RESONANCE EXPANSIONS OF SCATTERED WAVES

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1. Introduction

The purpose of this paper is to describe expansions of solutions to the wave equation on $\mathbb{R}^n$ with a compactly supported perturbation present. We show that under a separation condition on resonances, the solutions can be expanded in terms of resonances close to the real axis, modulo an error rapidly decaying in time. To avoid the discussion of particular aspects of potential, gravitational or obstacle scattering, the results are stated using the abstract "black box" formalism of Sjöstrand and the second author [20].

When M is a compact Riemannian manifold, and Δ, its Laplacian, then we have a generalized Fourier expansion of the wave group:

$$\frac{\sin t\sqrt{\Delta}}{\sqrt{\Delta}} f(x) = \sum_{\lambda_j^2 \in \operatorname{Spec}(\Delta)} e^{i\lambda_j t} w_j(x) \,, \quad \Delta w_j = \lambda_j^2 w_j \,, \tag{1.1}$$

and the convergence is absolute in the case of smooth data.

The simplest case of a non-compact spectral problem is given by taking $\mathbb{R}^n$ with the usual Laplacian outside a compact set – we can for instance "glue" any compact Riemannian manifold to $\mathbb{R}^n$ or consider the obstacle problem with the Dirichlet Laplacian on $\mathbb{R}^n \setminus \mathcal{O}$. Since *resonances* or *scattering poles* constitute a natural replacement of discrete spectral data for problems on exterior domains, we expect a similar expansion involving them in place of eigenvalues – this point of view was emphasized early by Lax-Phillips [10] (see [19],[33] and [35] for overviews of recent results).

The resonances are defined as poles of the meromorphic continuation of the resolvent or of the scattering matrix but despite the stationary nature of these definitions, they are fundamentally a dynamical concept: the real part of a resonance describes the rest energy of a state and the imaginary part its rate of decay. Consequently they should be understood in terms of long time behaviour of solutions to evolution equations, and in particular, to the wave equation. For the Schrödinger evolution equation we refer to the recent paper by Soffer-Weinstein [23] and references given there; in that case one considers resonances which come from perturbation of embedded eigenvalues.

Under a *quantum* dynamical condition that the obstacle $\mathcal{O}$ is non-trapping (that is a condition on the behaviour of solutions of the evolution equation – see Sect.3), Lax-Phillips [10] and

Vainberg [27] showed that for n odd

$$
\left.\begin{aligned}
&(\partial_t^2 - \Delta) u(t,x) = 0\,, \quad x \in \mathbb{R}^n \setminus \mathcal{O}\,, \quad u|_{\mathbb{R}\times\partial\mathcal{O}} = 0\,,\\
&u|_{t=0} = f \in \mathcal{C}^\infty_{\rm c}(\mathbb{R}^n \setminus \overline{\mathcal{O}})\\
&\tfrac{1}{i}\partial_t u|_{t=0} = g \in \mathcal{C}^\infty_{\rm c}(\mathbb{R}^n \setminus \overline{\mathcal{O}})
\end{aligned}\right\} \Longrightarrow \tag{1.2}
$$

$$
u(t,x) = \sum_{\operatorname{Im}\lambda_l \leq C} \sum_{j=1}^{m_{\mathcal{O}}(\lambda_l)} w_{\lambda_l,j}(x) e^{it\lambda_l} t^{j-1} + \mathcal{O}(e^{-(C-\epsilon)t})\,, \ x \in K\,,
$$

$K \subset \mathbb{R}^n \setminus \mathcal{O}$ compact, where $m_{\mathcal{O}}(\lambda_l)$ is the multiplicity of the resonance. Here we took the convention that $\operatorname{Im}\lambda_l \geq 0$, that is that the resonances lie in the upper half-plane. That the quantum non-trapping condition follows from the *classical* non-trapping condition was shown in the works of Andersson, Melrose, Morawetz, Ralston, Strauss, Sjöstrand and Taylor – see the appendix to the second edition of [10] and references given there. For completeness and motivation we review Vainberg's argument in the generality of "black box" scattering, in Sect.3.

A comparison between (1.2) and the ultimately trapping case of a compact manifold (1.1), suggests that an expansion should be valid for all perturbations. That however is far from clear.

We present here a conditional result for strongly trapping perturbations (for which there are resonances converging to the real axis – see (4.1)). Under a generically reasonable but unverifiable separation assumption for resonances close to the real axis (4.2), we show that (1.2) remains valid when we sum over resonances satisfying $\operatorname{Im}\lambda_j \leq |\lambda_j|^{-K}$, K sufficiently large, and when we replace the error by $\mathcal{O}(t^{-N})$ for any N – see Theorem 2 in Sect.4.

As far as we know the only unconditional results in trapping situations are discussed in [6]. Two hyperbolic examples: the modular surface and the hyperbolic cylinder are shown to admit expansion of waves in terms of resonances. However, even there, the strongest expected version of the expansions is not available as it would require too much knowledge about the Riemann zeta function (simplicity of zeros and lower bounds on derivatives). We consider the difficulty present already in that extremely well understood case as a justification for presenting a conditional result.

Finally, and to motivate expansions of the type (1.2) further, we mention the relation to more applied aspects of scattering theory. In some situations it seems plausible to measure the signal at different times. A fitting algorithm would then allow finding the resonances λ_j which would provide new useful invariants, especially when radar or sound pulse techniques are ineffective. That this could work is shown by computations of resonances based on solving the wave equation – see [11].

In physical chemistry, it is understood that the *transition state theory* rate of decay is often lower than the decay rate given by the resonance width of a individual resonance. This is related to the understanding of solutions of evolution equations in terms many overlapping resonance states [17]. Expansions given in Theorem 2 below are a step in that direction: the error term is larger than a contribution from any individual resonance as $t \to \infty$. Yet, due to the density of resonances close to the real axis, and on appropriate time scales, the leading term is believed to dominate.

In the paper, C will stand for a constant whose value may change from line to line. The notation $\langle\lambda\rangle$ means $(1 + |\lambda|^2)^{\frac{1}{2}}$.

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2. Preliminaries

We start by recalling the framework of "black box" scattering from [20]. Thus we consider a complex Hilbert space $\mathcal{H}$ with an orthogonal decomposition

$$\mathcal{H} = \mathcal{H}_{R_0} \oplus L^2(\mathbb{R}^n \setminus B(0, R_0))$$

where $R_0 > 0$ is fixed and $B(x, R) = \{y \in \mathbb{R}^n : |x - y| < R\}$. We assume that P is a self-adjoint operator, $P : \mathcal{H} \longrightarrow \mathcal{H}$, with domain $\mathcal{D} \subset \mathcal{H}$, satisfying the following conditions:

$$\begin{gathered} 1\!\!1_{\mathbb{R}^n\setminus B(0,R_0)} \mathcal{D} = H^2(\mathbb{R}^n \setminus B(0, R_0)) \\ 1\!\!1_{\mathbb{R}^n\setminus B(0,R_0)} P = -\Delta|_{\mathbb{R}^n\setminus B(0,R_0)} \\ (P + i)^{-1} \text{ is compact} \\ P \geq -C, \ C \geq 0 \,. \end{gathered}$$

Under the above conditions, it known that the resolvent $R(\lambda) = (P - \lambda^2)^{-1} : \mathcal{H} \longrightarrow \mathcal{D}$ meromorphically continues from $\{\lambda : \mathrm{Im}\lambda < 0, \lambda^2 \notin \sigma(P)\}$ to the whole complex plane $\mathbb{C}$ when n is odd or to the logarithmic plane Λ when n is even, as an operator from $\mathcal{H}_{\mathrm{comp}}$ to $\mathcal{D}_{\mathrm{loc}}$ with poles of finite rank.

The poles are called *resonances* of P. We will denote the set of resonances by $\mathrm{Res}(P)$ and will always include them according to their multiplicity, $m(\lambda)$. In order to gurantee a polynomial bound on the counting function of resonances, we need to add an additional condition on P.

To formulate it, we use P to construct a self-adjoint reference operator $P^\sharp$ on

$$\mathcal{H}^\sharp = \mathcal{H}_{R_0} \oplus L^2(M \setminus B(0, R_0))$$

as in [20] where $M = (\mathbb{R}^n/R\mathbb{Z})^n$ for some $R > R_0$. Let $N(P, I)$ denote the number of eigenvalues of $P^\sharp$ in the interval I, the assumption we need is then

$$N(P^\sharp, [-C, \lambda]) = \mathcal{O}(\lambda^{n^\sharp/2}), \ \text{for } \lambda \geq 1$$

for some number $n^\sharp \geq n$. As was shown in [20],[29],[30] and in a yet greater generality in [19], this implies that

$$N(r) = \sum_{\substack{\lambda \in \mathrm{Res}(P) \\ |\lambda| \leq r, \arg \lambda < \theta}} m(\lambda) \leq C_\theta r^{n^\sharp} \,. \tag{2.1}$$

In addition we have a polynomial bound for the logarithm on the norm of the cutoff resolvent in the whole complex plane away from the resonance set in odd dimensions and in neighbourhoods of the real axis in even dimensions: for any $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, $\chi = 1$ near $B(0, R_0)$,

$$\|\chi R(\lambda) \chi\|_{\mathcal{H} \to \mathcal{H}} \leq C_\theta e^{C_\theta |\lambda|^{n^\sharp + \epsilon}}, \quad \lambda \in \{\lambda \ : \ \arg \lambda < \theta\} \setminus \bigcup_{\lambda_j \in \mathrm{Res}(P)} D(\lambda_j, \langle \lambda_j \rangle^{-n^\sharp - \epsilon}) \,. \tag{2.2}$$

see Proposition 4.1 below, [31], and references given there.

The wave group of a black box perturbation can be defined abstractly as in Proposition 2.1 of [22]:

$$\mathcal{U}(t) = \exp it \begin{pmatrix} 0 & I \\ P & 0 \end{pmatrix} \,. \tag{2.3}$$

The entries of the matrix representation of $\mathcal{U}(t)$ are

$$\mathcal{U}(t) = \begin{pmatrix} D_t U(t) & U(t) \\ D_t^2 U(t) & D_t U(t) \end{pmatrix},$$

and the strongly continuous family of operators $U(t) : \mathcal{D}^\alpha \to \mathcal{D}^{\alpha+1}$, can be identifed as the solution operator of the following initial value problem:

$$\left\{ \begin{array}{rcll} (D_t^2 - P)U(t)g & = & 0 & \text{for } t \in \mathbb{R} \\ U(0)g & = & 0 & \\ D_t U(0)g & = & g & \end{array} \right. \tag{2.4}$$

where $g \in \mathcal{H}$. In other words,

$$U(t) = i\frac{\sin t\sqrt{P}}{\sqrt{P}}.$$

Since $1\!\!1_{\mathbb{R}^n \setminus B(0,R_0)} U(t) 1\!\!1_{\mathbb{R}^n \setminus B(0,R_0)} : \mathcal{C}^\infty(\mathbb{R}^n) \to \mathcal{C}^\infty(\mathbb{R}^n)$ we can describe it by its Schwartz kernel:

$$1\!\!1_{\mathbb{R}^n \setminus B(0,R_0)} U(t) 1\!\!1_{\mathbb{R}^n \setminus B(0,R_0)} = U(t,x,y) \in \mathcal{D}'(\mathbb{R} \times (\mathbb{R}^n \setminus B(0,R_0)) \times (\mathbb{R}^n \setminus B(0,R_0))) . \tag{2.5}$$

Here, as usual, $\mathcal{D}^\alpha = (P+i)^{-\alpha}\mathcal{H}$ and $\mathcal{D}^\infty = \cap_{\alpha \geq 0} \mathcal{D}^\alpha$

3. The nontrapping case

The *quantum non-trapping condition* in the black box setting takes the following form. Let $\chi \in C_0^\infty(\mathbb{R}^n)$ be identically one near $B(0,R_0)$.

$$\begin{gathered} \forall\, a > R_0\,,\ |y| \leq a \ \ \exists\, C_a > 0 \\ U(t,x,y) \in C^\infty \text{ if } |x| < t - C_a, |y| > R_0 \\ \chi U(t)\chi : \mathcal{H} \to \mathcal{D}^\infty \end{gathered} \tag{3.1}$$

This abstract definition of nontrapping condition is motivated by the validity of the generalized Huyghen's Principle (or of the results on the propagation of singularities) in the case when the scatterer is "classically nontrapping" – see the [10, Appendix],[35] and references given there.

