FRACTAL WEYL LAWS FOR QUANTUM RESONANCES

MACIEJ ZWORSKI

1. Introduction

We present results of recent work with Johannes Sjöstrand [18] on upper bounds of the number of semiclassical resonances for systems with chaotic classical dynamics. These upper bounds are interpreted as "fractal Weyl laws for resonances" since the exponent is now related to the dimension of the trapped set of the classical system. Despite some numerical evidence, for models based on partial differential equations there are no rigorous results showing that these bounds are optimal. However, recent joint work with Stéphane Nonnenmacher [14] shows that the bounds are optimal for some discrete models of chaotic scattering based on open quantum maps.

Here, some of the ideas of [18] are explained in detail by proving a simpler result about the number of (complex) eigenvalues of a chaotic potential with a complex absorbing barrier. That corresponds to a model popular in computational chemistry – see the work of Stefanov [19] for a recent mathematical treatment and references. The energy interval we consider has a fixed length, rather than the length Ch, which leads to further, more serious, simplifications.

Thus let $V \in \mathcal{C}^\infty_{\rm c}(\mathbb{R}^n; \mathbb{R})$ be supported in $B(0, R_0)$. The complex absorbing barrier is given by $W \in \mathcal{C}^\infty(\mathbb{R}^n)$, satisfying $W \geq 0$, $W = 0$ in $B(0, R_0)$, and $W > 1$ in $B(0, R_1)$, $R_1 > R_0$. We consider

$$P \stackrel{\rm def}{=} -h^2 \Delta + V(x) - iW(x)\,, \tag{1.1}$$

The absorbing barrier created by W is a model of infinity since it produces no reflection in semiclassical propagation. We say that the flow of the flow of H_p, $p = |\xi|^2 + V(x)$, is hyperbolic near energy E we mean it in the standard sense of (1.7), or the weaker sense given in §3.1.

Theorem. *Suppose that $P(h)$ is given by* (1.1) *with* $\operatorname{supp} V \subset B(0, R_0)$, *and assume that the classical flow near energy E is hyperbolic, and that the union of trapped sets* (1.6) *with energies $|\widetilde{E} - E| < 2\delta$ has upper Minkowski dimension m.*

Then for any $\widetilde{m} > m$, and $C_0 > 0$ there exists C_1 such that

$$|\operatorname{Spec}(P(h)) \cap [E - \delta, E + \delta| - i[0, C_0 h]| \leq C_1 h^{-\widetilde{m}/2}\,. \tag{1.2}$$

When the trapped is set is of pure dimension, *$\widetilde{m}$ can be replaced by m.*

To motivate this theorem and the results of [18] reviewed below we first recall well known results about discrete spectra of selfadjoint semiclassical operators. Thus, let $P = -h^2\Delta_g + V(x)$ be a self-adjoint Schrödinger operator on a compact Riemannian n-manifold, (X,g), $V \in \mathcal{C}^\infty(X;\mathbb{R})$. The spectral asymptotics as $h \to 0$ are given by the celebrated *Weyl law* – see [4] and [7] for recent advances and numerous references. If we assume that the zero energy surface is nondegenerate,

$$p \stackrel{\text{def}}{=} |\xi|_g^2 + V(x) = E \implies dp \neq 0\,,$$

then

$$|\mathrm{Spec(P)} \cap [E - Ch, E + Ch)| = \mathcal{O}(h^{-n+1})\,. \tag{1.3}$$

Let H_p be the Hamilton vector field of p on T^*X, locally given by

$$H_p = \sum_{j=1}^n \frac{\partial p}{\partial \xi_j}\partial_{x_j} - \frac{\partial p}{\partial x_j}\partial_{\xi_j}\,, \quad (x,\xi) \in T^*\mathbb{R}^n\,.$$

When the flow, $\exp tH_p \ :\ p^{-1}(E) \to p^{-1}(E)$, has the property that the set of its closed orbits has Liouville measure zero on $p = E$, then we have the infinitesimal version of the Weyl law:

$$|\mathrm{Spec(P)} \cap [E - Ch, E + Ch]| = \frac{2Ch}{(2\pi h)^n}\int_{p(x,\xi)=E} d\mathcal{L}(x,\xi) + o(h^{-n+1})\,, \tag{1.4}$$

where $d\mathcal{L}$ is the Liouville measure on $p = E$, that is $d\mathcal{L}dp = dxd\xi$. This result is the mathematical starting point of many recent investigations, mostly in physical literature, of the finer structure of the spectrum and its relation to classical dynamics – see [1] and references given there.

When the manifold is non-compact the situation is dramatically different. The simplest case is that of a manifold which is Euclidean outside of a compact set and $V \in \mathcal{C}^\infty_{\mathrm{c}}(X;\mathbb{R})$. The discrete eigenvalues of P are replaced by *quantum resonances* which are defined as the poles of the meromorphic contin uation of

$$(P - z)^{-1} \ :\ \mathcal{C}^\infty_{\mathrm{c}}(X) \longrightarrow \mathcal{C}^\infty(X)\,, \quad \operatorname{Im} z > 0\,,$$

and we denote the set of resonances by $\mathrm{Res}(P(h))$.

In [18] we provide upper bounds for the number of resonances for a much larger class of operators P in $D(0,Ch)$. The main result [18, Theorem 3] states that for classical Hamiltonians p with hyperbolic flow on $p = 0$ (see (1.7) and §3.1 below):

$$|\operatorname{Res} P(h) \cap D(0,Ch)| = \mathcal{O}(h^{-\nu})\,, \tag{1.5}$$

where $2\nu + 1$ is essentially the dimension of the trapped (non-wandering) set in $p^{-1}(0)$,

$$K_E \stackrel{\text{def}}{=} \{(x,\xi) \in T^*X \ :\ p(x,\xi) = E\,,\ \exp(tH_p)(x,\xi) \not\to \infty\,, \quad t \to \pm\infty\}\,. \tag{1.6}$$

In the case of a compact manifold $\nu = n-1$ so that (1.5) reduces to (1.3). By dimension we always mean the upper Minkowski dimension

$$m_0 = 2n - 1 - \sup\{d \ : \ \limsup_{\epsilon \to 0} \epsilon^{-d} \operatorname{vol}(\{\rho \in p^{-1}(0) \ : \ d(\rho, K) < \epsilon\}) < \infty\} .$$

A simple example is provided by a three bump potential shown in Fig.1.

