonzero prime ideals of $k[x]$ and of $k[y]$. For example, the prime ideal $(x-y)$ cannot be extended to a prime ideal whose intersections with $k[x]$ and $k[y]$ are specified primes $(x-a)$ and $(y-b)$, unless $a=b$. This phenomenon is related to the fact that $(x-y)$ is not minimal among prime ideals meeting $k[x]$ and $k[y]$ in the zero ideal; it appears that to lift general arrays of primes $(P_i^{(0)})$, $(P_i^{(1)})$ to $R^{(0)} \otimes_k R^{(1)}$, one should look at minimal primes containing the ideals $P_i^{(0)} \otimes_k R^{(1)} + R^{(0)} \otimes_k P_i^{(1)}$. These can be studied by forming the algebraic closure $\bar{k}$

of k, looking at primes of $R^{(0)} \otimes_k \bar{k}$ that intersect $R^{(0)}$ in the $P_i^{(0)}$ and primes of $\bar{k} \otimes_k R^{(1)}$ that intersect $R^{(1)}$ in the $P_i^{(1)}$, and using the facts noted earlier about tensor products over an algebraically closed field. I suspect this method can be used to extend Corollary 18 to the case where I is a general finite tree order; and in fact, to show that given a family of primes in $R^{(0)}$ indexed by a tree order $I^{(0)}$, and a family in $R^{(1)}$ indexed by another tree order $I^{(1)}$, one can lift these to a family of primes in $R^{(0)} \otimes_k R^{(1)}$ indexed by $I^{(0)} \times I^{(1)}$, although the latter is not in general a tree order. But I will not pursue these ideas.

8. Examples concerning algebraic extension of the base field

In this last section we shall give a counterexample, and a general technique for constructing examples, on the behavior of arrays of prime ideals under algebraic extension of the base field.

Let k be a field of characteristic $\neq 2$ containing an element c which is not a square. We shall give below a k-algebra containing a "diamond" of prime ideals (four primes with the order relations of the left-hand diagram in (12)), such that on extending scalars to $k(\sqrt{c}\,)$, each of these primes splits into exactly two primes, and the resulting array has the order structure (13). Hence the original "diamond" of primes cannot be lifted to $R \otimes_k k(\sqrt{c}\,)$.

The idea will be to work backwards: Start with a family of prime ideals of the form (13) in a $k(\sqrt{c}\,)$-algebra R' having an automorphism θ of order 2 which interchanges $\sqrt{c}$ and $-\sqrt{c}$, and also interchanges each pair of ideals P_i and P_i' in that diagram. The fixed ring of θ will then be a k-algebra R which, on extension of scalars to $k(\sqrt{c}\,)$, gives R', and each of those pairs of primes will be represented by a single prime in R, giving the desired "diamond" configuration.

Let us apply this idea using the instance of (13) given in (15). In our discussion of that example we referred to our pairs of primes as interchanged by the automorphism over the base-field that sent the three indeterminates to their negatives. Now if we take that base-field to be $k(\sqrt{c}\,)$, then since the descriptions of those primes do not involve the element $\sqrt{c}$, the k-algebra automorphism that not only changes the signs of x, y, and z but also that of $\sqrt{c}$ will permute these primes in the same way. The fixed ring of this automorphism is the polynomial ring $k[\sqrt{c}\,x, \sqrt{c}\,y, \sqrt{c}\,z]$. Renaming $\sqrt{c}\,x$, $\sqrt{c}\,y$ and $\sqrt{c}\,z$ as x, y, z, and letting $R = k[x, y, z]$, we get from (15) the array of prime ideals in R:

$$\begin{array}{ccc} & (x, z, y^2 - c) & \\ / & & \backslash \\ (x - z, y^2 - c) & & (x + z, y^2 - c) \\ \backslash & & / \\ & (x^2y^2 - cz^2) & \end{array}. \qquad (17)$$

In $R \otimes_k k(\sqrt{c}\,)$, the bottom prime lifts to the two primes $(xy + \sqrt{c}\,z)$ and $(xy - \sqrt{c}\,z)$, the prime on the left to $(x - z, y + \sqrt{c}\,)$ and $(x - z, y - \sqrt{c}\,)$, etc., and these have the order structure (13). (The reader can verify these assertions now, or wait and see that they are instances of general results that will be recalled in the proof of the next lemma.)