Our aim is to write down an asymptotic expansion of the solution of the wave equation in terms of resonances whe the nontrapping assumption holds and when the dimension is odd. This is a result due to Lax and Phillips and it fits naturally in their framework of scattering theory based on the Lax-Phillips semigroup. In the even dimensional case, we can no longer describe the asymptotic behaviour of the solution of the wave equation in terms of resonances, with exponentially small error in time, as in the odd dimensional case, because that is not true even in the free case. Instead, one can obtain a polynomial decay (t^{1-n} as in the free case) of the local energy of the solution of the wave group as t goes to infinity – see the work of Melrose [12].

As stated in the introduction we will follow a method due to Vainberg [27], [28]. We note that in more particular cases the method was very clearly explained in [18] and [4]. The main step in the argument is an estimate of the cutoff resolvent under some logarithmic curve near the real axis. We state it in the following

Proposition 3.1. *Assume that the nontrapping condition* (3.1) *holds. Then for any $N > 0$, we have, for some constants C and T,*

$$\|\chi R(\lambda)\chi\|_{\mathcal{H}\to\mathcal{H}} \leq C|\lambda|^{-1} e^{T|\operatorname{Im}\lambda|}$$

for $|\operatorname{Im}\lambda| < N\log|\lambda|$ and $|\operatorname{Re}\lambda| > C(N)$ for some $C(N)$ depending on N.

Remarks.
1. The above estimate implies that the cutoff resolvent is holomorphic in the region under the logarithmic curve $\{\lambda : \operatorname{Im}\lambda < N\log|\lambda|, |\operatorname{Re}\lambda| > C(N)\}$
2. The power $|\lambda|^{-1}$ appearing in the theorem, is optimal, as this is the power in the estimate for the cutoff free resolvent.
3. The theorem implies the estimate

$$\|\chi R(\lambda)\chi\|_{\mathcal{D}^1\to\mathcal{H}} \le C|\lambda|^{-2}e^{T|\operatorname{Im}\lambda|} \text{ for } \operatorname{Im}\lambda < N\log|\lambda|, |\operatorname{Re}\lambda| > C(N). \tag{3.2}$$

Indeed, we have,

$$\begin{aligned}\chi^2 I &= \chi R(\lambda)(P^2-\lambda^2)\chi \\ &= \chi R(\lambda)\chi P + \chi R(\lambda)[-\Delta,\chi] - \lambda^2\chi R(\lambda)\chi\end{aligned}$$

Thus, for $\tilde\chi \in C_0^\infty$ such that $\tilde\chi \equiv 1$ on supp χ, we get

$$\begin{aligned}\|\chi R(\lambda)\chi\|_{\mathcal{D}^1\to\mathcal{H}} &\le |\lambda|^{-2}(\|\chi R(\lambda)\chi\|_{\mathcal{H}\to\mathcal{H}}\|P\|_{\mathcal{D}\to\mathcal{H}} \\ &\quad +\|\chi R(\lambda)\tilde\chi\|_{\mathcal{H}\to\mathcal{H}}\|[-\Delta,\chi]\|_{\mathcal{D}^1\to\mathcal{H}} + \|\chi^2 I\|_{\mathcal{D}^1\to\mathcal{H}}) \\ &\le C|\lambda|^{-2}e^{T|\operatorname{Im}\lambda|},\end{aligned}$$

for $\operatorname{Im}\lambda < N\log|\lambda|$, $|\operatorname{Re}\lambda| > C(N)$, which proves the claim.
We will need the following

Lemma 3.1. *Suppose that H_1 and H_2 are Hilbert spaces, and that $N(t) : H_1 \to H_2$ is a family of bounded operators that depend smoothly on t when $t \in \mathbb{R}$, depend analytically on t when $\operatorname{Re} t > T > 0$ and are equal to zero when $t < 0$. Suppose that there are constants j_0 and C_j such that for all $j \ge 0$, we have*

$$\left\|\frac{\partial^j}{\partial t^j}N(t)\right\| \le C_j|t|^{j_0-j}, \text{ for } \operatorname{Re} t > T.$$

Then the operator

$$\widehat{N}(\lambda) := \mathcal{F}_{t\to\lambda}N(t) : H_1 \to H_2,\ \operatorname{Im}\lambda < 0$$

can be continued analytically to the domain $-\frac{3\pi}{2} < \arg\lambda < \frac{\pi}{2}$ and when $|\lambda| \ge 1$, it satisfies the estimate

$$\|\widehat{N}\| \le C_j|\lambda|^{-j}e^{T|\lambda|},\ j \ge 0.$$

Proof. This follows from a straightforward contour deformation and integration by parts – see [28], Lemma 4 on p.346. □

Proof of Proposition 3.1: Fix a as in the nontrapping condition and let $\zeta_a = \zeta_a(t,x) \in C^\infty(\mathbb{R}^+ \times \mathbb{R}^n)$ be a cut-off function satisfying $0 \le \zeta_a(t,x) \le 1$, with $\zeta_a(t,x)$ not depending on x when $|x| \le R_0$ and

$$\zeta_a(t,x) = \begin{cases} 1 \text{ when } t \le |x| + C_a \\ 0 \text{ when } t \ge |x| + C'_a \end{cases}$$

for some $C'_a > C_a$.
Then, the nontrapping condition implies that for $|y| \le a$, we have

$$(1-\zeta_a)U(t)\chi_a : \mathcal{H} \to \mathcal{D}^\infty$$

where $\chi_a \in C_0^\infty(\mathbb{R}^n)$ is a cutoff function such that $\chi_a \equiv 1$ near $B(0,R_0)$ and supp $\chi_a \subset B(0,a)$.

Now, for any $g \in \mathcal{H}$, $\zeta_a U(t)\chi_a g$ satisfies

$$\left\{ \begin{array}{rcl} (D_t^2 - P)(\zeta_a U(t)\chi_a g) & = & -(D_t^2 - P)((1-\zeta_a)U(t)\chi_a g) \text{ for } t > 0 \\ U(0)\chi_a g & = & 0 \\ D_t U(0)\chi_a g & = & \chi_a g \end{array} \right. \tag{3.3}$$

We define

$$F_a(t)g \stackrel{\text{def}}{=} -(D_t^2 - P)((1-\zeta_a)U(t)\chi_a g) \,,$$

and we note that because of the nontrapping condition, $F_a(t)g \in C^\infty(\mathbb{R}_t, \mathcal{D}^\infty)$. Also, because of the property of the cutoff function ζ_a, $F_a(t)g \equiv 0$, for $t < C_a$, and it has a compact support in t for each x with $|x| > R_0$. The size of the support is, however, not uniform in x.

We let

$$\tilde{R}_a(\lambda) = -i\mathcal{F}_{t\to\lambda}\left(\zeta_a t_+^0 U(t)\chi_a\right) .$$

where t_+^0 denotes the Heaviside function. Then, we have

$$\begin{array}{rcl} (P-\lambda^2)\tilde{R}_a(\lambda)g & = & i\mathcal{F}_{t\to\lambda}\left((D_t^2 - P)(\zeta_a t_+^0 U(t)\chi_a g)\right) \\ & = & \chi_a g + i\widehat{F_a(\bullet)}g(\lambda) \end{array} \tag{3.4}$$

Let us write $F_a(t)g$ as $\chi_b F_a(t)g + (1-\chi_b)F_a(t)g$ where $\chi_b \in C_0^\infty(\mathbb{R}^n)$ is identically 1 near $B(0,R_0)$ and satisfies $\chi_a\chi_b = \chi_b$. By solving the free wave equation, we can find $V_a(t)g$ satisfying the following problem

$$\left\{ \begin{array}{c} (D_t^2 + \Delta)V_a(t)g = (1-\chi_b)F_a(t)g \text{ on } \mathbb{R}\times\mathbb{R}^n \\ V_a(0)g = D_t V_a(0)g = 0 \end{array} \right.$$

We can now write

$$\begin{array}{rcl} (1-\chi_b)F_a(t)g & = & (D_t^2+\Delta)(\chi_c V_a(t)g) + (D_t^2+\Delta)((1-\chi_c)V_a(t)g) \\ & = & \chi_c(1-\chi_b)F_a(t)g + [\Delta,\chi_c]V_a(t)g + (D_t^2 - P)((1-\chi_c)V_a(t)g) \end{array}$$

where $\chi_c \in C_0^\infty(\mathbb{R}^n)$ is identically 1 near $B(0,R_0)$ and satisfies $\chi_c\chi_b = \chi_c$. Let

$$R_a^\sharp(\lambda) = \tilde{R}_a(\lambda) - \mathcal{F}_{t\to\lambda}\left((1-\chi_c)V_a(t)\right) .$$

We can then rewrite (3.4) as follows

$$\begin{array}{rcl} (P-\lambda^2)R_a^\sharp(\lambda)g & = & \chi_a g + i\mathcal{F}_{t\to\lambda}\left(\chi_b F_a(t)g + \chi_c(1-\chi_b)F_a(t)g + [\Delta,\chi_c]V_a(t)g\right) \\ & = & \chi_a\left(I + i\mathcal{F}_{t\to\lambda}(\chi_b F_a(t) + \chi_c(1-\chi_b)F_a(t) + [\Delta,\chi_c]V_a(t))\right)g \end{array}$$

where the last line we used the support properties of the cutoff functions.

Applying $R(\lambda)$ to both sides of the equation, we get

$$R_a^\sharp(\lambda)g = R(\lambda)\chi_a\left(I + i\mathcal{F}_{t\to\lambda}(\chi_b F_a(t) + \chi_c(1-\chi_b)F_a(t) + [\Delta,\chi_c]V_a(t))\right)g$$

We will see below that the operator

$$I + i\mathcal{F}_{t\to\lambda}(\chi_b F_a(t) + \chi_c(1-\chi_b)F_a(t) + [\Delta,\chi_c]V_a(t))$$

is a small perturbation of I, when λ is large and lies under a logarithmic curve. Thus, we can invert it and obtain an expression for $R(\lambda)\chi_a$ in such a range of λ's. The estimate of the operator norm of $\chi_a R(\lambda)\chi_a$ will then follow an estimate for $R_a^\sharp(\lambda)$.

Now let us state the more precise estimates that will justify the above claims.

$$\|\chi\tilde{R}_a(\lambda)\|_{\mathcal{H}\to\mathcal{D}^j} \leq C|\lambda|^{2j-1}e^{T|\operatorname{Im}\lambda|} \text{ for } j=0,1/2 \tag{3.5}$$

$$\|\widehat{\chi F_a(\bullet)}\|_{\mathcal{H}\to\mathcal{H}} \leq C_j|\lambda|^{-j}e^{T|\operatorname{Im}\lambda|} \text{ for } j\geq 0 \tag{3.6}$$

$$\|\chi\widehat{(1-\chi_c)V_a(\bullet)}\|_{\mathcal{H}\to\mathcal{D}^j} \leq C|\lambda|^{2j-1}e^{T|\operatorname{Im}\lambda|} \text{ for } j=0,1/2 \tag{3.7}$$

$$\|\widehat{[\Delta,\chi_c]V_a(\bullet)}\|_{\mathcal{H}\to\mathcal{H}} \leq C|\lambda|^{-j}e^{T|\operatorname{Im}\lambda|} \text{ for } j\geq 0. \tag{3.8}$$

where χ above is, as usual, a smooth function with compact support on $\mathbb{R}^n$ which is identically 1 near $B(0,R_0)$, and T is some large constant.