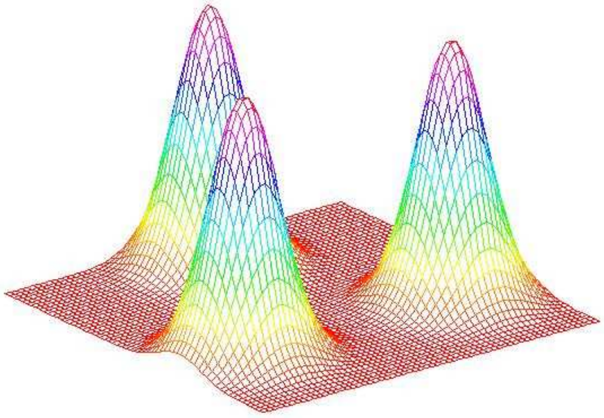

FIGURE 1. A three bump potential exhibiting hyperbolic dynamics at an interval of energies.

The basic hyperbolicity assumption at an energy E can be stated as follows: for $\rho \in p^{-1}(E)$ lying in a neighbourhoood of the trapped set K_E we have,

$$\begin{aligned}
&T_\rho(p^{-1}(E)) = \mathbb{R}H_p(\rho) \oplus E_+(\rho) \oplus E_-(\rho) \ , \quad \dim E_\pm(\rho) = n-1 \, , \\
&p^{-1}(E) \ni \rho \longmapsto E_\pm(\rho) \subset T_\rho(p^{-1}(E)) \ \text{ is continuous,} \\
&d(\exp tH_p)_\rho(E_\pm(\rho)) = E_\pm(\exp tH_p(\rho)) \, , \\
&\exists \, \lambda > 0 \quad \|d(\exp tH_p)_\rho(X)\| \leq Ce^{\pm \lambda t}\|X\| \, , \quad \text{for all } X \in E_\pm(\rho), \ \mp t \geq 0.
\end{aligned} \tag{1.7}$$

An example of a potential satisfying this assumption at a range of non-zero energies is given in Fig.1 – see [13] and [16, Appendix c]. Following [16] we will formulate a weaker dynamical hypothesis in §3.

The first estimate involving the dimension of K was proved by the first author in [16, Theorems 4.6, 5.5, and 5.7]: there exists constants $C_0, C_1 > 0$, such that for $\delta_0 > 0$ fixed and small enough

$$
\begin{gathered}
|\operatorname{Res}(P(h)) \cap \{z \;:\; |z| < \delta\,, \;\; \operatorname{Im} z > -\mu\}| \leq C_1 \delta \left(\frac{h}{\mu}\right)^{-n} \mu^{-\frac{1}{2}\widetilde{m}}\,, \\
C_0 h \leq \mu \leq 1/C_0\,, \;\; C_0 h^{\frac{1}{2}} \leq \delta \leq \delta_0\,, \;\; 0 < h < 1/C_0\,,
\end{gathered} \tag{1.8}
$$

where now $\widetilde{m}$ is any number greater than the dimension of the union of trapped set with energies $|\widetilde{E}-E| < 2\delta_0$. In homogeneous situations, such as for instance obstacle scattering, $\widetilde{m} = m+1$. When $\mu = C_0 h$, the improvement in (1.5) lies in allowing $\delta \simeq h$, which is the natural limit for this type of spectral estimates.

Earlier, non-geometric, bounds on the number of resonances (scattering poles) were obtained by Melrose [11],[12] and the second author [21],[22]. In the case of convex co-compact Schottky quotients (and any convex co-compact quotients in dimension two) the analogue of (1.5) was proved in [6] using zeta function techniques, improving earlier estimates of [23] the proof of which was largely based on [16]. These technique gave similar results for the zeros of zeta functions of rational maps [3],[20], in which case the dimension of the trapped set becomes essentially the dimension of the Julia set.

Numerical investigations in different settings of semiclassical three bump potentials [8],[9], Schottky quotients [6], three disc scattering [10], and Cantor-like Julia sets for $z \mapsto z^2 + c$, $c < -2$ [20], suggest that for $\mu \simeq Ch$ and $\delta \simeq 1$ the estimate (1.8) is optimal. A different model was recently considered in [14]: quantum resonances were defined using an open quantum map with a classical "trapped set" corresponding to K intersected with a hypersurface transversal to the flow. The numerical results and a simple linear algebraic toy model suggest that the fine estimate (1.5) is optimal. A similar model was also used in [15] where the fractal Weyl law gave corrections to the applications of random matrix theory to open quantum systems.

We should stress that the simplification provided in Theorem above avoids one of the more delicate aspects of [18]: second microlocalization with respect to a hypersurface in the $\mathcal{C}^\infty$ case. We refer to [18, §2] for an outline of the proof of (1.5).

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2. Preliminaries

In this section we present various results of semiclassical microlocal analysis needed in the proof of the theorem in §1. We provide proofs of all the results which cannot be in the standard reference [4].

2.1. **Review of semiclassical pseudodifferential calculus.** We recall the definition of semiclassical symbols on $\mathbb{R}^n$:

$$S^{m,k}(T^*\mathbb{R}^n) = \{a \in \mathcal{C}^\infty(T^*\mathbb{R}^n \times (0,1]) : |\partial_x^\alpha \partial_\xi^\beta a(x,\xi;h)| \leq C_{\alpha,\beta} h^{-m} \langle \xi \rangle^{k-|\beta|}\},$$

where outside X_0 we take the usual $\mathbb{R}^n$ coordinates in this definition. The corresponding class pseudodifferential operators is denoted by $\Psi_h^{m,k}(\mathbb{R}^n)$, and we have the usual Weyl quantization formula:

$$\mathrm{Op}_h^w(a)u(x) = \frac{1}{(2\pi h)^n} \int\int a\left(\frac{x+y}{2}, \xi\right) e^{i\langle x-y,\xi\rangle/h} u(y) dy d\xi \,,$$

and we refer to [4] for a detailed discussion. We remark only that when we consider the operators acting on half-densities we can define the surjective symbol map,

$$\sigma_h \; \Psi^{m,k}(\mathbb{R}^n) \; : \longrightarrow S^{m,k}(T^*\mathbb{R}^n)/S^{m-2,k-2}(T^*\mathbb{R}^n)\,,$$

see [17, Appendix]. We keep this in mind but for notational simplicity we supress the half-density notation.