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S0021-8693(02)00666-X/FLA AID:9425 Vol.●●●(●●●) Yjabr9425 P.21 (1-22)
ELSGMLTM(JABR):m1 2002/12/16 Prn:19/12/2002; 8:50 by:Rima p. 21

Again, to visualize these properties, I find it helpful to look at the corresponding subvarieties of affine 3-space, shown below. When the base field is k, each set shown below represents the set of $\bar{k}$-valued points of an irreducible variety; but over $k(\sqrt{c}\,)$, each represents two such varieties, one for each choice of sign. The reader can start with one choice of signs in the bottom variety, note the choices of sign that allow one to traverse the figure upward and downward, and verify that one must go around twice to return to the original variety.

$$\begin{array}{ccc} & \{(0, \pm\sqrt{c}, 0)\} & \\ \diagup & & \diagdown \\ \{(s, \pm\sqrt{c}, s)\} & & \{(s, \pm\sqrt{c}, -s)\}. \\ \diagdown & & \diagup \\ & \{(s, t, \pm st/\sqrt{c})\} & \end{array} \qquad (18)$$

The above technique can be applied to a quite general class of situations:

Lemma 19. *Let K/k be a finite Galois field extension with Galois group G, let I be a finite partially ordered set, given with an action of G on it by order automorphisms, and let I/G be the orbit set of this action, with its natural induced partial ordering, under which $[i] < [j]$ if and only if i is $<$ some member of $[j]$, equivalently, if and only if some member of $[i]$ is $< j$.*

Then there exists a finitely generated k-algebra R, and a family of prime ideals of R, $(P_{[i]})_{[i]\in I/G}$, which has precisely the order structure of I/G, and such that in the extension ring $R \otimes_k K$, the set of primes which lie over primes in the above family can be indexed $(P_i)_{i\in I}$, in such a way that this family, ordered by inclusion, has precisely the order structure of I, the map $- \cap R$ takes each ideal P_i to $P_{[i]}$, i.e., corresponds to the canonical map $I \to I/G$, and the action of $G = \mathrm{Gal}(K/k)$ on $\{P_i\}$ induced by its action on the second tensor factor of $R \otimes_k K$ corresponds to the given action of G on I.

Moreover, R can be taken to be a polynomial ring over k in $|I|$ indeterminates, and each P_i to be generated by a subspace of the K-vector space in $R \otimes_k K$ spanned by the indeterminates.

Proof. As in the preceding example, we will start with the K-algebra that is to be $R \otimes_k K$ and the I-tuple of primes that are to be the P_i, and obtain R as the fixed ring of an appropriate action of $\mathrm{Gal}(K/k)$.

Let us construct our K-algebra using Lemma 12, as a polynomial algebra

$$S = K[x_i]_{i\in I},$$

and for each $i \in I$, take P_i to be the ideal of S generated by the set of indeterminates $\{x_j \mid j \leqslant i\}$. Letting G act on the indeterminates x_i via the given action on the index set I, and on K as its Galois group over k, we get an action of G on the above ring S by k-algebra automorphisms, which clearly acts as desired on the P_i. Let R be the fixed k-algebra of this action.

Now when we regard S as a K-vector space, the action of G is "semilinear;" i.e., for $g \in G$, $c \in K$, $s \in S$ one has $g(cs) = g(c)g(s)$. By A. Speiser's Theorem ([13], cf. [9,

S0021-8693(02)00666-X/FLA AID:9425 Vol.●●●(●●●) Yjabr9425 P.22 (1-22)
ELSGMLTM(JABR):m1 2002/12/16 Prn:19/12/2002; 8:50 by:Rima p. 22

Proposition 1.3], [3, Proposition 5.7.1, p. 202]) S has a K-vector-space basis B consisting of G-invariant elements, i.e., elements of R. If we write elements of S in terms of this basis, then the action of G on S is induced by its action on the coefficients; hence our fixed ring R is precisely the k-linear span of B, and $S \cong R \otimes_k K$. Since the subspace $K\{x_i\}$ of S spanned by the indeterminates x_i is G-invariant, it likewise has a K-basis $\{y_\alpha\}$ of G-invariant elements, which necessarily has the same cardinality $|I|$ as the original basis of indeterminates. Thus S is also the polynomial algebra over K in these G-invariant elements y_α, so the k-subalgebra generated by these elements will be the fixed ring R; so R is a polynomial algebra over k in $|I|$ indeterminates, as claimed.

The prime ideals P_i of S belonging to each orbit of the action of G on such ideals will contract to a common prime ideal $P_{[i]}$ of the fixed ring R, and the members of the given orbit will be the only primes contracting to $P_{[i]}$ [1, §V.2.2, Theorem 2]. It is not hard to deduce (e.g., using [1, §V.2.1, Corollary 2 to Theorem 1]) that the partial ordering of these contracted primes is that of I/G, as desired.

Since the K-subspace $K\{y_\alpha\}$ of S spanned by the y_α is the same as the K-subspace $K\{x_i\}$ spanned by the original indeterminates x_i, each prime P_i, being generated by a subset of $\{x_i\}$, is generated by a subset, equivalently, by a K-subspace, of $K\{y_\alpha\}$. □

We could have shortened the above proof slightly by skipping the choice of the basis B, simply choosing $\{y_\alpha\}$ as above and noting that $S = K[y_\alpha]$, so $R = k[y_\alpha]$; but the present proof makes it clear that a large part of the argument goes over to the case of a family of prime ideals of any commutative K-algebra that is permuted by an action of the group $\mathrm{Gal}(K/k)$ extending its action on K.

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