The first two estimates can be easily obtained by writing out the Fourier transform: for the estimate (3.5) we make use of the energy estimate for the wave group, and for the estimate (3.6), of the smoothness of $F_a(t)$, and of the fact that $\chi F_a(t)$ is uniformly compactly supported in t .

It remains to show the estimates (3.7) and (3.8). They will follow from the properties of the fundamental solution of the free wave group and Lemma 3.1.

To this end, rewrite V_a as

$$V_a(t)g = (1-\chi_b)\zeta_a t_+^0 U(t)\chi_a g - t_+^0 U_0(t)(1-\chi_b)\chi_a g + q(t,x)\,, \tag{3.9}$$

where $U_0(t)$ is the free group, that is, it solves the same problem as $U(t)$, with P replaced by $-\Delta$. We can then check that $q(t,x)$ solves the following problem

$$\begin{cases} (D_t^2+\Delta)q(t,x) &= [\Delta,\chi_b]\zeta_a t_+^0 U(t)\chi_a g \text{ in } \mathbb{R}\times\mathbb{R}^n \\ q(0,x) &= 0 \text{ in } \mathbb{R}^n \\ D_t q(0,x) &= 0 \text{ in } \mathbb{R}^n \end{cases}$$

Note that the forcing term $[\Delta,\chi_b]\zeta_a t_+^0 U(t)\chi_a g$ has compact support in both space and time variables and that it is identically 0 for $t<0$. The second fact implies of course that $q(t,x)$ is also identically 0 for $t<0$.

The function $\chi(x)q(t,x)$ can be written down explicitly in terms of the forward fundamental solution $E_+(t,x)$ of the wave operator as

$$\chi(x)q(t,x) = \int_0^t \chi E_+(t-s) * [\Delta,\chi_b]\zeta_a U(s)\chi_a g ds$$

When n is odd, $E_+(t,x)$ is supported on the forward light cone over the origin. Since we have compact support in x, we see that for t sufficiently large, the forcing term will never intersect the backward lightcone over (t,x). Hence $\chi q(t,x)=0$ when t is sufficiently large.

When n is even, $E_+(t,x)$ does not vanish inside the forward lightcone, and we can no longer conclude that $\chi q(t,x)=0$ for sufficiently large t. However, since the forcing term does not intersect the singular support of the fundamental solution when t is sufficiently large. This allows us to write, for large t

$$\chi(x)q(t,x) = c_n \int_0^\infty \int_{\mathbb{R}^n} \left((t-s)^2+|x-y|^2\right)^{\frac{1-n}{2}} ([\Delta,\chi_b]\zeta_a U(s)\chi_a g)(y)dyds.$$

Clearly, this implies that $\chi(x)q(t,x)$ is analytic for $\operatorname{Re} t$ sufficiently large and that it satisfies the estimate

$$\|D_t^j(\chi(x)q(t,x))\|_{\mathcal{D}^{\frac{1}{2}}} \leq Ct^{\frac{1-n}{2}-j}\|g\|_{\mathcal{H}} \text{ for } j\geq 0.$$

Similarly, we can prove that the same result holds for the free wave group $U_0(t)$:

$$U_0(t)(1-\chi_b)\chi_a g = E_+(t) * (1-\chi_b)\chi_a g\,.$$

To see the estimate (3.8), we start by observing that the first term in (3.9) is eliminated as $[\Delta, \chi_c](1-\chi_b) \equiv 0$. We then combine the estimates on $\chi q(x,t)$ and $U_0(t)$ for large values of t with the known global smoothess of $V_a(t)$, and apply Lemma 3.1. For (3.7) we note that

$$\|\chi(1-\chi_c)(1-\chi_b)\zeta_a t_+^0 U(t)\chi_a\|_{\mathcal{H}\to\mathcal{D}^j} \leq C|\lambda|^{2j-1} e^{T|\operatorname{Im}\lambda|} \text{ for } j = 0, 1/2\,.$$

To complete the proof, we only have to check that for any $N > 0$ and

$$\lambda \in \{\lambda \in \mathbb{C} : |\operatorname{Im}\lambda| \leq N \log|\lambda|\},$$

there exists $C(N) > 0$ such that for $|\operatorname{Re}\lambda| > C(N)$, we have

$$C_j |\lambda|^{-j} e^{T|\operatorname{Im}\lambda|} < 1/2.$$

This completes the proof of Proposition 3.1. □

We can now use the bound of the cutoff resolvent to desribe the the asymptotic behaviour of the solution of the wave equation in terms of resonances. The relation comes from the following standard formula:

$$\chi U(t)\chi g = \frac{i}{2\pi}\int_{-\infty-i\alpha}^{\infty-i\alpha} e^{it\lambda}\chi R(\lambda)\chi g\, d\lambda\,, \quad t > 0\,, \quad g \in \mathcal{D}^1 \tag{3.10}$$

for some positive constant α such that $R(\lambda)$ is holomorphic in the half plane $\{\lambda \in \mathbb{C} : \operatorname{Im}\lambda \leq \alpha\}$. Such α exists as P is bounded from below. Note that the above integral converges absolutely in $\mathcal{H}$ in view of the bound of $\chi R(\lambda)\chi$ given in (3.2). As mentioned above, an expansion is expected in odd dimensions only:

Theorem 1. *Assume the nontrapping condition* (3.1) *and let* $\chi \in C_0^\infty(\mathbb{R}^n)$ *be identically* 1 *in a neighbourhood of* $B(0, R_0)$*. Then, for any* $A > 0$ *and small* $\epsilon > 0$*, we have for* $t > 0$ *sufficiently large, and* $g \in \mathcal{D}^1$

$$\chi U(t)\chi g = \sum_{\substack{\lambda_j \in \operatorname{Res}(P)\\ \operatorname{Im}\lambda < A}} \sum_{m=0}^{M_j} e^{it\lambda_j} t^m w_{j,m} + E_A(t)g, \;\; g \in \mathcal{D}^1$$

where $w_{j,m} \in \mathcal{D}^\infty$ *are the cut-off resonance states corresponding to* λ_j*, and* $E_A(t)g$ *is the error term satisfying the following estimate*

$$\|E_A(t)\|_{\mathcal{D}^1\to\mathcal{H}} \leq C(\epsilon)e^{-(A-\epsilon)t}$$

Proof. First of all, note that in the odd dimensional case, the cutoff resolvent has a meromorphic continuation to the complex plane. The theorem can then be proved by deforming the contour of integration in (3.10) from the line $\operatorname{Im} = -\alpha$ in the lower half plane to the line $\operatorname{Im} = A - \delta$ in the upper half plane where δ is a small positive number chosen to make sure that the line is bounded away from the resonance set. Since δ will finally be absorbed by ϵ in the statement of the theorem, we will simply assume that the line $\operatorname{Im} = A$ is already bounded away from the resonance set. Now such a deformation of the contour of integration is allowed because of the bound of cutoff resolvent proved in Proposition 3.1 which also gives the stated bound of the error term. In the following, we will only use the estimate of Proposition 3.1 with $N = 1$.

More precisely, let us fix $M > 0$, and consider the contour Γ given by the following four line segments running in the counterclockwise direction: $\Gamma_1 = \{x - i\alpha : -M < x < M\}$, $\Gamma_2 = \{M + ix : -\alpha < x < A\}$, $\Gamma_3 = \{x + iA : -M < x < M\}$, $\Gamma_4 = \{-M + ix : -\alpha < x < A\}$.

By using the residue theorem, we have the following

$$\begin{aligned}\int_{-M-i\alpha}^{M-i\alpha} e^{it\lambda}\chi R(\lambda)\chi g\, d\lambda &= \int_{\Gamma_1} e^{it\lambda}\chi R(\lambda)\chi g\, d\lambda \\ &= \sum_{\substack{\lambda_j \in \operatorname{Res}(P) \\ |\operatorname{Re}\lambda|<M, \operatorname{Im}\lambda<A}} \sum_{m=0}^{M_j} e^{it\lambda_j} t^m w_{j,m} - \int_{\Gamma_2} e^{it\lambda}\chi R(\lambda)\chi g\, d\lambda \\ &\quad - \int_{\Gamma_3} e^{it\lambda}\chi R(\lambda)\chi g\, d\lambda - \int_{\Gamma_4} e^{it\lambda}\chi R(\lambda)\chi g\, d\lambda\end{aligned}$$

For M sufficiently large, we may assume that both Γ_2 and Γ_4 lie inside the logarithmic region. Hence, we have

$$\|\int_{\Gamma_2} e^{it\lambda}\chi R(\lambda)\chi g\, d\lambda\|_{\mathcal{H}} \leq C\int_{-\alpha}^{A} e^{-tx} M^{-2} e^{T|x|}\|g\|_{\mathcal{D}^1} dx$$

which goes to zero when we let $M \to \infty$. Similar estimate is true for the integration over Γ_4. Thus the error term in the Theorem is entirely due to the integral over Γ_3 (with $M = \infty$) which we estimate as follows

$$\begin{aligned}\|\int_{\Gamma_3} e^{it\lambda}\chi R(\lambda)\chi g\, d\lambda\|_{\mathcal{H}} \leq & \int_{-e^A}^{e^A} e^{-tA}\|\chi R(x+iA)\chi\|_{\mathcal{D}^1\to\mathcal{H}}\|g\|_{\mathcal{D}^1} dx \\ & + C\int_{e^A}^{\infty} e^{-tA}|x+iA|^{-2} e^{TA}\|g\|_{\mathcal{D}^1} dx \\ & + C\int_{-\infty}^{-e^A} e^{-tA}|x+iA|^{-2} e^{TA}\|g\|_{\mathcal{D}^1} dx \\ \leq & C(\epsilon) e^{-(A-\epsilon)t}\|g\|_{\mathcal{D}^1}\,,\end{aligned}$$

for t sufficiently large (depending on A). This completes the proof of the theorem. $\square$

Remark: From the argument above it is clear that g could be taken in $\mathcal{D}^\epsilon$, for any $\epsilon > 0$. As was pointed out to us by Nicolas Burq, an argument of Morawetz [14] gives the optimal class $g \in \mathcal{H}$. It is essentially based on a transformation of an initial value problem into a non-homogeneous problem.

4. The trapping case

The *quantum trapping condition* used in this section postulates the existence of resonances converging to the real axis faster than any polynomial. More precisely, we assume

$$\exists\, \lambda_j \in \operatorname{Res}(P) \quad \operatorname{Im}\lambda_j = \mathcal{O}(|\lambda_j|^{-N}) \quad \text{for any } N > 0. \tag{4.1}$$

Remarks:
1. The quantum trapping condition holds in the case of *classical trapping* caused by the presence of elliptic orbits as was shown in various levels of precision and generality in [26],[31],[24].
2. The trapping and nontrapping conditions do not complement each other. If (4.1) condition

holds then (3.1) does not, in view of Remark 1 following it. However, when (3.1) does not hold we do not necessarily have (4.1) as shown by Ikawa [8] in the example of two convex bodies.