For $a \in S^{m,k}(T^*\mathbb{R}^n)$ we define

$$\text{ess-supp}_h \; a \subset T^*\mathbb{R}^n \sqcup S^*\mathbb{R}^n \,, \quad S^*\mathbb{R}^n \stackrel{\text{def}}{=} (T^*\mathbb{R}^n \setminus 0)/\mathbb{R}_+ \,,$$

where the usual $\mathbb{R}_+$ action is given by multiplication on the fibers: $(x,\xi) \mapsto (x,t\xi)$, as

$$\begin{aligned} \text{ess-supp}_h \; a = \complement\{(x,\xi) \in T^*\mathbb{R}^n \; : \; \exists\, \epsilon > 0 \; \partial_x^\alpha \partial_\xi^\beta a(x',\xi') = \mathcal{O}(h^\infty)\,,\; d(x,x') + |\xi - \xi'| < \epsilon\} \\ \sqcup \; \complement\{(x,\xi) \in T^*\mathbb{R}^n \setminus 0 : \exists\, \epsilon > 0 \; \partial_x^\alpha \partial_\xi^\beta a(x',\xi') = \mathcal{O}(h^\infty \langle \xi' \rangle^{-\infty})\,, \\ d(x,x') + 1/|\xi'| + |\xi/|\xi| - \xi'/|\xi'|| < \epsilon\}/\mathbb{R}_+ \,, \end{aligned}$$

where the second complement is in $S^*\mathbb{R}^n$. For $A \in \Psi_h^{m,k}(\mathbb{R}^n)$, then define

$$\mathrm{WF}_h(A) = \text{ess-supp}_h \; a \,, \quad A = \mathrm{Op}_h^w(a)\,,$$

noting that, as usual, the definition does not depend on the choice of Op_h^w. For

$$u \in \mathcal{C}^\infty((0,1]_h; \mathcal{C}^\infty(\mathbb{R}^n))\,, \quad \forall\, K \Subset \mathbb{R}^n\,, N \in \mathbb{N} \;\exists\; P\,, h_0\,, \quad \|u\|_{C^N(K)} \leq h^{-P}\,, \; h < h_0\,.$$

we define

$$\mathrm{WF}_h(u) = \left(\bigcup \left\{\mathrm{WF}_h(A) \; : \; A \in \Psi^{0,0}(\mathbb{R}^n) \; : \quad Au \in h^\infty \mathcal{C}^\infty((0,1]_h; \mathcal{C}^\infty(\mathbb{R}^n))\right\}\right)^\complement,$$

where the complement is taken in $T^*\mathbb{R}^n \sqcup S^*\mathbb{R}^n$. Here we will be concerned with a purely semiclassical theory and deal only with *compact* subsets of $T^*\mathbb{R}^n$.

To illustrate the h-pseudodifferential calculus at work we prove two simple lemmas which will be used later. We say that $A \in \Psi^{m,k}(\mathbb{R}^n)$ is elliptic on $K \Subset T^*\mathbb{R}^n$ if $|\sigma(A)\restriction_K| > h^{-m}/C$.

Lemma 2.1. *Suppose $Q \in \Psi^{0,m}(\mathbb{R}^n)$ is elliptic at (x_0,ξ_0), $\|u\|_{L^2} = 1$, and $\mathrm{WF}_h(u)$ is contained in a sufficiently small neighbourhood of (x_0,ξ_0). Then for h small enough,*

$$\|Qu\|_{L^2} \geq 1/C\,.$$

Lemma 2.2. *Suppose that* $\psi_j \in \mathcal{C}^\infty_{\rm b}(T^*\mathbb{R}^n)$, $\psi_1^2 + \psi_2^2 = 1$, $\operatorname{supp}\psi_1 \subset \{(x,\xi) \; : \; |\xi| \leq C\}$. *Then, there exist* $\Psi_1 \in \Psi^{0,-\infty}(\mathbb{R}^n)$ *and* $\Psi_2 \in \Psi^{0,0}(\mathbb{R}^n)$, *with principal symbols* ψ_1 *and* ψ_2 *respectively, such that*

$$\Psi_1^2 + \Psi_1^2 = I + R \,, \quad R \in \Psi^{-\infty,-\infty}(\mathbb{R}^n) \,, \quad \Psi_j^* = \Psi_j \,.$$

Proof. Functional calculus gives

$$(\psi_1^w)^2 + (\psi_2^w)^2 = I + r_1^w \,, \quad r_1 \in S^{-1,-\infty}(T^*\mathbb{R}^n) \,,$$

in particular $r = \mathcal{O}(h) : H^{-M}(\mathbb{R}^n) \to H^M(\mathbb{R}^n)$. If h is small enough we put

$$\Psi_j^1 = (1 + r_1^w)^{-\frac14} \psi_j^w (1 + r_1^w)^{-\frac14} \,,$$

so that

$$(\Psi_1^1)^2 + (\Psi_2^1)^2 = I + r_2^w \,, \quad r_2 \in S^{-2,-\infty}(T^*\mathbb{R}^n) \,, \quad (\Psi_j^1)^* = \Psi_j^1 \,.$$

and we can then proceed by iteration. □

The semiclassical Sobolev spaces, $H^s_h(\mathbb{R}^n)$ are defined by choosing a globally elliptic, self-adjoint operator, $A \in \Psi^{0,1}(\mathbb{R}^n)$ (that is an operator satisfying $\sigma(A) \geq \langle \xi \rangle / C$ everywhere) and putting

$$\|u\|_{H^s_h} = \|A^s u\|_{L^2(\mathbb{R}^n)} \,.$$

When $\mathbb{R}^n = \mathbb{R}^n$,

$$\|u\|^2_{H^s_h} \sim \int_{\mathbb{R}^n} \langle h\xi \rangle^{2s} |\mathcal{F}u(\xi)|^2 d\xi \,, \quad \mathcal{F}u(\xi) \stackrel{\rm def}{=} \int_{\mathbb{R}^n} u(x) e^{-i\langle x, \xi\rangle} dx \,.$$

The following lemma will also be useful:

