Combinatorial Game Theory Workshop

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A main aim of the workshop was to bring together the two camps, mathematicians working in combinatorial game theory and computer scientists interested in algorithmics and Artificial intelligence.

The Workshop attracted a mix of people from both communities (17 from mathematics, 16 from computer science and 2 undergraduates) as well as a mixture of new and established researchers. The oldest was Richard Guy, turning 90 in 2006 and the youngest was in 3rd year University. There were attendees from Europe, Asia as well as North America.

The Workshop succeeded in its primary goal and more. New collaborations were struck. There was quick dissemination and evaluation of major new results and new results were developed during the Workshop. Part of the success was due to the staff and facilities at BIRS.

The facilities at BIRS were appreciated by all the participants. The main room allowed lectures to mix computer presentations with overheads and chalkboard calculations. (No prizes for guessing which community used which technology.) The coffee lounges and break away rooms allowed discussions to continue on, in comfort, until late in the night. Our thanks go to all the staff who made the stay such a wonderful experience and to the BIRS organization for hosting the workshop.

The elder statesmen of the community, Berlekamp, Conway, Fraenkel and Guy, all took active roles in the proceedings. The first three gave survey talks on various topics and all were involved in discussions throughout the days and the evenings. The younger (established) generation were represented by the likes of Demaine, Grossman, Müller and Siegel.

As befits a workshop on combinatorial games, games were invented, played and analyzed. Philosophers Football (Phutball) was much in evidence. There was a Konane tournament played over three evenings. Much effort went into attempting an analysis of Sticky Towers of Hanoi, a game invented at the Workshop by Conway, spearheaded by Conway and the youngest attendee, Alex Fink. There were many representatives of the Go community quite a few of whom had never met each other.

Collaboration is very important in this community. For example, David Wolfe presented a progress report on work of G. A. Mesdal on a class of partizan splitting games, answering questions first raised over 30 years ago. Mesdal is a joint effort of eleven co-authors from North America and New Zealand. Eight of the eleven attended the workshop and the number of co-authors had risen to twelve by the time the Workshop ended.

In the end, between the talks and the discussions, there was simply too much to absorb in such a short time. The talks, surveys and several consequent papers are slated to appear as (tentatively titled) *Games of No Chance 3* in the MSRI book series.

All the presentations were at a high standard and all had lively discussions during and after the time allotted. Some highlights were: Conway's talk on lexicodes; Berlekamp's overview *Today's View of Combinatorial Game Theory*; and Fraenkel's *What hides beyond the curtain separating Nim from non-Nim games*; Demaine's talk on Dyson Telescopes and Moving Coin Puzzles also showed the complexity in some very new and some very old puzzles.

However the highlights were the reports by:

1. *Plambeck on a breakthrough in the analysis of impartial misèr games;*
2. *Siegel on extending the analysis of loopy games;*
3. *Friedman and Landsberg on applying renormalization techniques from physics to combinatorial games—this paper was both controversial and thought provoking and lead to the most discussions, including ones on the nature of truth and of proof;*
4. *Nakamura on the use of 'cooling by 2' to determine the winner in 'races to capture' in Go. One of the goals of the Workshop was to bring together researchers from mathematics and computer science. This was one of the talks that helped bridge the gap and engendered much discussion. The 30 minutes allotted to the talk was too short, and most participants stayed an extra hour (into the dinner-time) so as to hear the details.*

All of these were very new, very important results, produced only months before the Workshop.

1: Misère Games: On page 146 of *On Numbers and Games*, in Chapter 12, "How to Lose When You Must," John Conway writes:

> *Note that in a sense, [misére] restive games are* ambivalent Nim-heaps*, which choose their size (g_0 or g) according to their company. There are many other games which exhibit behaviour of this type, and it would be very interesting to have some general theory for them.*

Questions about the analysis of misére impartial octal games were raised in [3, 6] and no good general analytical techniques have been developed apart from finding the genus sequence [3]. (See [1, 22], see also [8, 20]). In his presentation, Plambeck provided such a general theory, cast in the language of commutative semigroups.