As discussed in the introduction, we present here an expansion of the wave group analogous to the expansion presented in Sect.3 but now involving only the resonances close to the real axis. We need to assume that for some large K,

$$\text{(4.2)} \quad \exists\, l \quad \lambda, \mu \in \mathrm{Res}(P) \cap \{\zeta \,:\, \mathrm{Im}\,\zeta < \langle \zeta \rangle^{-K}\}\,,\ \lambda \neq \mu\,,\ \Longrightarrow\ |\lambda - \mu| \geq C(\max\{|\lambda|, |\mu|\})^{-l}\,.$$

We also assume that, in the same region,

$$\text{(4.3)} \qquad \exists\, \alpha \quad \text{algebraic multiplicity of } \lambda \leq \alpha \text{ for all } \lambda \in \mathrm{Res}(P)$$

The separation condition and its plausibility will be discussed at the end of this section and a basic but already quite involved example will be described in Sect.5. The condition (4.3) is quite reasonable as generically all resonances are simple [9].

We can now state our main theorem

Theorem 2. *Assume that P satisfies the trapping condition (4.1), the separation condition (4.2), and the condition on the uniform bound on the multiplicity of resonances (4.3). Let $\chi \in C_0^\infty(\mathbb{R}^n)$ be identically equal to 1 near the ball $B(0, R_0)$. Then, for any sufficiently large M, we have*

$$\text{(4.4)} \qquad \chi U(t) \chi g = \sum_{\substack{\lambda_j \in \mathrm{Res} P \\ \mathrm{Im}\,\lambda_j \leq \langle \lambda_j \rangle^{-(K)}}} \sum_{m=1}^{m_j} e^{it\lambda_j} t^m w_{j,m} + E(t) g\,, \quad g \in \mathcal{D}^M\,,$$

for some K depending on k, α and $n^\sharp$. As in the nontrapping case, $w_{j,m}$'s are the cut-off resonance states associated with the resonance λ_j and $E(t)$ is the error term satisfying the following estimate

$$\text{(4.5)} \qquad \|E(t)\|_{\mathcal{D}^M \to \mathcal{H}} = \begin{cases} C_N t^{-N} & \text{when } n \text{ is odd} \\ C t^{-n+1} & \text{when } n \text{ is even}\ , \end{cases} \quad .$$

for any large constant N depending on M, α, l and $n^\sharp$.

We refer the reader to the remark after the proof of Theorem 2, equations (4.18) and (4.19), for an exact relation of the constants mentioned above. The double sum in (4.4) should be understood as follows: the convergence of the outer sum is absolute in $\mathcal{L}(\mathcal{D}^M, \mathcal{H})$. However we *cannot* control the absolute convergence of the double sums. In principle, there could be cancellations in the inner sum.

We also note that as $\mathrm{Im}\,\lambda_j > 0$ (though we do not exclude the possibility of embedded eigenvalues $\mathrm{Im}\,\lambda_j = 0$), the individual terms are smaller than than the error term when $t \gg \langle \lambda_j \rangle^{K+1}$. What counts is the cumulative effect of all the resonances and having a large density of them converging to the real axis (see [24],[31]). At the moment, however, we do not have good lower bounds for $w_{j,m}$'s.

The proof of this theorem follows from a contour deformation, just as in the non-trapping case. The distinction comes from the needed resolvent estimates which are now of a completely different nature.

Under the separation condition and the uniform bound of the multiplicity of resonance, we can obtain a polynomial bound on the cutoff resolvent in a region very close to the real axis. This depends on a semiclassical maximum principle of our earlier paper [31, Lemma 2]. We will state a modified form of this principle below and its proof is a direct modification of the one in [31]. As in that paper it is more convenient to work in the semiclassical setting.

We let $P(h) = h^2 P$ where $0 < h < 1$ is a semiclassical parameter and let

$$\Omega \subset \{z : \operatorname{Re} z > 0\} \subset \mathbb{C},$$

be an open neighbourhood of some energy level $E \in \mathbb{R}^+$. We denote by $\mathcal{R}_P$ the set of semi-classical resonances of $P(h)$, that is the poles of the analytic continuation of the cutoff resolvent $\chi R(z,h)\chi = \chi(P(h)-z)^{-1}\chi$ of $P(h)$ in Ω. Thus, $z \in \mathcal{R}_P \Rightarrow \lambda = h^{-1}z^{1/2} \in \operatorname{Res}(P)$. Then the separation condition (4.2) takes the following form

$$z_i, z_j \in \mathcal{R}_P \,,\; z_i \neq z_j \Longrightarrow |z_i - z_j| \geq C h^{\tilde{k}} \tag{4.6}$$

($\tilde{k}$ can be chosen to be $2l+1$).

We start with the semi-classical maximum principle:

Lemma 4.1. *Fix an integer $n^\sharp$ and let $0 < h < 1$. Suppose $F(z,h)$ is a family of holomorphic function (in z) defined in a neighbourhood of*

$$\Omega(h) = [E(h) - 5h^{\tilde{k}}, E(h) + 5h^{\tilde{k}}] + i[-4h^{k+n^\sharp+2}, 4h^{k+1}]$$

for $k > n^\sharp$, $\tilde{k} \leq k - n^\sharp$, and a real number $E(h)$. If $F(z,h)$ satisfies

$$\begin{aligned} |F(z,h)| &\leq Ae^{Ah^{-n^\sharp}\log\frac{1}{h}} \quad \text{on } \Omega(h) \\ |F(z,h)| &\leq \frac{1}{\operatorname{Im} z} \quad \text{on } \Omega(h) \cap \{\operatorname{Im} z < 0\}, \end{aligned}$$

then, there exist $h(k) > 0$ and $C > 0$ such that

$$|F(z,h)| \leq \frac{C}{h^{k+n^\sharp+2}} \quad \text{for } z \in \tilde{\Omega}(h) = [E(h) - h^{\tilde{k}}, E(h) + h^{\tilde{k}}] + i[-2h^{k+n^\sharp+2}, 2h^{k+n^\sharp+2}] \tag{4.7}$$

Proof. For the convenience of the reader, we sketch the proof of this modification of [31, Lemma 2]. We first construct an auxillary function $f(z,h)$ given as follow:

$$f(z,h) = (\pi\alpha^2)^{-1/2} \int \exp\left(-\frac{(x-z)^2}{\alpha^2}\right) \psi_h(x)\, dx \,, \quad \alpha = h^{k+1}$$

where $\psi_h \in C_0^\infty(\mathbb{R})$ satifies $0 \leq \psi_h \leq 1$ and

$$\psi_h(x) = \begin{cases} 0 \text{ when } |x - E(h)| \geq 3h^{\tilde{k}} \\ 1 \text{ when } |x - E(h)| \leq 2h^{\tilde{k}} \end{cases}$$

It can be checked readily that, for h sufficiently small, $f(z,h)$ has the following properties,

1. $f(z,h)$ is holomorphic in $\Omega(h)$
2. $|f(z,h)| \leq e$ in $\Omega(h)$
3. $|f(z,h)| \geq \frac{1}{2}$ for $z \in [E(h) - h^{\tilde{k}}, E(h) + h^{\tilde{k}}] + i[-\alpha, \alpha]$.
4. $|f(z,h)| \leq Ce^{-Ch^{2\tilde{k}}/\alpha^2}$ on $\Omega(h) \cap \{|\operatorname{Re} z - E(h)| > 4h^{\tilde{k}}\}$ for some constant $C > 0$.

Next, we apply the maximum principle to the function $G(z,h) = e^{iz\sigma} f(z,h)F(z,h)$ on $\Omega(h)$, where

$$\sigma = \frac{\log(S(h)A\exp(Ah^{-n^\sharp}\log\frac{1}{h}))}{S(h) + \alpha} > 0\,, \quad S(h) = 2h^{k+n^\sharp+2}\,.$$

Using the bounds of $F(z,h)$, and the properties of the auxillary function $f(z,h)$, we can estimate $|G(z,h)|$ on the boundary of $\Omega(h)$ by

$$|G(z,h)| \leq Ce^{S(h)\sigma}\frac{1}{S(h)}.$$

Now, applying the maximum principle, we have

$$\begin{aligned}\max_{\Omega(h)}|G(z,h)| &\leq Ce^{S(h)\sigma}\frac{1}{S(h)}\\ &\leq C\frac{1}{S(h)}\end{aligned}$$

as $e^{S(h)\sigma} \leq C$ when h is sufficiently small.

Hence, on $\tilde{\Omega}(h) = [E(h) - h^{\tilde{k}}, E(h) + h^{\tilde{k}}] + i[-S(h), S(h)]$, we have

$$\begin{aligned}\frac{C}{S(h)} &\geq \max_{z\in\tilde{\Omega}(h)} |e^{iz\sigma}||f(z,h)||F(z,h)|\\ &\geq Ce^{-S(h)\sigma}\max_{z\in\tilde{\Omega}(h)}|F(z,h)|\end{aligned}$$

which implies

$$|F(z,h)| \leq \frac{C}{S(h)}, \quad \forall z \in \tilde{\Omega}(h)$$

proving the lemma. □

We also recall the bound on the cut-off resolvent (see [31] and references given there). It holds in greater generality ([31, Lemma 1]), but we will present a direct argument for the case at hand, based on the work of Melrose [13], Sjöstrand [19],[20], Vodev [29], and the second author [32],[34]:

Proposition 4.1. *Let $P(h)$ be the semi-classical operator introduced above and $\Omega \Subset \{z : \operatorname{Re} z > 0\}$, a pre-compact neighbourhood of $E \in \mathbb{R}_+$. Then for $0 < h < h_0$ we have*

$$\begin{gathered}\|\chi R(z,h)\chi\|_{\mathcal{H}\to\mathcal{H}} \leq \exp(C_\Omega h^{-n^\sharp}\log(1/g(h)))\,,\\ z \in \Omega \setminus \bigcup_{z_j\in\mathcal{R}_P} D(z_j, g(h))\,, \quad 0 < g(h) \ll 1\,,\end{gathered}$$

where $R(z,h)$ is the meromorphically continued resolvent and χ is as in Theorem 2.