Lemma 2.3. *Suppose that* P_t, *is a family of operators such that*

$$P_t \; : \; H^s_h(\mathbb{R}^n) \longrightarrow H^{s-m}_h(\mathbb{R}^n) \,,$$
$$\forall \; A \in \Psi^{0,-\infty}(\mathbb{R}^n) \,, \quad \operatorname{ad}_{P_t} A = \mathcal{O}(h) \; : \; L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n) \,, \quad 0 < h < h_0(t), .$$

Let Ψ_j *be as in Lemma 2.2 and suppose that*

$$\|P_t \Psi_j u\| \geq th \|\Psi_j u\| - \mathcal{O}(h/t)\|u\| \,, \quad j = 1, 2 \,, \quad u \in \mathcal{C}^\infty_{\rm c}(\mathbb{R}^n) \,.$$

Here the constants in $\mathcal{O}$ *are independent of* h *and* t. *Then for* $t > t_0 \gg 1$ *and* $0 < h < h_0(t)$,

$$\|P_t u\| \geq th\|u\|/2 \,.$$

Proof. We recall from Lemma 2.2 that

$$\|\Psi_1 v\|^2 + \|\Psi_2 v\|^2 = \|v\|^2 + \langle Rv, v\rangle = \|v\|^2 + \mathcal{O}(h^\infty)\|v\|_{H_h^{-N}} \,, \tag{2.1}$$

and hence with $v = P_t u$,

$$
\begin{aligned}
\|P_t u\|^2 &= \|\Psi_1 P_t u\|^2 + \|\Psi_2 P_t u\|^2 - \mathcal{O}(h^\infty)\|u\|^2 \\
&\geq \|P_t \Psi_1 u\|^2 + \|P_t \Psi_2 u\|^2 - \|[\Psi_2, P_t]u\|^2 - \|[\Psi_2, P_t]u\|^2 \\
&\quad - \ 2\left(\|\Psi_1 P_t u\|\|[\Psi_1, P_t]u\|^2 + \|\Psi_2 P_t u\|\|[\Psi_2, P_t]u\|^2\right)^{\frac{1}{2}} - \mathcal{O}(h^\infty)\|u\|^2 \\
&\geq \|P_t \Psi_1 u\|^2 + \|P_t \Psi_2 u\|^2 \\
&\quad - 2C(\|[\Psi_1, P_t]u\|^2 + \|[\Psi_2, P_t]u\|^2) - \|P_t u\|^2/C - \mathcal{O}(h^\infty)\|u\|^2 \\
&\geq \|P_t \Psi_1 u\|^2 + \|P_t \Psi_2 u\|^2 - C' h^2 \|u\|^2 - \|P_t u\|^2/C \,.
\end{aligned}
$$

We now use the hypothesis of the lemma and (2.1) with $v = u$ to obtain

$$
\begin{aligned}
\|P_t u\|^2 &\geq t^2 h^2 (\|\Psi_1 u\|^2 + \|\Psi_2 u\|^2) - C' h^2 \|u\|^2 - \|P_t u\|^2/C \\
&\geq t^2 h^2 \|u\|^2 - C' h^2 \|u\|^2 - \|P_t u\|^2/C
\end{aligned}
$$

and the lemma follows. □

2.2. $S_{\frac{1}{2}}$ **spaces with two parameters.** We define the following symbol class:

$$
a \in S^{m,\widetilde{m},k}_{\frac{1}{2}}(T^*\mathbb{R}^n) \iff |\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq C_{\alpha\beta} h^{-m} \tilde{h}^{-\widetilde{m}} \left(\frac{\tilde{h}}{h}\right)^{\frac{1}{2}(|\alpha|+|\beta|)} \langle \xi \rangle^{k-|\beta|} \,, \tag{2.2}
$$

where in the notation we supress the dependence of a on h and $\tilde{h}$. We define the Weyl quantization of a in the usual way

$$
a^w(x, hD_x)u = \frac{1}{(2\pi h)^n} \int a\left(\frac{x+y}{2}, \xi\right) e^{\frac{i}{h}\langle x-y, \xi\rangle} u(y) dy d\xi \,,
$$

and the standard results (see [4]) show that if $a \in S^{m,\widetilde{m},k}_{\frac{1}{2}}(T^*\mathbb{R}^n)$ and $b \in S^{m',\widetilde{m}',k'}_{\frac{1}{2}}(T^*\mathbb{R}^n)$ then

$$
a(x, hD_x) \circ b(x, hD_x) = c(x, hD_x) \quad \text{with} \quad c \in S^{m+m', \widetilde{m}+\widetilde{m}', k+k'}_{\frac{1}{2}}(T^*\mathbb{R}^n) \,.
$$

The presence of the additional parameter $\tilde{h}$ allows us to conclude that

$$
c \equiv \sum_{|\alpha|<M} \frac{1}{\alpha!} \partial_\xi^\alpha a D_x^\alpha b \mod S^{m+m', \widetilde{m}+\widetilde{m}'-M, k+k'-M}_{\frac{1}{2}}(T^*\mathbb{R}^n) \,,
$$

that is, we have a symbolic expansion in powers of $\tilde{h}$. We could also consider an expansion in the Weyl quantization – see (2.4).

We denote our class of operators by $\Psi^{m,\widetilde{m},k}_{\frac{1}{2}}(T^*\mathbb{R}^n)$. For simplicity we will only state the characterization à la Beals for a simpler class of symbols:

Lemma 2.4. *Suppose that* $A : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$. *Then* $A = \mathrm{Op}_h^w(a)$ *with*

$$
\partial_x^\alpha \partial_\xi^\beta a = \mathcal{O}(h^{-m} \tilde{h}^{-\widetilde{m}}) \left(\frac{\tilde{h}}{h}\right)^{\frac{1}{2}(|\alpha|+|\beta|)} \,, \tag{2.3}
$$

if and only if for any sequence $\{\ell_j\}_{j=1}^N$ *of linear functions on* $T^*\mathbb{R}^n$ *we have*

$$\| \operatorname{ad}_{\operatorname{Op}_h^w(\ell_1)} \circ \cdots \operatorname{ad}_{\operatorname{Op}_h^w(\ell_N)} Au\|_{L^2(\mathbb{R}^n)} \leq Ch^{-m+N/2}\tilde h^{-\widetilde m+N/2}\|u\|_{L^2(\mathbb{R}^n)} \,,$$

for any $u \in \mathcal{S}(\mathbb{R}^n)$.