The misère analysis of a combinatorial game often proves to be far more difficult than its normal play version. In fact it is an open question (Plambeck) if there is a misére impartial game whose analysis is simpler than the normal play version and there is no know way of analyzing misére partizan games ([15], Problem 9).

To take a typical example, the normal play of Dawsons Chess was solved as early as 1956 by Guy and Smith [16], but even today, a complete misère analysis has not been found. Guy tells the story [15]:

"[Dawsons chess] is played on a $3 \times n$ board with white pawns on the first rank and black pawns on the third. It was posed as a losing game (last-player-losing, now called misère) so that capturing was obligatory. Fortunately, (because we still don't know how to play Misère Dawsons Chess) I assumed, as a number of writers of that time and since have done, that the misère analysis required only a trivial adjustment of the normal (last-player-winning) analysis. This arises because Bouton, in his original analysis of Nim [5], had observed that only such a trivial adjustment was necessary to cover both normal and misère play..."

But even for impartial games, in which the same options are available to both players, regardless of whose turn it is to move, Grundy & Smith [14] showed that the general situation in misère play soon gets very complicated, and Conway [6], (p. 140) confirmed that the situation can only be simplified to the microscopically small extent noticed by Grundy & Smith.

In Chapter 13 of [3], the genus theory of impartial misère disjunctive sums is extended significantly from its original presentation in chapter 7 (How to Lose When You Must) of Conways On Numbers and Games [6]. But excluding the tame games that play like Nim in misère play, theres a remarkable paucity of example games that the genus theory completely resolves. For example, the section Misère Kayles ([3], pg 411) promises, “Although several tame games arise in Kayles (see Chapter 4), wild games abounding and well need all our [genus-theoretic] resources to tackle it...” However, it turns out Kayles wasn‘t tackled at all. It was left to the amateur William L. Sibert to settle misère Kayles using completely different methods. One finds a description of his solution at end of the updated Chapter 13 in [4], and also in [22].

When normal play is in effect, every game with nimber $G^+(G) = k$ can be thought of as the nim heap k. No information about best play of the game is lost by assuming that G is in fact precisely the nim heap of size k. Moreover, in normal play, the nimber of a sum is just the nim-sum of the nimbers of the summands. In this sense every normal play impartial game position is simply a disguised version of Nim (see [3], Chapter 4, for a full discussion).

Genera. When misère play is in effect, nimbers can still be defined but many inequivalent games are assigned the same nimber, and the outcome of a sum is not determined by nimber of the summands. These unfortunate facts lead directly to the apparent great complexity of many misère analyses. Nevertheless progress can be made. The key definition, taken directly from [6], now at the bottom of page 141: In the analysis of many games, we need even more information than is provided by either of these values [G^+ and G^-], and so we shall define a more complicated symbol that we call the G^o-value or genus. This is the symbol

$$g_{\cdot} g_0 g_1 g_2 \cdots$$

where $g = G^+(G)$, $g_0 = G^-(G)$, $g_1 = G^-(G+2)$, $g_2 = G^-(G+2+2)$, ..., where, in general, g_n is the G^--value of the sum of G with n other games all equal to [the nim-heap of size] 2.

At first sight, the genus symbol looks to be an potentially infinitely long symbol in its exponent. In practice, it can be shown that the g_is always fall into an eventual period two pattern. By convention, the symbol is written down with a finite exponent with the understanding that its final two values repeat indefinitely.

Evidently the exponent of a genus symbol of a game G is closely related to the outcome of sums of G with all multiples of misère nim heaps of size two.

The genus computations are intended to illustrate the complexities of a misère analysis when the only tools available to be applied are those described in Chapter 13 of Winning Ways.