Proof. We will prove a more general statement by taking $P(h)$ to be any semi-classical compactly supported perturbation of $-h^2\Delta$, that is an operator satisfying semi-classical version of the assumptions in Sect.2 (see [31, Sect.2]). In particular, we now demand that

$$\begin{gathered}1\!\!1_{\mathbb{R}^n\setminus B(0,R_0)}P(h) = -h^2\Delta|_{\mathbb{R}^n\setminus B(0,R_0)}\\ N(P^\sharp(h), [-C,\lambda]) = \mathcal{O}\left(\left(\frac{\lambda}{h^2}\right)^{n^\sharp/2}\right) \text{ for } \lambda \geq 1\,.\end{gathered} \tag{4.8}$$

Let $R_0(z,h)$ be the meromorphic continuation of the free resolvent $(-h^2\Delta - z)^{-1}$ from $\operatorname{Im} z < 0$ to $\widetilde{\Omega}$, $\overline{\Omega} \Subset \widetilde{\Omega} \Subset \mathbb{C}$. Let us also consider the following cut-off functions $\chi_i \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, $i = 0, 1, 2$, $\chi_0 \equiv 1$ near $\overline{B}(0,R_0)$, $\chi_i \equiv 1$ near $\operatorname{supp}\chi_{i-1}$ and $\chi \equiv 1$ near $\operatorname{supp}(\chi_2)$. We then define

$$\begin{gathered}Q_0 = Q_0(z,h) = (1-\chi_0)R_0(z,h)(1-\chi_1)\,,\\ Q_1 = Q_1(z_0,h) = \chi_2 R(z_0,h)\chi_1\,,\ \operatorname{Im} z_0 < 0\,,\end{gathered}$$

so that

$$\begin{aligned}&(P(h)-z)(Q_0(z,h)+Q_1(z_0,h)) = I + K_0(z,h) + K_1(z_0,z,h)\\ &K_0(z,h) = -[h^2\Delta,\chi_0]R_0(z,h)(1-\chi_1)\\ &K_1(z_0,z,h) = -[h^2\Delta,\chi_2]R(z_0,h)\chi_1 + \chi_2(z_0-z)R(z_0,h)\chi_1\,.\end{aligned}$$

We now put

$$K = K(z_0, z, h) = K_1(z_0, z, h)\chi + K_0(z, h)\chi$$

which is a compact operator $\mathcal{H} \to \mathcal{H}$ and the norm of $K(z_0, z_0, h)$ is $\mathcal{O}(h)$. Hence $(I + K(z_0, z_0, h))^{-1}$ exists for h small enough and consequently (via analytic Fredholm theory) $(I + K(z_0, z, h))^{-1}$ is meromorphic in z (under our assumptions, on the Riemann surface of $z = w^2$ for n odd and $z = e^w$ for n even). Hence

$$\chi R(z, h)\chi = (Q_0(z, h)\chi + Q_1(z_0, z, h)\chi)(I + K(z_0, z, h))^{-1} ,$$

and we have essentially reviewed the proof of the meromorphic continuation of the resolvent from [20]. To estimate the resolvent we now use [7, Chapter V, Theorem 5.1]:

$$\|\chi R_0(z, h)\chi\| \leq (\|Q_0\chi\| + \|Q_1\chi\|)\frac{\det(I + (K^*K)^{\frac{n^\sharp+1}{2}})}{\det(I + K^{n^\sharp+1})} ,$$

where $n^\sharp$ is as in (4.8) and, as we will see below, its choice justifies the application of the determinants.

Hence we have to estimate

$$M(h) = \sup_{z\in\widetilde{\Omega}} \det(I + (K^*K)^{\frac{n^\sharp+1}{2}}) ,$$

and then, from below, the holomorhic function (of z):

$$h(z, h) = \det(I + K^{n^\sharp+1}) ,$$

where by Weyl inequalities (see [7]), $|h(z, h)| \leq M(h)$, $z \in \widetilde{\Omega}$.

To estimate $M(h)$ we need to estimate the eigenvalues of $(K^*K)^{\frac{1}{2}}$, that is the characteristic values of K, $\mu_j(K)$. From the standard properties of characteristic values it is enough to estimate the characteristic values of various summands.

We start with

$$\mu_j([h^2\Delta, \chi_2]R(z_0, h)\chi_1) \leq Ch\left(\frac{j}{h}\right)^{-\frac{1}{n^\sharp}} ,$$

$$\mu_j(\chi_2 R(z_0, h)\chi_1) \leq C\left(\frac{j}{h}\right)^{-\frac{2}{n^\sharp}} ,$$

by noting that

$$\forall\ N, M\ \ \chi_2 R(z_0, h)\chi_1 - \chi_2(P^\sharp(h) - z_0)^{-1}\chi_1 = \mathcal{O}(h^N)\ :\ \mathcal{H} \longrightarrow \mathcal{D}^M ,$$

(see the proof of [20, Proposition 5.4]) and, that the estimates for the characteristic values of $\chi_2(P^\sharp(h) - z_0)^{-1}\chi_1$ follow from (4.8).

To handle the $K_0\chi$ term, we observe that for $\operatorname{Im} z \leq 0$,

$$\mu_j([h^2\Delta, \chi_2]\chi R_0(z, h)\chi) \leq Ch\left(\frac{j}{h}\right)^{-\frac{1}{n^\sharp}}$$

(see, for instance, [32, Lemma 4]). For $\operatorname{Im} z > 0$ we write

$$\chi R_0(z, h)\chi = \chi(-h^2\Delta - z)^{-1}\chi + \chi(R_0(z, h) - (-h^2\Delta - z)^{-1})\chi ,$$

where $R_0(z, h)$ is the meromorphic continuation of the resolvent from $\operatorname{Im} z < 0$ and $(-h^2\Delta - z)^{-1}$ is the resolvent, holomorphic on L^2 for $z \in \mathbb{C} \setminus \mathbb{R}_+$. Hence we only have to estimate the characteristic values of $\chi(R_0(z,h)-(-h^2\Delta-z)^{-1})\chi$, and for that we rewrite it using the standard representation of the spectral projection (see for instance the proof of [32, Lemma 1]):

$$R_0(z,h)-(-h^2\Delta-z)^{-1} = {}^t\mathbb{E}(\lambda,h)\mathbb{E}(-\lambda,h)\,,\quad \mathbb{E}(\lambda,h)\ :\ \mathcal{C}^\infty_{\rm c}(\mathbb{R}^n)\to\mathcal{C}^\infty(\mathbb{S}^{n-1})\,,$$
$$\mathbb{E}(\lambda,h)(\omega,x) = c_n\lambda^{\frac{n-2}{2}}e^{i\frac{\lambda}{h}\langle x,\omega\rangle}\,,\quad \lambda^2=z\,,\ \operatorname{Im}\lambda>0\,.$$

Hence,

$$\begin{aligned}\mu_j(\chi(R_0(z,h)-(-h^2\Delta-z)^{-1}\chi) &= \mu_j(\chi^t\mathbb{E}(\lambda,h)(I-\Delta_{\mathbb{S}^{n-1}})^{-k}(I-\Delta_{\mathbb{S}^{n-1}})^k\mathbb{E}(-\lambda,h)\chi)\\ &\leq \|\chi^t\mathbb{E}(\lambda,h)\|\mu_j((I-\Delta_{\mathbb{S}^{n-1}})^{-k})\|(I-\Delta_{\mathbb{S}^{n-1}})^k\mathbb{E}(-\lambda,h)\chi\|\\ &\leq C^kj^{-\frac{2k}{n-1}}(2k)!\exp(C\langle\lambda\rangle/h)\\ &\leq C\exp(C\langle\lambda\rangle/h-j^{\frac{1}{n-1}/C})\,,\end{aligned}$$

where we used the Cauchy inequalities and then optimized in k. By summing up the contributions from different terms in K, we obtain the following estimate on the determinant

$$M(h)=\mathcal{O}(e^{Ch^{-n^\sharp}})\,.$$

We also note that the same estimates hold for $I-K(z_0,z_0,h)(I-K(z_0,z_0,h))^{-1}$ which, as we already observed, can be inverted by Neumann series. Hence, $|h(z_0)|>e^{-Ch^{-n^\sharp}}$. Let us now put $\Omega_0=D(z_0,r)$, $\overline{\Omega}_0\subset\widetilde{\Omega}$, $|\operatorname{Im}z_0|<r<\operatorname{Re}z_0$ (choosing z_0 appropriately for that). Let $N(h)$ be the number of zeros, $z_j(h)$, of $h(z,h)$ in $\overline{D(z_0,r+\epsilon)}\subset D(z_0,r+2\epsilon)\subset\widetilde{\Omega}$. Then by the Jensen inequality

$$N(h)\leq C_\epsilon(\max_{D(z_0,r+2\epsilon)}\log|h(z,h)|-\log|h(z_0,h)|)\leq C_\epsilon(\log M(h)-\log|h(z_0)|)\leq Ch^{-n^\sharp}\,,$$

and in $D(z_0,r+\epsilon)$ we can write

$$h(z,h)=g(z,h)\prod_{j=1}^{N(h)}(z-z_j(h))\,,\quad g(z,h)\neq0\,,\quad z\in D(z_0,r+\epsilon)\,.$$

We can now use some essentially standard methods of complex analysis (averaging and Harnacks's inequality [19, Sect.8] or Cartan's Lemma and Carathéodory's inequality [34]) to conclude that

$$\exp(-Ch^{-n^\sharp})\leq|g(z,h)|\leq\exp(Ch^{-n^\sharp})\,,\quad z\in D(z_0,r)\,,$$

and this gives us the conclusion of the Proposition for $z\in D(z_0,r)\setminus\bigcup_{z_j\in\mathcal{R}_P}D(z_j,g(h))$, $g(h)<\epsilon$. Since the argument for lower bounds for h gives existence points near $|z-z_0|=r$, where $|h(z,h)|>\exp(-Ch^{-n^\sharp})$ we can iterate this argument and obtain the estimate in all of Ω (we remark that when Ω is in a small conic neighbourhood of the real axis we can also do it by varying z_0). $\square$

We will apply Proposition 4.1 and the semiclassical maximum principle to the modification of the operator valued meromorphic family $F(z,h)=\chi R(z,h)\chi$. This gives the following following

Lemma 4.2. *Assume the separation condition* (4.6). *Let $z_j\in\mathcal{R}_P$ satisfy $\operatorname{Im}z_j\leq h^{k+n^\sharp+2}$ and put*

$$\chi R(z,h)\chi=\sum_{l=0}^{m_j}\frac{A_{j,l}(h)}{(z-z_j)^l}+\tilde{A}(z,h,z_j)\ \ \textit{on}\ \Omega(h) \tag{4.9}$$

where m_j is the multiplicity of z_j and $\Omega(h)$ is defined as in Lemma 4.1 with $E(h) = \operatorname{Re} z_j$. Then, on $\widetilde{\Omega}(h)$, we have

$$\begin{aligned} \|A_{j,l}(h)\|_{\mathcal{H}\to\mathcal{H}} &\leq \mathcal{O}(h^{-(m_j-l+1)(k+n^\sharp+2)}) \\ \|\tilde{A}(z,h,z_j)\|_{\mathcal{H}\to\mathcal{H}} &\leq \mathcal{O}(h^{-(m_j+1)(k+n^\sharp+2)}) \end{aligned}$$

where again $\widetilde{\Omega}(h)$ is defined as in Lemma 4.1 with $E(h) = \operatorname{Re} z_j$.