Proof. We can assume that $m = \widetilde m = 0$. The statement follows from the proof in [4, Chapter 8] and a rescaling:

$$(\tilde x, \tilde \xi) = (\tilde h/h)^{\frac12}(x,\xi)\,.$$

In fact, we define the following unitary operator on $L^2(\mathbb{R}^n)$:

$$U_{h,\tilde h}u(\tilde x) = (\tilde h/h)^{\frac n4}u((h/\tilde h)^{\frac12}\tilde x)\,,$$

for which we can check that

$$a(x,hD_x) = U_{h,\tilde h}^{-1}a_{h,\tilde h}(\tilde x, \tilde h D_{\tilde x})U_{h,\tilde h}\,, \quad a_{h,\tilde h}(\tilde x,\tilde\xi) = a((h/\tilde h)^{\frac12}(\tilde x,\tilde \xi))\,.$$

Clearly a satisfies (2.3) if and only if $a_{h,\tilde h} \in \mathcal{C}^\infty_{\rm b}(T^*\mathbb{R}^n)$. The Beals condition for $\tilde h$-pseudodifferential operators is

$$\| \operatorname{ad}_{\tilde\ell_1(\tilde x,\tilde hD_{\tilde x})} \circ \cdots \circ \operatorname{ad}_{\tilde\ell_N(\tilde x,\tilde hD_{\tilde x})} a_{h,\tilde h}(\tilde x,\tilde hD_{\tilde x})u\|_{L^2} \leq C\tilde h^N\|u\|_{L^2}\,.$$

But this is the condition in the lemma since we should take

$$\tilde\ell_j = (\ell_j)_{h,\tilde h} = (\tilde h/h)^{\frac12}\ell_j\,,$$

and this completes the proof. □

We will also need the following application of the semi-classical calculus:

Lemma 2.5. *Suppose that* $\partial^\alpha a$, $\partial^\alpha b = \mathcal{O}_\alpha((\tilde h/h)^{|\alpha|/2})$, *and that* $c^w(x,hD) = a^w(x,hD)\circ b^w(x,hD)$. *Then*

$$c(x,\xi) = \sum_{k=0}^N \frac{1}{k!}\left(\frac{ih}{2}\sigma(D_x,D_\xi;D_y,D_\eta)\right)^k a(x,\xi)b(y,\eta)\restriction_{x=y,\xi=\eta} + e_N(x,\xi)\,, \tag{2.4}$$

where for some M

$$\begin{gathered} |\partial^\alpha e_N| \leq C_N h^{N+1}\times \\ \sum_{\alpha_1+\alpha_2=\alpha} \sup_{\substack{(x,\xi)\in T^*\mathbb{R}^n\\(y,\eta)\in T^*\mathbb{R}^n}} \sup_{|\beta|\leq M,\beta\in\mathbb{N}^{2n}} \left|(h^{\frac12}\partial_{(x,\xi;y,\eta)})^\beta (i\sigma(D)/2)^{N+1}\partial^{\alpha_1}a(x,\xi)\partial^{\alpha_2}b(y,\eta)\right|\,, \end{gathered} \tag{2.5}$$

where $\sigma(D) = \sigma(D_x,D_\xi;D_y,D_\eta)$.

Proof. This follows from from the standard estimates of symbolic calculus (see [4, Proposition 7.6]): suppose that $A(D)$ is a non-degenerate real quadratic form. Then there exists M such that

$$|\partial^\alpha \exp(iA(D))a(x,\xi)| \leq C\sum_{|\beta|\leq M}\sup_{(x,\xi)\in T^*\mathbb{R}^n}|\partial^{\alpha+\beta}a(x,\xi)|\,.$$

We observe that a rescaling $\tilde{x} = x/\sqrt{s}$, $s > 0$, shows that

$$|\partial^\alpha \exp(isA(D))a(x,\xi)| \leq C \sum_{|\beta|\leq M} \sup_{(x,\xi)\in T^*\mathbb{R}^n} |\partial^\alpha(\sqrt{s}\partial)^\beta a(x,\xi)| \,.$$

To obtain an expansion we use the Taylor expansion:

$$\exp(ihA(D)) = \sum_{k=0}^N \frac{(ihA(D))^k}{k!} + \frac{1}{N!}\int_0^1 (1-t)^N \exp(ithA(D))(ihA(D))^{N+1} dt \,.$$

In the notation of the lemma and with $A(D) = \sigma(D_x, D_\xi; D_y, D_\eta)/2$,

$$c(x,\xi) = \exp(iA(D))a(x,\xi)b(y,\eta)\restriction_{x=y,\eta=\xi} \,,$$

and the lemma follows. □

As a particular consequence we notice that if $a \in S^{0,0,-\infty}_{\frac{1}{2}}(T^*\mathbb{R}^n)$ and $b \in S^{0,-\infty}(T^*\mathbb{R}^n)$ then

$$a^w(x,hD) \circ b^w(x,hD) = c^w(x,hD) \,, \quad c(x,\xi) =$$

$$\sum_{k=0}^N \frac{1}{k!} \left(ih\sigma(D_x, D_\xi; D_y, D_\eta)\right)^k a(x,\xi)b(y,\eta)\restriction_{x=y,\xi=\eta} + \mathcal{O}(h^{\frac{N+1}{2}}\tilde{h}^{\frac{N+1}{2}}) \,,$$

and the usual pseudodifferential calculus allows a remainder improvement to

$$\mathcal{O}(h^{\frac{N+1}{2}}\tilde{h}^{\frac{N+1}{2}}\langle\xi\rangle^{-\infty}) \,.$$

The following proposition will provide estimates on the number of eigenvalues:

Proposition 2.6. *Suppose that* $a \in S^{0,0,-\infty}_{\frac{1}{2}}(T^*\mathbb{R}^n)$ *and*

$$\operatorname{supp} a \subset W_{h,\tilde{h}} \,,$$

where $W_{h,\tilde{h}}$ *satisfies*

$$W_{h,\tilde{h}} \subset \bigcup_{k=1}^{K(h)} B_k \,, \quad \operatorname{diam} B_k \leq C_1 (h/\tilde{h})^{\frac{1}{2}} \,.$$

Then for $0 < h < h_0$, *there exists a finite rank operator* $R(h)$ *such that for*

$$\operatorname{Op}_h(a) - R(h) \in \Psi^{0,-\infty,}_{\frac{1}{2}}(\mathbb{R}^n) \,, \quad \operatorname{rank} R(h) = C_2 \tilde{h}^{-n} K(h) \,.$$