Plambeck’s breakthrough was to introduce a *quotient semigroup* structure on the set of all positions of an impartial game with fixed rules. The basic construction is the same for both normal and misère play. In normal play, it leads to the familiar Sprague-Grundy theory. In misère play, when applied to the set of all sums of positions played according to a particular game’s rules, it leads to a quotient of a free commutative semigroup by the game’s indistinguishability congruence. Playing a role similar to the one that *nim sequences* do for normal play, mappings from single-heap positions into a game’s misère quotient semigroup succinctly and necessarily encode all relevant information about its best misère play. Plambeck showed examples of wild misère games that involve an infinity of ever-more complicated canonical forms amongst their position sums that may nevertheless possess a relatively simple, even finite misère quotient. Suppose Γ is a taking and breaking game whose rules have been fixed in advance. Let h_i be a distinct, purely formal symbol for each $i \geq 1$. We will call the set $H = \{h_1, h_2, h_3, \ldots\}$ the *heap alphabet.* A particular symbol h_i will sometimes be called a *heap of size i*. The notation H_n stands for the subset $H_n = \{h_1, \ldots, h_n\} \subseteq H$ for each $n \geq 1$. Let $\mathcal{F}_H$ be the free commutative semigroup on the heap alphabet H. The semigroups $\mathcal{F}_H$ and $\mathcal{F}_{H_n}$ include an identity Λ, which is just the empty word. There’s a natural correspondence between the elements of $\mathcal{F}_H$ and the set of all position sums of a taking and breaking game Γ. In this correspondence, a finite sum of heaps of various sizes is written multiplicatively using corresponding elements of the heap alphabet H. This multiplicative notation for sums makes it convenient to take the convention that the empty position $\Lambda = 1$. It corresponds to the *endgame*—a position with no options. Fix the rules and associated *play convention* (normal or misère) of a particular taking and breaking game Γ. Let $u, v \in \mathcal{F}_H$ be game positions in Γ. We’ll say that *u is indistinguishable from v over $\mathcal{F}_H$*, and write the relation $u \; \rho \; v$, if for every element $w \in \mathcal{F}_H$, uw and vw are either both P-positions, or are both N-positions.

Lemma 1 The relation ρ is a congruence on $\mathcal{F}_H$.

Suppose the rules and play convention of a taking and breaking game Γ are fixed, and let ρ be the indistinguishability congruence on $\mathcal{F}_H$, the free commutative semigroup of all positions in Γ. The *indistinguishability quotient* $\mathcal{Q} = \mathcal{Q}(\Gamma)$ is the commutative semigroup

$$Q = \mathcal{F}_H/\rho.$$

Notice that the indistinguishability quotient can be taken with respect to either play convention (normal or misère). The details of the indistinguishability congruence then determine the structure of the indistinguishability quotient. Since the word "indistinguishability" is quite a mouthful, $\mathcal{Q}$ is called the *quotient semigroup* of Γ. When Γ is a normal play game, its quotient semigroup $\mathcal{Q} = \mathcal{Q}(\Gamma)$ is more than just a semigroup. *A re-interpretation of the Sprague-Grundy theory says that these are always groups, each isomorphic to a direct product of a (possibly infinite) set of Z_2's (cyclic groups of order two).* If u is a position in $\mathcal{F}_H$ with normal play nim-heap equivalent $*k$, the members of a particular congruence class $u\rho \in \mathcal{F}_H/\rho$ will be precisely all positions that have normal-play nim-heap equivalent $*k$. The identity of $\mathcal{Q}$ is the congruence class of all positions with nim-heap equivalent $*0$. The "group multiplication" corresponds to nim addition. For misère play, the quotient structure is a *semigroup*. Surprisingly, it's often a finite object, even for a game that has an infinite number of different canonical forms occurring amongst its sums. The elements of a particular congruence class all have the same outcome. Each class can be thought of as carrying a big stamp labelled "P" (previous player wins in best play for all positions in this class) or "N" (next player wins). In normal play, there's only one equivalence class labelled "P"—these are the positions with nim heap equivalent $*0$. In misère play, for all but the trivial games with one position $*0$, or two positions $\{*0, *1\}$, there is always more than one "P" class—one corresponding to the position $*1$, and at least one more, corresponding to the position $*2 + *2$.

At the time of the presentation, Plambeck had 20 games each of whose octal description was short but whose analysis had defied his attempts. Plambeck offered varying amounts of money for their solutions. During the Workshop, Aaron Siegel solved four of them and, in conjunction with Plambeck, has solved all of the games and produced a computer program that helps with representations of the quotient semi-groups.