Proof. First of all, let us note that z_j is the only pole of $\chi R(z,h)\chi$ in $\Omega(h)$ because of the separation condition. As a result, we can write $\chi R(z,h)\chi$ in the form given in (4.9) with $\tilde{A}(z,h,z_j)$ holomorphic in $\widetilde{\Omega}$. Now, we have

$$(z-z_j)^{m_j}\chi R(z,h)\chi = A_{j,m_j}(h) + (z-z_j)A_{j,m_{j-1}} + \cdots + (z-z_j)^{m_j}\tilde{A}(z,h,z_j)$$

Put $F_1(z,h) = (z-z_j)^{m_j}\chi R(z,h)\chi$. Then $F_1(z,h)$ is a holomorphic family of operators satisfying the assumptions of Lemma 4.1. That it satisfies the estimate (4.7) follows from Proposition 4.1. Hence, we get

$$\|F_1(z,h)\|_{\mathcal{H}\to\mathcal{H}} = \mathcal{O}(h^{-(k+n^\sharp+2)}) \text{ on } \widetilde{\Omega}(h)\,. \tag{4.10}$$

Putting $z = z_j$, we have

$$\|A_{j,m_j}(h)\|_{\mathcal{H}\to\mathcal{H}} = \mathcal{O}(h^{-(k+n^\sharp+2)})$$

We now put

$$\begin{aligned} F_2(z,h) &= \frac{(z-z_j)^{m_j}\chi R(z,h)\chi - A_{j,m_j}(h)}{z-z_j} \\ &= A_{j,m_j-1}(h) + \cdots + (z-z_j)^{m_j-1}\tilde{A}(z,h,z_j) \end{aligned}$$

Then $F_2(z,h)$ is holomorphic on $\widetilde{\Omega}(h)$ and

$$\|F_2(z,h)\|_{\mathcal{H}\to\mathcal{H}} \leq \mathcal{O}(h^{-(k+n^\sharp+2)})/h^{k+n^\sharp+2} \leq \mathcal{O}(h^{-2(k+n^\sharp+2)}) \text{ on } \partial\widetilde{\Omega}(h).$$

By the maximum principle,

$$\|F_2(z,h)\|_{\mathcal{H}\to\mathcal{H}} = \mathcal{O}(h^{-2(k+n^\sharp+2)}) \text{ on } \widetilde{\Omega}(h).$$

Again, putting $z = z_j$, we have

$$\|A_{j,m_j-1}(h)\|_{\mathcal{H}\to\mathcal{H}} = \mathcal{O}(h^{-2(k+n^\sharp+2)})\,.$$

Proceeding inductively, we obtain the required estimates in the lemma. $\square$

The following lemma is essentially a semi-globalization of the previous one

Lemma 4.3. *Assume the separation condition* (4.2) *and the condition on the uniform boundedness of the multiplicities of the resonances* (4.3). *Let*

$$\widetilde{\Omega}_j(h) = [\operatorname{Re} z_j - h^{\tilde{k}}, \operatorname{Re} z_j + h^{\tilde{k}}] + i[-2h^{k+n^\sharp+2}, 2h^{k+n^\sharp+2}]$$

and

$$\widetilde{\Omega} = \bigcup_{\substack{z_j \in \mathcal{R}_P \cap \Omega \\ \operatorname{Im} z_j < h^{k+n^\sharp+2}}} \widetilde{\Omega}_j(h) \cup \left[(\mathbb{R} + i[-h^{k+n^\sharp+2}, h^{k+2n^\sharp+3}]) \cap \Omega\right]\,,$$

$k = \tilde{k} + n^\sharp$. *Then, we have, in* $\widetilde{\Omega}$,

$$\chi R(z,h)\chi = \sum_{\substack{z_j \in \mathcal{R}_P \cap \Omega \\ \operatorname{Im} z_j < h^{k+n^\sharp+2}}} \sum_{l=1}^{m_j} \frac{A_{j,l}(h)}{(z-z_j)^l} + \tilde{A}(z,h)$$

with

$$\|A_{j,l}(h)\|_{\mathcal{H}\to\mathcal{H}} = \mathcal{O}(h^{-\alpha(k+n^\sharp+2)}) \tag{4.11}$$

$$\|\tilde{A}(z,h)\|_{\mathcal{H}\to\mathcal{H}} = \mathcal{O}(h^{-((\alpha+1)(k+n^\sharp+2)+n^\sharp)}), \tag{4.12}$$

where α *is as in* (4.3).

Proof. By Lemma 4.2, the estimate (4.11) holds for z_j's in

$$\Omega^\sharp \stackrel{\text{def}}{=} \bigcup_{\substack{z_j \in \mathcal{R}_P \cap \Omega \\ \operatorname{Im} z_j < h^{k+n^\sharp+2}}} \widetilde{\Omega}_j(h).$$

Also, since $\chi R(z,h)\chi$ is holomorphic on

$$\Omega^\flat \stackrel{\text{def}}{=} (\mathbb{R} + i[-h^{k+n^\sharp+2}, h^{k+n^\sharp+2}]) \cap \Omega \setminus \Omega^\sharp,$$

another application of the semiclassical maximum principle (Lemma 4.1) shows that

$$\|\chi R(z,h)\chi\| = \mathcal{O}(h^{-(k+2n^\sharp+3)}) \tag{4.13}$$

also holds in $\Omega^\flat \cap (\mathbb{R} + i[-h^{k+n^\sharp+2}, h^{k+2n^\sharp+3}])$: every point there has a neighbourhood of the form

$$\Omega(h) = [E(h) - h^{\tilde{k}}, E(h) + h^{\tilde{k}}] + i[-h^{k+n^\sharp+2}, h^{k+n^\sharp+2}]$$

and contained in $\Omega^\flat$.

The same argument can be applied to the remainder term $\tilde{A}(z,h)$, to obtain the estimate (4.12) where the extra factor $n^\sharp$ is added to account for the total number of resonances that may exist in $\widetilde{\Omega}$. In fact, we need to prove the estimate on boxes of the form

$$\Omega(h) = [E(h) - h^{\tilde{k}}, E(h) + h^{\tilde{k}}] + i[-h^{k+n^\sharp+2}, A(h)],$$
$$A(h) = h^{n^\sharp+k+2} \quad \text{or} \quad A(h) = h^{2n^\sharp+k+3},$$

where in the first case, the box contains exactly one resonance, z_i, $E(h) = \operatorname{Re} z_i$, and in the second case, no resonance, with the nearest resonance in $\Omega^\sharp$, a distance $h^{\tilde{k}}$ away. In the first case we use the estimate obtained in the previous lemma, (4.10):

$$\|(z-z_i)^\alpha \chi R(z,h)\chi\| \leq C h^{-(k+n^\sharp+2)} \tag{4.14}$$

and the estimate on the sum of polar parts given by (4.11)

$$\sum_{\substack{z_i \neq z_j \in \mathcal{R}_P \cap \Omega \\ \operatorname{Im} z_j < h^{k+n^\sharp+2}}} \sum_{l=1}^{m_j} \frac{A_{j,l}(h)}{(z-z_j)^l} = \mathcal{O}(1) h^{-n^\sharp} h^{-\alpha(k+n^\sharp+2)} h^{-\alpha\tilde{k}} \; : \; \mathcal{H} \longrightarrow \mathcal{H},$$

which, gives (4.12) in that box. In the second case, we use the estimate on $\chi R(z,h)\chi$ and the estimate on the same estimate on the sum of the polar parts, now including all resonances in $\widetilde{\Omega}$. This concludes the proof of the lemma. □

Restricting to a subregion of $\widetilde{\Omega}$ which is bounded away from the poles of $\chi R(z,h)\chi$, we obtain a polynomial bound of the cutoff resolvent:

Proposition 4.2. *Under the separation condition* (4.2) *and the condition on the uniform bound of the multiplicity of the resonances* (4.3), *the cutoff resolvent has a polynomial bound. That is there exists an integer N, which can be choosen as $(\alpha+1)(k+n^\sharp+2)$, such that*

$$\|\chi R(z,h)\chi\|_{\mathcal{H}\to\mathcal{H}} = \mathcal{O}(h^{-N}) \tag{4.15}$$

in the region between the curves Γ_1 and Γ_2, shown in Fig.1.

Proof. The estimate can be obtained by combining (4.13) and (4.14). □

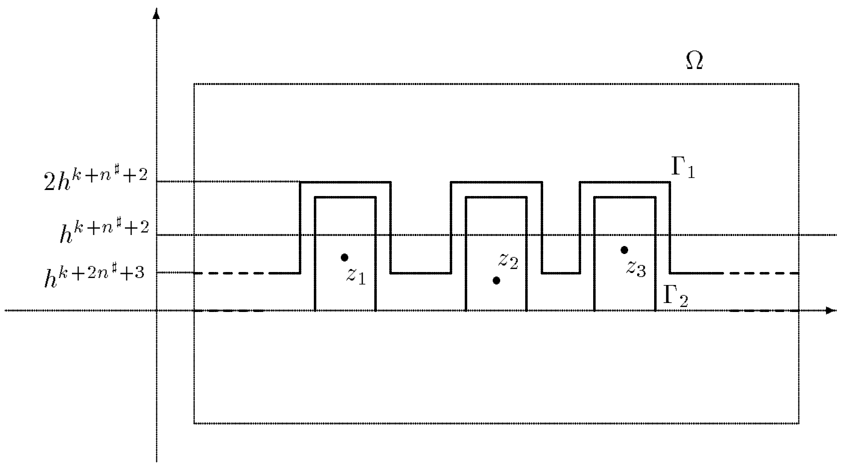

FIGURE 1. The region in which the bound (4.15) is satisfied.

By using the standard scaling procedure, we can translate the semiclassical estimate into a classical one as

$$\|\chi R(z)\chi\|_{\mathcal{H}\to\mathcal{H}} \le C|\lambda|^{N-2}$$

on a zig-zag curve around the curve

$$\operatorname{Im}\lambda \le C|\lambda|^{-\frac{k+n^\sharp+2}{2}},$$

as well as in some small vertical strips about the real axis which are bounded away from the resonances, as shown in Fig.2.

We now write $K = \frac{k+2n^\sharp+3}{2}$ and $N' = N-2$. Note that we actually have a better bound for the cutoff resolvent in the above region if we restrict its domain to 'smoother functions', i.e. $\mathcal{D}^j$ for j large. Indeed, by using the same computation as in the derivation of (3.2), we have, in the above region

$$\|\chi R(\lambda)\chi\|_{\mathcal{D}^1\to\mathcal{H}} \le C|\lambda|^{-2}|\lambda|^{N'}.$$

By iterating the argument, we obtain

$$\|\chi R(\lambda)\chi\|_{\mathcal{D}^M\to\mathcal{H}} \le C|\lambda|^{-2} \quad \text{for } M \ge \frac{N'}{2}+1. \tag{4.16}$$

With this bound on the resolvent, we can perform the contour deformation to the integral

$$\chi U(t)\chi g = \frac{i}{2\pi}\int_{-\infty-i\alpha}^{\infty-i\alpha} e^{it\lambda}\chi R(\lambda)\chi g\, d\lambda, \quad g \in \mathcal{D}^M\,, \tag{4.17}$$

and that will give the asymptotic expansion of the cutoff wave group in the trapping case.

Proof of Theorem 2: We will first consider the simpler case of odd dimensions. Essentially by changing the cutoff function used in the Theorem, we may assume that the data g is of compact support and χ is identically 1 near the support of g. We start with a contour deformation similar to that in the proof of Theorem 1 with the modification that Γ_3 is now given by a curve fitting into the region shown in the Fig.2. By (4.16), we can move Γ_2 and Γ_4 to infinity and obtaining the resonance expansion as stated in the Theorem with the error term given by the integration over Γ_3. To estimate this integral, we first note that, for C large enough

$$\begin{aligned} \|\int_{\Gamma_3} e^{it\lambda}\chi R(-\lambda)\chi g\, d\lambda\|_{\mathcal{H}} &= \|\sum_{\substack{\mu_j^2\in\mathrm{Spec}(P)\\ \mathrm{Im}\,\mu_j<0}} e^{-it\mu_j}v_j + \int_{\Gamma_3+iC} e^{it\lambda}\chi R(-\lambda)\chi g\, d\lambda\|_{\mathcal{H}} \\ &= \mathcal{O}(e^{-At})\|g\|_{\mathcal{D}^1} \quad A = \min(\mathrm{Im}\,\mu_j)\,, \end{aligned}$$

where we have used the holomorphy of $R(-\lambda)$ in the upper half plane (except at finitely many poles corresponding to eigenvalues) and the fact that a similar bound as (4.16), with $M = 1$, holds for $R(-\lambda)$. Hence we can replace $R(\lambda)$ by $R(\lambda) - R(-\lambda)$ in the integral over Γ_3 by introducing an exponentially decaying error. Now, by noting that $(R(\lambda) - R(-\lambda))\langle P\rangle^{-M} = \langle\lambda^2\rangle^{-M}(R(\lambda) - R(-\lambda))$ and that for M even, $\langle P\rangle^{-M}\chi g = \chi\langle P\rangle^{-M} g$ by using the support properties of g and χ stated above, and that the same estimates as above hold for $\chi(R(\lambda) - R(-\lambda))\chi$, we have

$$\begin{aligned} \|E(t)g\|_{\mathcal{H}} &= \|\int_{\Gamma_3} e^{it\lambda}\langle\lambda^2\rangle^{-M}\chi(R(\lambda) - R(-\lambda)\chi g d\lambda\|_{\mathcal{H}} + \mathcal{O}(e^{-At})\|g\|_{\mathcal{D}^1} \\ &\leq C\int_1^\infty e^{-tx^{-K}}|x|^{-2M+N-2}\, dx\|g\|_{\mathcal{D}^M} + \mathcal{O}(e^{-At})\|g\|_{\mathcal{D}^1} = \mathcal{O}\left(t^{-\frac{2M-N+2}{K}}\right)\|g\|_{\mathcal{D}^M} \end{aligned}$$