Proof. We take a partition of unity on $W_{h,\tilde{h}}$,

$$\sum_{k=1}^{K'(h)} \chi_k = 1 \quad \text{on } W_{h,\tilde{h}}, \quad \operatorname{supp} \chi_k \subset U_k \,, \quad \chi_k \in S^{0,0,-\infty,-\infty}_{\Sigma,\frac{1}{2}}(T^*\mathbb{R}^n) \,.$$

If $\psi = 1 - \sum_k \chi_k \in S^{0,0,-\infty}_{\frac12}$ then the condition on the support of a shows that, for all $\alpha, \beta \in \mathbb{N}^n$, $\partial^\alpha a \, \partial^\beta \psi \equiv 0$. Consequently, $\mathrm{Op}_h(\psi) A \in \Psi^{0,-\infty,-\infty}_{\frac12}(\mathbb{R}^n)$. Hence it suffices to show that for each k there exists an operator R_k such that

$$\mathrm{Op}_h(\chi_k) A - R_k \in \Psi^{0,-\infty,-\infty}_{\frac12}(\mathbb{R}^n) \,, \quad \mathrm{rank}(R_k) \leq C \tilde{h}^{-n} \,,$$

with C independent of k. By taking a finer cover of $W_{h,\tilde{h}}$ (with a number of elements $K''(h) \leq C'' K(h)$) we can assume that $\mathrm{Op}_h(\chi_k) A = \mathrm{Op}_h(a_k)$, where

$$\mathrm{supp}\, a_k \subset \{ (x, \xi) \ : \ |x - x_k| + |\xi - \xi_k| \leq C (h/\tilde{h})^{\frac12} \}$$

We then consider the following operators

$$Q_k = \mathrm{Op}_h(q_k) \,, \quad q_k = |x - x_k|^2 + |\xi - \xi_k|^2 \,.$$

If $\chi \in \mathcal{C}^\infty_{\rm c}(\mathbb{R})$, $\chi(t) = 1$ for $t \leq \widetilde{C}$, $\chi(t) = 0$ for $t > \widetilde{2C}$, then

$$\chi(\tilde{h} Q_k / h) A_k - A_k \in \Psi^{0,-\infty,-\infty}_{\frac12} \,.$$

The standard analysis of the spectrum of harmonic oscillators shows that $\chi(\tilde{h} Q_k / h)$ is a finite rank operator and its rank is bounded by $C' \tilde{h}^{-n}$. Hence we can take $R_k = \chi(\tilde{h} Q_k / h) A_k$. □

2.3. **One parameter groups of elliptic operators.** We recall a special case of a result of Bony and Chemin [2, Théoreme 6.4]. Let $m(x, \xi)$ be an order function in the sense of [4]:

$$m(x, \xi) \leq C m(y, \eta) \langle (x - y, \xi - \eta) \rangle^N \,. \tag{2.6}$$

The class of symbols, $S(m)$, corresponding to m is defined as

$$a \in S(m) \iff |\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq C_{\alpha\beta} m(x, \xi) \,.$$

If m_1 and m_2 are order functions in the sense of (2.6), and $a_j \in S(m_j)$ then (we put $h = 1$ here),

$$a_1^w(x, D) a_2^w(x, D) = b^w(x, D) \,, \quad b \in S(m_1 m_2) \,,$$

with b given by the usual formula,

$$\begin{aligned} b(x, \xi) &= a_1 \# a_2(x, \xi) \\ &\stackrel{\rm def}{=} \exp(i \sigma(D_{x^1}, D_{\xi^1}; D_{x^2}, D_{\xi^2})/2) a_1(x^1, \xi^1) a_2(x^2, \xi^2) \restriction_{x^1 = x^2 = x, \xi^1 = \xi^2 = \xi} \,. \end{aligned} \tag{2.7}$$

A special case of [2, Théoreme 6.4] gives

Proposition 2.7. *Let m be an order function in the sense of (2.6) and suppose that $G \in \mathcal{C}^\infty_{\rm c}(T^* \mathbb{R}^n; \mathbb{R})$ satisfies*

$$G(x, \xi) - \log m(x, \xi) = \mathcal{O}(1) \,, \quad \partial_x^\alpha \partial_\xi^\beta G(x, \xi) = \mathcal{O}(1) \,, \quad |\alpha| + |\beta| \geq 1 \,. \tag{2.8}$$

Then

$$\exp(tG^w(x,D)) = B_t^w(x,D)\,, \quad B_t \in S(m^t)\,. \tag{2.9}$$

Here $\exp(tG^w(x,D))$ *is constructed using spectral theory of bounded self-adjoint operators. The estimates on* $B_t \in S(m^t)$ *depend* only *on the constants in* (2.8) *and in* (2.6). *In particular they are independent of the support of* G.

Proof. □

The hypotheses on G in (2.8) are equivalent to the statement that $\exp(tG) \in S(m^t)$, for all $t \in \mathbb{R}$. We start with

Lemma 2.8. *Let* $U(t) \stackrel{\text{def}}{=} (\exp tG)^w(x,D) \;:\; \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$. *For* $|t| < \epsilon_0(G)$, *the operator* $U(t)$ *is invertible, and*

$$U(t)^{-1} = B_t^w(x,D)\,, \quad B_t \in S(m^{-t})\,.$$

Proof. We apply the composition formula (2.7) to obtain

$$U(-t)U(t) = Id + E_t^w(x,D)\,, \quad E_t \in S(1)\,.$$

More explicitely we write (see [4, Proposition 7.7] and Lemma 2.5 here)

$$\begin{aligned} E_t(x_1,\xi) &= \int_0^s e^{sA(D)} A(D) (e^{-tG(x_1,\xi_1)+tG(x_2,\xi_2)})\restriction_{x_2=x_1=x,\xi_2=\xi_1=\xi}\, ds \\ &= \int_0^s (it/2) e^{sA(D)} (D_{\xi_1} G D_{x_2} G - D_{x_1} G D_{\xi_2} G) e^{-tG(x_1,\xi_1)+tG(x_2,\xi_2)}\restriction_{x_2=x_1=x,\xi_2=\xi_1=\xi}\, ds\,, \end{aligned}$$

where $A(D) = i\sigma(D_{x_1}, D_{\xi_1}; D_{x_2}, D_{\xi_2})/2$.