2: Loopy Games. Aaron Siegel reported on two parts of the work contained in his PhD thesis, this particular presentation concerned loopy games. The traditional theory of combinatorial games assumes that no position may be repeated. This restriction guarantees that arbitrary sums of games will terminate; the result is a clean, recursive, and computationally efficient theory. However, there are many interesting games that allow repetition, including Fox and Geese, Hare and Hounds, Backsliding Toads and Frogs, Phutball and Checkers. Go is a peculiar example: the ko rule forbids most repeated positions, but local repetition is extremely important when the board must be decomposed to effect a tractable analysis.

Every game that permits repeated positions faces the possibility of nonterminating play. This is typically resolved by declaring infinite plays drawn (as in Checkers and Chess), but alternative resolutions are not uncommon. For example, Hare and Hounds declares infinite plays wins for the Hare, and some dialects of Go rules forbid them altogether. The disjunctive theory, in its most general form, assumes that in sums within finite play, the game is drawn unless the same player wins on every component in which play is nonterminating. This is vacuously true for games where infinite play is drawn to begin with, and it applies equally well to games such as Hare and Hounds. Go, with its unique ko rule, does not fit so cleanly into the theory.

The general disjunctive theory was first considered by Robert Li [17], who in the mid-1970s focused on games where it is a disadvantage to move, including a variant of Hackenbush. Shortly thereafter, Conway, together with his students Clive Bach and Simon Norton, generalized and codified the theory and coined the term loopy game. Their results, including the fundamental concepts of stoppers and sides, appeared first in a 1978 paper [6] and were reprinted in Winning Ways. At roughly the same time, Shaki [21] and Fraenkel and Tassa [13] studied approximations and reductions of partizan loopy games under a slightly different set of assumptions. Despite this flurry of initial activity, there were few advances in the two decades following the first publication of Winning

Ways. Moews generalization of sidling was a rare exception: Published in his 1993 thesis [18, 19],it constituted the first real advance in the disjunctive theory since the late 1970s.

Various authors have studied loopy games in other contexts. Generalizations of the Sprague-Grundy theory to impartial loopy games were introduced by Smith [23] a full decade before Li invented the partizan theory. They were studied in the 1970s by Fraenkel and Perl [11] and Conway[3], and much more recently by Fraenkel and Rahat [12]. James Flanigan, in his 1979 thesis and two subsequent papers [9, 10], analyzed conjunctive and selective sums of partizan loopy games.

Meanwhile, the greatest advance of the 1990s came from an entirely different quarter, the study of kos in Go. The interplay between local cycles and the global state of the position gives rise to a rich and fascinating temperature theory, which appears to differ from Conways disjunctive theory in striking ways. The theory was first realized by Berlekamp, following his analysis of loop free Go positions with his student David Wolfe (see [2]. Many others have since investigated the theory of kos, including Fraser, Müller, Nakamura, Spight and Takizawa. (See [25, 24, 27, 28], for some examples.)

Siegel showed how to calculate canonical forms of loopy games and gave some of their characteristics. One of his remarkable achievements is the software package CGSuite (for the "computationally efficient theory" of finite disjunctive sums) and then and its extension to be able to calculate the canonical form of loopy games.

Siegel, Ottaway and Nowakowski showed how rich the canonical forms of small games can be when they considered 1-dimensional Phutball played on boards of length 7, 8, 9, 10, and 11.

3: Cooling and Go. The applications of combinatorial game theory to the game of Go have, so far, been focused on endgames and eyespace values. A *capturing race* is a particular kind of life and death problem in which both of the two adjacent opposing groups are fighting to capture the opponent's group each other. Skills in winning races are very important factor to the strength of Go as well as openings and endgames techniques. In order to win the complicated capturing races, techniques of counting liberties, taking away the opponent's liberties and extending own liberties in addition to wide and deep reading are necessary. Nakamura, " "On Counting Liberties in Capturing Races of Go" showed that the 'counting' required can be regarded as combinatorial game with a score. Within this framework, he showed how to analyze capturing races that have no shared liberty or have just simple shared liberties using combinatorial game values of external liberties and an evaluation formula to find out the outcome of the capturing races. Essentially, the evaluation formula is by cooling. All applications of cooling so far have been chilling (cooling by 1) but in this case, one must cool by 2 !

4: Renormalization techniques.