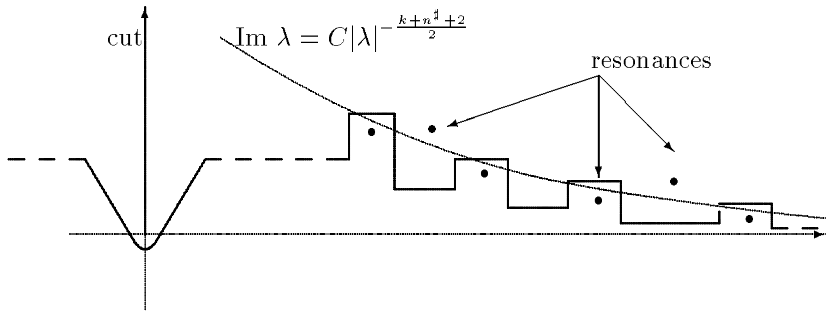

FIGURE 2. The deformation contour for large energies and the deformation near zero in the even dimensional case.

by the change of variable $y = tx^{-K}$ where K is given by $\frac{k+2n^\sharp+3}{2}$. In the even dimensional case the same argument is applied, except near zero where we have to deform the contour in a different way – see Fig.2. As is well known, the contribution from zero gives the same error term as in the free case, $\mathcal{O}(t^{-n+1})$. Note that (4.11) gives the absolute convergence of the the sum in the statement of the Theorem. □

By tracing the constants used in the proof above more careful, we see that we can take K, N in the Theorem 2 to be given by

$$K = (l + n^\sharp + 2) \tag{4.18}$$

$$N = \frac{2M - (\alpha + 1)(2l + 2n^\sharp + 3) + 2}{K} \tag{4.19}$$

We conclude this section by making a few comments about the assumption (4.2). As will be explained in Sect.5 (see Lemma 5.1), the assumption follows from the same assumption for eigenvalues of a reference operator $P^\sharp$, in the construction of which we have a free choice outside of the "black box". Hence, for instance, we could consider (4.2) for eigenvalues of the Laplacian on a compact Riemannian manifold. It is expected, or at least it is natural to conjecture, that (4.2) holds generically in that case. However, once the condition $\lambda \neq \mu$ is dropped, no example which is not of "product type" (see (4.20) below) and in which separation holds, is known to the authors (when we assume $\lambda \neq \mu$ then it trivially holds for spheres).

In a setting different from ours, an example of an operator for which the separation condition is satisfied is given by a generic harmonic oscillator:

$$P = \frac{1}{2}\sum_{i=1}^{n} \left(D_{x_i}^2 + \alpha_i^2 x_i^2\right) , \quad \alpha_i > 0 . \tag{4.20}$$

The spectrum of P is given by $\sum_{i=1}^n \alpha_i(k_j + 1/2) = \langle \alpha, \mathbf{k}\rangle + \beta$, $\alpha = (\alpha_1, \cdots, \alpha_n)$. Since for $\mathbf{k}, \mathbf{l} \in \mathbb{N}_0^n$, $|\mathbf{k} - \mathbf{l}| \leq \max(|\mathbf{k}|, |\mathbf{l}|)$, the separation condition will follow from the generically valid Diophantine condition:

$$\exists\, N\, , C > 0\; \forall\, \mathbf{n} \in \mathbb{N}^n \quad \langle \alpha, \mathbf{n}\rangle > \frac{1}{C}|\mathbf{n}|^{-N} ,$$

Finally, we mention that in some numerical computations of scattering poles, the *a priori* assumption of separation is required – see [15].

5. A spherically symmetric example

Although we cannot verify the assumption (4.2) in any interesting situation, we can apply the method of proof to a family of spherically symmetric examples. It is quite clear that the argument should work for a large class of examples but we will consider only a very specific one similar to [21, Example 3] (see [21, Fig.1] for a picture in dimension two).

Let us define the following metric, g, on $\mathbb{R}^n$, $\mathbb{R}^n \setminus 0 \simeq \mathbb{R}_+ \times \mathbb{S}^{n-1}$:

$$g = dr^2 + a(r)^2 h_{\mathbb{S}^{n-1}} , \quad a(r) = r\, , \; r > r_0\, , \; a'(0) = 1, a^{(2p)}(0) = 0\, . \tag{5.1}$$

Here, $h_{\mathbb{S}^{n-1}}$ is the standard metric on the sphere. The first condition on $a(r)$ implies that the metric is the Euclidean metric outside of a ball, and the second, that the metric on $\mathrm{R}^n \setminus 0$ extends to a smooth metric on $\mathbb{R}^n$ – see for instance [1].

The Laplace operator is given by

(5.2)
$$-\Delta_g = D_r^2 - i(n-1)\frac{a'(r)}{a(r)}D_r - \frac{1}{a(r)^2}\Delta_{\mathbb{S}^{n-1}}$$
$$= a(r)^{-\frac{n-1}{2}}\left(D_r^2 - \frac{1}{a(r)^2}\Delta_{\mathbb{S}^{n-1}} - \frac{2(n-1)a''(r)a(r)+(n-1)(n-3)(a'(r))^2}{4a(r)^2}\right)a(r)^{\frac{n-1}{2}},$$

and we will consider it semi-classically using $h = (\lambda_k)^{-1/2}$, $\lambda_k \in \sigma(-\Delta_{\mathbb{S}^{n-1}})$, as the semi-classical parameter. This gives us a family of one dimensional operators:

$$Q(h) = (hD_r)^2 + V(r) + h^2 V_2(r)\,,\ r \in (0,\infty)\,,$$
$$V(r) = \frac{1}{a(r)^2}\,,\ V_2(r) = -\frac{2(n-1)a''(r)a(r)+(n-1)(n-3)(a'(r))^2}{4a(r)^2}\,, \tag{5.3}$$

and, in the notation of the previous section,

$$\lambda \in \mathrm{Res}(P) \iff (h\lambda)^2 \in \mathcal{R}_{Q(h)}\,,\ h^{-2} \in \sigma(-\Delta_{\mathbb{S}^{n-1}})\,. \tag{5.4}$$

Let us now assume that

$$a'(r) > 0 \ \text{ for } \ 0 \leq r < r_m \ \text{ and for } \ r > r_M$$
$$a'(r_m) = a'(r_M) = 0\,,\quad a''(r_m) < 0\,,\quad a''(r_M) > 0\,, \tag{5.5}$$

see Fig.3 where the graph of $a(r)^{-2}$ is plotted. In that case, energy levels $E > a(r_M)^{-2}$ and $E < a(r_m)^{-2}$ are non-trapping for the operator $Q(h)$. From that we can conclude [3],[16] that

$$\min(a(r_0)^{-2}, a(r_m)^{-2}) - \epsilon < \mathrm{Re}\, z < a(r_M)^{-2} + \epsilon\,,\ z \in \mathcal{R}_{Q(h)}\,,\implies \mathrm{Im}\, z > Ch\log(1/h)\,, \tag{5.6}$$

$C > 0$. To reduce the problem further we state the following essentially well known lemma (see Proposition 2.1 of [5] and [25]) which is particularly simple in the one dimensional case:

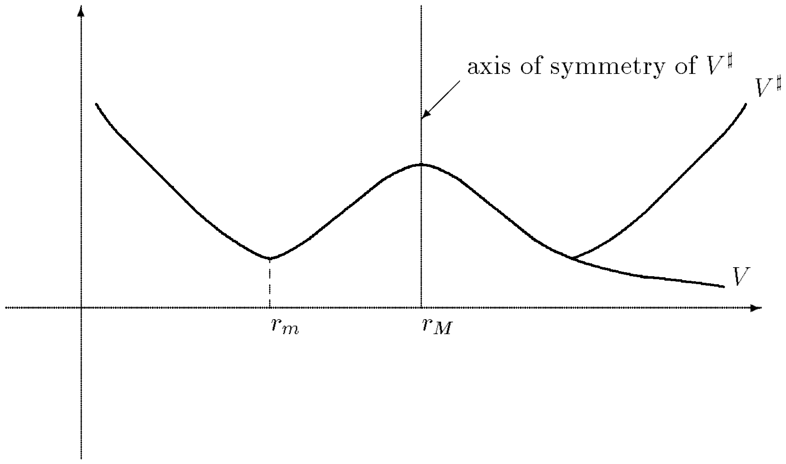

FIGURE 3. The one dimensional potential corresponding to the spherically symmetric metric.

Lemma 5.1. *Suppose u is a resonant state of $Q(h)$ corresponding to a simple $z \in \mathcal{R}_{Q(h)}$. Then*

$$\int_{r_0+\epsilon}^{r_0+1} |u(r)|^2 dr \leq C \max(\operatorname{Im} z, e^{-c/h}) \int_0^{r_0+1} |u(r)|^2 dr \,. \tag{5.7}$$

For $\operatorname{Im} z < h^K$ we can consequently reduce the separation condition to the condition on a reference operator $Q^\sharp(h)$:

$$\begin{gathered} Q^\sharp(h) = (hD_r)^2 + V^\sharp(r) + h^2 V_2^\sharp(r) \,, \quad V^\sharp, V_2^\sharp \in \mathcal{C}^\infty((0, r_1); \mathbb{R}) \;\; r_1 > r_0 \,, \\ V^\sharp(r) = V(r) \,, \;\; V_2^\sharp(r) = V_2(r) \,, \quad \text{for} \quad r < r_0 \,, \quad V^\sharp(r) = (r - r_1)^{-2} \,, \quad \text{for} \quad r > r_1 - \epsilon \,. \end{gathered} \tag{5.8}$$

¿From Lemma 5.1 and from the spectral theorem we conclude that

$$z \in \mathcal{R}_{Q(h)} \,, \; \operatorname{Im} z < h^K \implies \exists \, \mu(z) \in \sigma(Q^\sharp(h)) \,, \; |\mu(z) - z| < h^K / C \,, \tag{5.9}$$

and we can choose $\mu(z)$ in a unique way by demanding that the corresponding eigenfunction is close to the quasi-mode obtained by truncating the resonant state u.

We will now construct $V^\sharp$ for which it will be easiest to verify the separation condition $|\mu(z) - \mu(w)| \geq h/(C \log(1/h))$, $z \neq w$. Using the results of Bleher [2], that is the case for a symmetric double well and the needed condition on a is

$$a(r_M + t) = a(r_M - t) \;\; \text{for} \;\; r_0 - r_M < t < T < r_M - r_m \,, \tag{5.10}$$

for some fixed T, see Fig.3. As $V^\sharp$ we can then take

$$V^\sharp(r) = \begin{cases} V(r) & \text{for } 0 < r \leq r_m \\ V(2r_M - r) & \text{for } r_M \leq r < 2r_M \end{cases} \tag{5.11}$$

where, as before, $V(r) = a(r)^{-2}$ and now $r_1 = 2r_M$.