Hence $E_t = t\widetilde{E}_t$ where $\widetilde{E}_t \in S(1)$ uniformly, and thus

$$E_t^w(x,D) = \mathcal{O}(t) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)\,.$$

This shows that for $|t|$ small enough $Id + E_t^w(x,D)$ is invertible, and Beals's lemma (see for instance [4, Proposition 8.3]) gives

$$(Id + E_t^w(x,D))^{-1} = C_t^w(x,D)\,, \quad C_t \in S(1)\,.$$

Hence $B_t = C_t \# \exp(-tG(x,\xi)) \in S(m^{-t})$. □

We now observe that

$$\begin{gathered} \frac{d}{dt}\left(U(-t)\exp(tG^w(x,D))\right) = V(t)\exp(tG^w(x,D))\,, \\ V(t) = A_t^w(x,D)\,, \quad A_t \in S(m^{-t})\,. \end{gathered} \tag{2.10}$$

In fact, we see that

$$\frac{d}{dt}U(-t) = -(G\exp(-tG))^w(x,D)\,, \quad U(-t)G^w(x,D) = (\exp(tG)\#G)^w(x,D)\,.$$

As before, the composition formula (2.7) gives

$$\exp(-tG)\#G - G\exp(-tG) =$$
$$\int_0^1 \exp(sA(D))A(D)\exp(-tG(x^1,\xi^1)G(x^2,\xi^2)\restriction_{x^1=x^2=x,\xi^1=\xi^2=\xi}\,,$$
$$A(D) = i\sigma(D_{x^1}, D_{\xi^1}; D_{x^2}, D_{\xi^2})/2\,.$$

The hypothesis on G shows that $A(D)\exp(tG(x^1,\xi^1))G(x^2,\xi^2)$ is a sum of terms of the form $a(x^1,\xi^1)b(x^2,\xi^2)$ where $a \in S(m^{-t})$ and $b \in S(1)$. The continuity of $\exp(A(D))$ on the spaces of symbols (see [4, Proposition 7.6]) gives (2.10).

If we put

$$C(t) \stackrel{\text{def}}{=} -V(t)U(-t)^{-1}\,,$$

then by Lemma 2.8, $C(t) = c_t^w$ where $c_t \in S(1)$. Symbolic calculus shows that c_t depends smoothly on t and

$$(\partial_t + C(t))(U(-t)\exp(tG^w(x,D))) = 0\,.$$

The proof of Proposition 2.7 is now reduced to showing

Lemma 2.9. *Suppose that $C(t) = c_t^w(x,D)$, where $c_t \in S(1)$, depends continuously on $t \in (-\epsilon_0, \epsilon_0)$. Then the solution of*

$$(\partial_t + C(t))Q(t) = 0\,, \quad Q(0) = q^w(x,D)\,, \quad q \in S(1)\,, \tag{2.11}$$

is given by $Q(t) = q_t(x,D)$, where $q_t \in S(1)$ depends continuously on $t \in (-\epsilon_0, \epsilon_0)$.

Proof. The Picard existence theorem for ODEs shows that $Q(t)$ is bounded on L^2. If $\ell_j(x,\xi)$ are linear functions on $T^*\mathbb{R}^n$ then

$$\frac{d}{dt}\operatorname{ad}_{\ell_1(x,D)} \circ \cdots \circ \operatorname{ad}_{\ell_N(x,D)} Q(t) + \operatorname{ad}_{\ell_1(x,D)} \circ \cdots \circ \operatorname{ad}_{\ell_N(x,D)}(C(t)Q(t)) = 0\,,$$
$$\operatorname{ad}_{\ell_1(x,D)} \circ \cdots \circ \operatorname{ad}_{\ell_N(x,D)} Q(0) \;:\; L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)\,.$$

If we show that for any choice of $\ell_j's$ and any N

$$\operatorname{ad}_{\ell_1(x,D)} \circ \cdots \circ \operatorname{ad}_{\ell_N(x,D)} Q(t) \;:\; L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)\,, \tag{2.12}$$

then Beals's lemma (see [4, Chapter 8]) concludes the proof. We proceed by induction on N:

$$\operatorname{ad}_{\ell_1(x,D)} \circ \cdots \circ \operatorname{ad}_{\ell_N(x,D)}(C(t)Q(t)) = C(t)\operatorname{ad}_{\ell_1(x,D)} \circ \cdots \circ \operatorname{ad}_{\ell_N(x,D)} Q(t) + R(t)\,,$$

where $R(t)$ is the sum of terms of the form

$$A_k(t)\operatorname{ad}_{\ell_1(x,D)} \circ \operatorname{ad}_{\ell_k(x,D)} Q(t)\,, \quad k < N\,, \quad A_k(t) = a_k(t)^w\,,$$

where $a_k(t) \in S(1)$ depend continuously on t (this statement can also be proved by induction using the derivation property of ad_ℓ: $\operatorname{ad}_\ell(CD) = (\operatorname{ad}_\ell C)D + C(\operatorname{ad}_\ell D)$). Hence by the induction hypothesis $R(t)$ is bounded on L^2, and depends continuously on t. Thus

$$\left(\frac{d}{dt}+C(t)\right)\mathrm{ad}_{\ell_1(x,D)}\circ\cdots\circ\mathrm{ad}_{\ell_N(x,D)}\,Q(t)=R(t)\;:\;L^2(\mathbb{R}^n)\;\longrightarrow\;L^2(\mathbb{R}^n)\,.$$

Since (2.12) is valid at $t=0$ we obtain it for all $t\in(-\epsilon_0,\epsilon_0)$. □

This proof comes from [18, Appendix]. We should stress that the main difficulties in [2] came from considering general Weyl calculi of pseudodifferential opearators. Here we need only the case of the simplest metric $g=dx^2+d\xi^2$.