Friedman & Landsberg presented a new approach to combinatorial games that unveiled connections between such games and nonlinear phenomena commonly seen in nature: scaling behaviors, complex dynamics and chaos, growth and aggregation processes. Using the game of Chomp (as well as variants of the game of Nim) as prototypes, they showed that the game possesses an underlying geometric structure that grows (reminiscent of crystal growth), and showed how this growth can be analyzed using a renormalization procedure. This approach not only obtains answers to some open questions about the game of Chomp, but opens a new line of attack for understanding (at least some) combinatorial games more generally through their underlying connection to nonlinear science.

Analysis of these two-player games has generally relied upon a few beautiful analytical results or on numerical algorithms that combine heuristics with look-ahead approaches ($\alpha - \beta$ pruning). Using Chomp as a prototype, this new geometrical approach unveils unexpected parallels between combinatorial games and key ideas from physics and dynamical systems, most notably notions of scaling, renormalization, universality, and chaotic attractors. Their central finding is that underlying the game is a probabilistic geometric structure that encodes essential information about the game, and that this structure exhibits a type of scale invariance: Loosely speaking, the geometry of small winning positions and large winning positions are the same after rescaling. (This general finding also holds for at least some other combinatorial games, as was explicitly demonstrated with a variant of Nim.) This geometric insight not only provides (probabilistic) answers to some open questions about Chomp, but suggests a natural pathway toward a new class of algorithms for more general combinatorial games, and hints at deeper links between such games and nonlinear science.

Chomp is an ideal candidate for the study, since in certain respects it appears to be among the simplest in the class of hard games. Its history is marked by some significant theoretical advances but it has yet to succumb to a complete analysis in the 30 years since its introduction by Gale and Schuh. The rules of Chomp are easily explained. Play begins with an N x M array of counters. On each turn a player selects a counter and removes it along with all counters to the north and east of it. Play alternates between the two players until one player takes the last counter, thereby losing the game. (An intriguing feature of Chomp, as shown by Gale, is that although it is very easy to prove that the player who moves first can always win, under optimal play, what this opening move should be has been an open question. The methodology provides a probabilistic answer to this question.)

For simplicity, consider the case of three-row (M=3) Chomp, a subject of recent study by Zeilberger [29] and Sun [26]. Generalizations to four-row and higher Chomp are analogous. To start, note that the configuration of the counters at any stage of the game can be described (using Zeilbergers coordinates) by the position p=[x,y,z], where x specifies the number of columns of height three, y specifies the number of columns of height two, and z the number with height one. Each position p may be classified as either a winner, if a player starting from that position can always force a win, or as a loser otherwise. The set of all losers contains the information for solving the game. One may conveniently group the losing positions according to their x values by defining a loser sheet Lx to be an infinite two-dimensional matrix whose (y,z)th component is a 1 if position [x,y,z] is a loser, and a 0 otherwise. (As noted by Zeilberger, one can express Lx in terms of all preceding loser sheets Lx-1, Lx-2, , L0.) Studies by Zeilberger [29, 30] and others have detected several numerical patterns along with a few analytical features about the losing positions, and their interesting but non-obvious properties have even led to a conjecture that Chomp may be chaotic in a yet-to-be-made-precise sense. However, many of the numerical observations to date have remained largely unexplained, and disjoint from one another.

To provide broader insight into the general structure of the game, the authors departed from the usual analytic/algebraic/algorithmic approaches. Instead showing how the analysis of the game can be recast and transformed into a type of renormalization problem commonly seen in physics (and later apply this methodology to other combinatorial games besides Chomp). Analysis of the resulting renormalization problem not only explains earlier numerical observations, but provides a unified, global description of the overall structure of the game. This approach will be distinguished by its decidedly geometric flavor, and by the incorporation of probabilistic elements into the analysis, despite the fact that the combinatorial games we consider are all games of no chance which lack any inherent probabilistic components to them whatsoever.

To proceed, consider the so-called instant-winner sheets, defined as follows: A position p=[x,y,z] is called an instant winner if from that position a player can legally move to a losing position with a smaller x- value. We therefore define an instant-winner sheet Wx to be the infinite, two- dimensional matrix consisting of all instant winners with the specified x-value, i.e., the (y,z)th component of matrix Wx is a 1 if position [x,y,z] is an instant winner, and a 0 otherwise. These instant-winner sheets will prove crucial for understanding the geometric structure of the game.