As an easy consequence of the results of [2] we now have

Lemma 5.2. *Let $E_M = a(r_M)^{-2}$. Then for any $C > 0$, the eigenvalues of $Q^\sharp(h)$ constructed using $V^\sharp$ given by* (5.11)*, satisfy*

$$\mu, \lambda \in \sigma(Q^\sharp(h)) \cap [E_M - Ch, E_M + Ch] \,, \quad \mu \neq \lambda \implies |\lambda - \mu| \geq \frac{1}{C'} \frac{h}{\log(1/h)} \,.$$

Proof. The lemma follows from [2, Theorem 1.2] applied, for the sake of simplicity, in the case of a symmetric double well. We can neglect the $h^2 V_2^\sharp$ term as it produces a negligible contribution to eigenvalues considered with the accuracy needed here. The eigenvalues of $Q^\sharp(h)$ in $(E_M - \epsilon, E_M + \epsilon)$ are given, up to an $\mathcal{O}(h^{\frac{4}{3}})$ error, as solutions to

$$\begin{gathered} I(\mu) = \pi k + \phi_\pm(\mu) \pm \frac{\pi}{4} \,, \\ I(\mu) = \frac{1}{2h} \int (E - V^\sharp(r))_+^{\frac{1}{2}} dr \,, \quad \phi_\pm(\mu) = \frac{\pi}{2} \gamma_\pm(h^{-1} |2V''(r_M)|^{-\frac{1}{2}} \mu) \,, \end{gathered}$$

where $\gamma_\pm$ are given by [2, (1.8)] and satisfy

$$\begin{gathered} 0 < |\gamma_\pm| < 1/2 \,, \quad \gamma_+(0) - \gamma_-(0) < 1 \,, \\ \lim_{t \to +\infty} \gamma_\pm(t) = 0 \,, \quad \lim_{t \to -\infty} \gamma_\pm(t) = \pm 1/2 \,. \end{gathered}$$

Putting $\mu = V(r_M) + t$ we have

$$I(\mu) = b_0(t) + t |\log|t|| b_1(t) \,, \quad b_0(0) > 0 \,, \; b_1(0) = |2V''(r_M)|^{-\frac{1}{2}} > 0 \,.$$

For simplicity we assume now that $2V''(r_M) = -1$ and make a change of variables $t = hs$, $|s| \leq C$. Substituting into the formulæ above we obtain the following equation for approximate eigenvalues:

$$h^{-1}b_0(hs) + \left(s\log\frac{1}{h} + s\log\frac{1}{|s|} \right)(1 + \mathcal{O}(sh)) = \pi k + \frac{\pi}{2}\gamma_\epsilon(s) + \epsilon\frac{\pi}{4}, \quad \epsilon = \pm 1 .$$

Let us denote by s_k^ϵ the solution to this equation and we will first consider the case when

$$\min(|s_k^\epsilon|, |s_{k'}^{\epsilon'}|) > \frac{1}{C}\left(\log\frac{1}{h}\right)^{-1} .$$

We then have

$$(s_k^\epsilon - s_{k'}^{\epsilon'})\log\frac{1}{h}(1 + \mathcal{O}(\log\log(1/h)/\log(1/h)) = \pi(k-k') + \frac{\pi}{2}(\gamma_\epsilon(s_k^\epsilon) - \gamma_{\epsilon'}(s_{k'}^{\epsilon'})) + \frac{\pi}{4}(\epsilon - \epsilon') .$$

Since $|s_k^\epsilon|, |s_{k'}^{\epsilon'}| \leq C$, we have $\delta < |\gamma_\epsilon(s_k^\epsilon) - \gamma_{\epsilon'}(s_{k'}^{\epsilon'})| < 1 - \delta$, $\delta = \delta(C)$ and hence when $k \neq k'$ or $\epsilon \neq \epsilon'$, that is when $s_k^\epsilon \neq s_{k'}^{\epsilon'}$, we have

$$|s_k^\epsilon - s_{k'}^{\epsilon'}| > \frac{1}{C_1}\left(\log\frac{1}{h}\right)^{-1} .$$

Let us now assume that

$$\max(|s_k^\epsilon|, |s_{k'}^{\epsilon'}|) < \frac{1}{C'}\left(\log\frac{1}{h}\right)^{-1} ,$$

with C' large (the mixed assumption implies separation). Then

$$\begin{aligned} &\frac{b_0}{h} + F = \pi k + \frac{\pi}{2}\gamma_\epsilon + \epsilon\frac{\pi}{4}, \quad |F| \leq 1/C' + \mathcal{O}(\log\log(1/h)/\log(1/h)) \\ &\frac{b_0}{h} + G = \pi k + \frac{\pi}{2}\gamma_{\epsilon'} + \epsilon'\frac{\pi}{4}, \quad |G| \leq 1/C' + \mathcal{O}(\log\log(1/h)/\log(1/h)) , \end{aligned}$$

that is,

$$|\pi(k-k') + \frac{\pi}{2}(\gamma_\epsilon(0) - \gamma_{\epsilon'}(0) + (\epsilon - \epsilon')/2)| \leq \frac{1}{C'} + \mathcal{O}\left(\frac{\log\log(1/h)}{\log(1/h)}\right) .$$

Since the multiple of $\pi/2$ on the left hand side has modulus strictly less than two, we conclude that for C' large enough and for h small enough, $k = k'$. We then use again the fact that $|\gamma_\epsilon(0) - \gamma_{\epsilon'}(0)| < 1$ to conclude that $\epsilon = \epsilon'$. Hence there can be only one s_k^ϵ in $(-1/C'\log(1/h), 1/C'\log(1/h))$ and this concludes the proof of the lemma. □

For eigenvalues of $Q^\sharp(h)$ in $[V(r_m), V(r_M) + \epsilon] \setminus (V(r_M) - Ch, V(r_M) + Ch)$ we can use the Bohr-Sommerfeld quantization rules as explained in [2]: (1.2) there for the energies above $V(r_M)$ and (1.4),(1.5) for those below. For eigenvalues in $[V(r_M) + \epsilon, V(r_M) + Ch]$ we obtain the separation condition

$$|\lambda - \mu| > \frac{1}{C}h(\log(1/h))^{-1}$$

arguing as in the proof of the lemma.

The situation is more complicated in $[V(r_m), V(r_M) - Ch]$. The approximate eigenvalues come in pairs $\lambda_k^\pm$ (corresponding to s_{k-1}^+ and s_k^- in the proof of the lemma) with λ_k^+ very close to λ_k^-. However we still have

$$|\lambda_l^\epsilon - \lambda_k^{\epsilon'}| > \frac{1}{C}h(\log(1/h))^{-1} , \quad k \neq l .$$

In the correspondence (5.9) only *one* of the pair of eigenvalues can correspond to simple resonance z, i.e. can appear as $\mu(z)$. In fact, [2, Lemma 2.3] and the proof of Theorem 1.2 there show that the approximate eigenfunction cannot concentrate in $|r - r_M| < |r_0 - r_M|$. On the other hand Lemma 5.1 shows that the resonant state corresponding to z gives rise to a quasimode concentrated in $r < r_M + Ch$. Reflection by $r \mapsto 2r_M - r$ gives another independent quasimode which corresponds to the other eigenvalues in the pair.

We combine this discussion with Lemma 5.2 to obtain

Proposition 5.1. *Suppose that* $0 < h < h_0$ *and that the resonances of* $Q(h)$ *in* $\{z \; : \; \operatorname{Im} z < h^K\}$ *are simple. Then they satisfy*

$$|z_i - z_j| > h(\log(1/h))^{-1}/C \,, \;\; z_i \neq z_j \,, \quad z_i, z_j \in \mathcal{R}_{Q(h)} \cap \{z \; : \; \operatorname{Im} z < h^K\} \,.$$

Remark: A finer analysis of the approximate eigenfunctions of $Q^\sharp(h)$ would lead to further restrictions on the resonances close to the real axis and would lead to an improvement which however is irrelevant here.

Using the method of [9, Remark 3.2] we can find a *generic* a satisfying (5.1),(5.5),(5.10) for which the resonances of $Q(h)$ are all simple for $h^{-2} \in \sigma(-\Delta_{\mathbb{S}^{n-1}})$. Hence the assumptions of the following theorem are frequently satisfied.

Theorem 3. *Suppose* $P = -\Delta_g$ *for the metric* (5.1) *on* $\mathbb{R}^n$, *with* a *satisfying* (5.5),(5.10), *and such that the resonances of* $Q(h)$ *are simple for* $h^{-2} \in \sigma(-\Delta_{\mathbb{S}^{n-1}})$ *(a generic condition). Then the conclusions of Theorem 2 hold.*

Proof. We need to show that the conclusion of Proposition 4.2 holds. The space $\mathcal{H}$ has the standard decomposition

$$\mathcal{H} = L^2(\mathbb{R}^n, \sqrt{g}) = L^2(\mathbb{R}_+ \times \mathbb{S}^{n-1}, a(r)^{n-1} dr d\omega) = L^2(\mathbb{S}^{n-1}, d\omega) \otimes L^2(\mathbb{R}_+, a(r)^{n-1} dr) \,.$$

If $\pi_k : L^2(\mathbb{S}^{n-1}, d\omega) \to L^2(\mathbb{S}^{n-1}, d\omega)$ is the orthogonal projection associated to $\lambda_k \in \sigma(-\Delta_{\mathbb{S}^{n-1}})$, then the decomposition above gives us the associated orthogonal projection

$$\Pi_k \; : \; \mathcal{H} \longrightarrow \mathcal{H} \,, \;\; \Pi_k \Pi_l = \Pi_{\delta_{lk} k} \,,$$

and

$$R(z,h) = \sum_{k=0}^{\infty} \Pi_k R(z,h) \Pi_k \,.$$

The cut-off in Proposition 4.2 can be chosen to be radial and hence

$$\|\chi R(z,h) \chi\|_{\mathcal{H} \to \mathcal{H}} = \max_k \|\Pi_k \chi R(z,h) \chi \Pi_k\|_{\mathcal{H} \to \mathcal{H}} \,.$$

We now observe that

$$\|\Pi_k \chi R(z,h) \chi \Pi_k\|_{\mathcal{H} \to \mathcal{H}} = \|\chi(\lambda_k h^2)(Q(\lambda_k^{-\frac{1}{2}}) - z)^{-1} \chi\|_{L^2(\mathbb{R}_+, dr) \to L^2(\mathbb{R}_+, dr)} \,,$$

and we want a uniform bound on the resolvent for $z \in \Omega$ (by abuse of notation we denote by $(Q - z)^{-1}$ the meromorphic continuation of the resolvent). From (5.6) we see that the poles close to $\mathbb{R}$ can occur only when

$$\frac{1}{C_\Omega} < \lambda_k h^2 < C_\Omega \,.$$

In view of Proposition 5.1, the separation condition holds and we can apply the method of proof of Proposition 4.2 to the family of operators $\mu Q(h)$, $1/C_\Omega < \mu < C_\Omega$, to obtain a uniform bound

$$\|\chi(\lambda_k h^2)(Q(\lambda_k^{-\frac{1}{2}}) - z)^{-1} \chi\|_{L^2(\mathbb{R}_+, dr) \to L^2(\mathbb{R}_+, dr)} \leq C_\Omega h^{-N} \,,$$

(strickly speaking we have to use [31, Lemma 1] here, in place of Proposition 4.1, since $Q(h)$ is a semi-classical Schrödinger operator with a non-compactly supported potential).

When $\lambda_k h^2 \ll 1$ or $\lambda_k h^2 \gg 1$, we can use the same type of argument, simplified by the absence of resonances close to the real axis. This establishes the analogue of Proposition 4.2 and we can apply the proof of Theorem 2. $\square$

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