Their first insight comes from numerical simulations. They numerically construct the instant winner sheets Wx for various x values using a recursive algorithm. Each sheet exhibits a nontrivial internal structure characterized by several distinct regions: a solid (filled) triangular region at the lower left, a series of horizontal bands extending to the right (towards infinity), and two other triangular regions of different densities. Most importantly, however, we observe that the set of instant-winner sheets Wx possess a remarkable scaling property: their overall geometric shape is identical up to a scaling factor! In particular, as x increases, all boundary-line slopes, densities, and shapes of the various regions are preserved from one sheet to the next (although the actual point-by-point locations of the instant winners within each sheet are different). Hence, upon rescaling, the overall geometric structure of these sheets is identical (in a probabilistic sense). The growth (with increasing x) of the instant-winner sheets is strikingly similar to certain crystal-growth and aggregation processes found in physics in each case, the structures grow through the accumulation of new points along current boundaries, and exhibit geometric invariance during this process. The loser sheets Lx can be numerically constructed in a similar manner; their characteristic geometry is revealed. It is found to consist of three (diffuse) lines: a lower line of slope mL and density of points L, an upper line of slope mU and density U, and a flat line extending to infinity. The upper and lower lines originate from a point whose height (i.e., z-value) is ax. The flat line (with density one) is only present with probability in randomly selected loser sheets. Like the instant-winner sheets, the loser sheets also exhibit this remarkable geometric scaling property: as x increases, the geometric structure of Lx grows in size, but its overall shape remains unchanged (the only caveat being that, as previously noted, the flat line seen in is sometimes absent in some of the loser sheets).

The second key finding is that there exists a well-defined, analytical recursion operator that relates one instant winner sheet to its immediate predecessor. Namely, one can write Wx+1 = R Wx, where R denotes the recursion operator. (The operator R can be decomposed as R=L(I+DM), where L is a left-shift operator, I is the identity operator, D is a diagonal element-adding operator, and M is a sheet-valued version of the standard mex operator which is often used for combinatorial games.) They point out that once a given instant-winner sheet Wx has been constructed, the corresponding loser sheer Lx can be found via Lx = M Wx.

The task is to determine an invariant geometric structure W such that if we act with the recursion operator followed by an appropriately-defined rescaling operator S, we get W back again: W = SR W (i.e., find a fixed point of the renormalization-group operator SR.) This can be done, but before doing so, even though the recursion operator R is exact and the game itself has absolutely no stochastic aspects to it, it is necessary to adopt a probabilistic framework in order to solve this recursion relation. Namely, the renormalization procedure will show that the slopes of all boundary lines and densities of all regions in the Wxs (and Lxs) are preserved not that there exists a point-by-point equivalence. In essence, bypassing consideration of the random-looking scatter of points surrounding the various lines and regions of Wx and Lx by effectively averaging over these fluctuations.

The key to implementing the renormalization analysis is to observe that the losers in Lx are constrained to lie along certain boundary lines of the Wx plot, and are conspicuously absent from the various interior regions of Wx (for all x). In other words, the interior regions of each Wx remain forbidden to the losers. Hence the geometry of Wxs must be very tightly constrained if it is to preserve these symmetries.

References

[1] D. T. Allemang. Generalized genus sequences for mis‘Internat. J. Game Theory, ere octal games. 30(4):539556 (2002), 2001.

[2] Elwyn Berlekamp and David Wolfe. Mathematical Go: Chilling Gets the Last Point. A K Peters, Ltd., Wellesley, Massachusetts, 1994.

[3] Elwyn R. Berlekamp, John H. Conway, and Richard K. Guy. Winning ways for your mathematical plays. Vol. 1. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], London, 1982. Games in general.

[4] Elwyn R. Berlekamp, John H. Conway, and Richard K. Guy. Winning ways for your mathematical plays. Vol. 1. A K Peters Ltd., Natick, MA, second edition, 2001.

[5] C. L. Bouton. Nim, a game with a complete mathematical theory. Annals of Mathematics, 3(2):3539, 1902.

[6] J. H. Conway. On numbers and games. Academic Press [Harcourt Brace Jovanovich Publishers], London, 1976. London Mathematical Society Monographs, No. 6.

[7] J. H. Conway. Loopy games. Ann. Discrete Math., 3:5574, 1978. Advances in graph theory (Cambridge Combinatorial Conf., Trinity College, Cambridge, 1977).

[8] Thomas S. Ferguson. Misère annihilation games. J. Combin. Theory Ser. A, 37(3):205230, 1984.

[9] J. A. Flanigan. Selective sums of loopy partizan graph games. Internat. J. Game Theory, 10(1):110, 1981.

[10] James A. Flanigan. Slow joins of loopy games. J. Combin. Theory Ser. A, 34(1):4659, 1983.

[11] A. S. Fraenkel and Y. Perl. Constructions in combinatorial games with cycles. In Infinite and finite sets (Colloq., Keszthely, 1973; dedicated to P. Erd. os on his 60th birthday), Vol. II, pages 667699. Colloq. Math. Soc. Janos Bolyai, Vol. 10. North-Holland, Amsterdam, 1975.

[12] Aviezri S. Fraenkel and Ofer Rahat. Infinite cyclic impartial games. 1999.

[13] Aviezri S. Fraenkel and Uzi Tassa. Strategies for compounds of partizan games. Math. Proc. Cambridge Philos. Soc., 92(2):193204, 1982.

[14] P. M. Grundy and C. A. B. Smith. Disjunctive games with the last player losing. Proc. Cambridge Philos. Soc., 52:527533, 1956.

[15] Richard Guy. Unsolved problems in combinatorial games. American Mathematical Society Proceedings of the Symposium on Applied Mathematics, 43, 1991.

[16] Richard K. Guy and Cedric A. B. Smith. The G-values of various games. Proceedings of the Cambridge Philosophical Society, 52:514526, 1956.

[17] S.-Y. R. Li. Generalized impartial games. Internat. J. Game Theory, 3:169184, 1974.

[18] David Moews. On Some Combinatorial Games Related to Go. PhD thesis, University of California, Berkeley, 1993.

[19] David Moews. Mathematical games coin-sliding and go. Theoretical Computer Science, 164:253276, September 1996.

[20] Thane E. Plambeck. Daisies, Kayles, and the Sibert-Conway decomposition in mis'ere octal games. Theoret. Comput. Sci., 96(2):361388, 1992.

[21] Ahiezer S. Shaki. Algebraic solutions of partizan games with cycles. Math. Proc. Cambridge Philos. Soc., 85(2):227246, 1979.

[22] W. L. Sibert and J. H. Conway. Mathematical Kayles. Internat. J. Game Theory, 20(3):237246, 1992.

[23] Cedric A. B. Smith. Graphs and composite games. J. Combinatorial Theory, 1:5181, 1966.

[24] W. L. Spight. Extended thermography for multiple kos in Go. Theoret. Comput. Sci., 252(1-2):2343, 2001. Computer and games (Tsukuba, 1998).

[25] W. L. Spight. Go thermography: the 4/21/98 Jiang-Rui endgame. In More games of no chance (Berkeley, CA, 2000), volume 42 of Math. Sci. Res. Inst. Publ., pages 89105. Cambridge Univ. Press, Cambridge, 2002.

[26] Xinyu Sun. Improvements on Chomp. Integers, 2:G1, 8 pp. (electronic), 2002.

[27] Takenobu Takizawa. An application of mathematical game theory to Go endgames: some width-two entrance rooms with and without kos. In More games of no chance (Berkeley, CA, 2000), volume 42 of Math. Sci. Res. Inst. Publ., pages 107124. Cambridge Univ. Press, Cambridge, 2002.

[28] Takizawa Takenobu. Mathematical game theory and its applications to go endgames. (Unknown where published. In Japanese.), 1999.

[29] Doron Zeilberger. Three-rowed Chomp. Adv. in Appl. Math., 26(2):168179, 2001.

[30] Doron Zeilberger. Chomp, recurrences and chaos. J. Difference Equations and its Applications 10:1281-1293, 2004.