3. The escape function for hyperbolic flows and its h dependent regularizations

In this section we modify [16, Sect.5] and construct a regularized escape function depending on a small parameter, essentially $h/\tilde h$. We assume that $p\in\mathcal{C}^\infty(T^*\mathbb{R}^n;\mathbb{R})$ satisfies

$$\begin{gathered}p=0\implies dp\neq 0\\ |x|\geq R\,,\quad |p(x,\xi)|<2\delta\implies \exp tH_p(x,\xi)\to\infty\ \text{ for either } t\to\infty \text{ or } t\to-\infty.\end{gathered}\tag{3.1}$$

In our case $p=\xi^2+V(x)-E$. We also recall the result of [5, Appendix]:

Proposition 3.1. *Suppose that* (3.1) *holds and that* $\widehat K$ *is the trapped set,*

$$\widehat K\stackrel{\text{def}}{=}\{\rho\in T^*\mathbb{R}^n\;:\;\exp(tH_p)(\rho)\not\to\infty\,,\;t\to\pm\infty\,,|p(\rho)|\leq\delta\}\Subset T^*\mathbb{R}^n\,.\tag{3.2}$$

Then for any two neighbourhoods, U,V*, of* $\widehat K$*,* $\overline U\subset V$ *there exists* $G_0\in\mathcal{C}^\infty(T^*\mathbb{R}^n)$ *such that*

$$\begin{gathered}\operatorname{supp}G_0\subset T^*\mathbb{R}^n\setminus U\,,\quad H_pG_0\geq 0\,,\quad H_pG\restriction_{p^{-1}([2\delta,2\delta])}\leq C\,,\\ H_pG_0\restriction_{p^{-1}([-\delta,\delta])\setminus V}\geq 1\,.\end{gathered}\tag{3.3}$$

3.1. **Dynamical assumptions.** We start with the hyperbolicity assumptions [16, §5] weaker than the more standard assumptions in §1. Let $\widehat K$ be the compact trapped set near zero energy given by (3.2). The trapped set at zero energy is given by $K=\widehat K\cap p^{-1}(0)$. We also have $\widehat K=\widehat\Gamma_+\cap\widehat\Gamma_-$, where

$$\widehat\Gamma_\pm\stackrel{\text{def}}{=}\{(x,\xi)\in T^*\mathbb{R}^n\;:\;|p(x,\xi)|\leq\delta\,,\quad\exp(tH_p)(x,\xi)\not\to\infty\,,\;t\to\mp\infty\}\,,\tag{3.4}$$

and the sets $\widehat K$, $\widehat\Gamma_\pm$ are clearly invariant under the flow,

$$\exp(tH_p)(\widehat K)\subset\widehat K\,,\quad\exp(tH_p)(\widehat\Gamma_\pm)\subset\widehat\Gamma_\pm\,.\tag{3.5}$$

We can now state the dynamical hypothesis.

- In a neighbourhood, Ω_{ρ_0} of any $\rho_0 \in K$,

$$\widehat{\Gamma}_\pm = \bigcup_{\rho \in \Omega_{\rho_0} \cap \widehat{\Gamma}_\pm} \widehat{\Gamma}_{\pm,\rho} \,, \quad \rho \in \widehat{\Gamma}_{\pm,\rho} \,,$$

$$\widehat{\Gamma}_{\pm,\rho} \cap \widehat{\Gamma}_{\pm,\rho'} = \emptyset \,, \quad \text{or} \quad \widehat{\Gamma}_{\pm,\rho} = \widehat{\Gamma}_{\pm,\rho'} \,.$$

- Each $\widehat{\Gamma}_{\pm,\rho}$ is a closed $\mathcal{C}^1$ manifold of dimension $n+d$, with $d \geq 0$ fixed, and the dependence

$$\Omega_{\rho_0} \cap \widehat{\Gamma}_\pm \ni \rho \longmapsto T_\rho \widehat{\Gamma}_{\pm,\rho}$$

is continuous.
- If $E^\pm_\rho \stackrel{\text{def}}{=} T_\rho \widehat{\Gamma}_{\pm,\rho}$, then $E^+_\rho + E^-_\rho = T_\rho p^{-1}(p(\rho)) \subset T_\rho(T^*\mathbb{R}^n)$, $\mathbb{R} H_p(\rho) \in E^\pm_\rho$, and

$$\|d(\exp tH_p)_\rho(X)\| \leq Ce^{\pm\lambda t}\|X\| \,, \quad \rho \in K \,, \quad \text{for all } X \in T_\rho(T^*\mathbb{R}^n)/E^\mp_\rho, \ \mp t \geq 0. \tag{3.6}$$

The above definition makes sense since by (3.5) $d(\exp tH_p)_\rho(E^\pm_\rho) = E_{\exp tH_p(\rho)}$, $\rho \in \widehat{\Gamma}_\pm$, we have

$$d(\exp tH_p)_\rho \; T_\rho(T^*\mathbb{R}^n)/E^\mp_\rho \longrightarrow T_{\exp tH_p(\rho)}(T^*\mathbb{R}^n)/E^\mp_{\exp tH_p(\rho)} \,, \quad \rho \in K \,,$$

and we choose continuously dependent norms in the last estimate in (3.6). We also note that $X \in T_\rho(T^*\mathbb{R}^n)/E^\mp_\rho$ implies that X can be identified with a vector tangent to $p^{-1}(p(\rho))$.

In [16, §5] it is shown that there exist two functions, $\varphi_\pm \in \mathcal{C}^{1,1}(T^*\mathbb{R}^n)$, $\varphi_\pm \geq 0$, $H_p^k \varphi_\pm \in \mathcal{C}^{1,1}(T^*\mathbb{R}^n)$, $k \in \mathbb{N}$, such that for ρ in a small neighbourhood of K,

$$\begin{gathered} \mp H_p \varphi_\pm(\rho) \sim \varphi_\pm(\rho) \,, \quad H_p^k \varphi_\pm(\rho) = \mathcal{O}(\varphi_\pm(\rho)) \,, \quad k \in \mathbb{N} \,, \,, \\ \varphi_\pm(\rho) \sim d(\rho, \widehat{\Gamma}_\pm) \,, \quad \varphi_+(\rho) + \varphi_-(\rho) \sim d(\rho, \widehat{K})^2 \,, \end{gathered}$$

and where $d(\bullet, \Gamma)$ is the distance to a closed set Γ. The notation $f \sim g$, means that there exists a constant $C > 0$ such

$$0 \leq g/C \leq f \leq Cg \,.$$

A local model for the simplest case of one trajectory is given by $p = \xi_1 + x_2\xi_2$, $(x,\xi) \in T^*\mathbb{R}^2$, so that

$$H_p = \partial_{x_1} + x_2 \partial_{x_2} - \xi_2 \partial_{\xi_2} \,, \quad \varphi_+ = \xi_2^2 \,, \quad \varphi_- = x_2^2 \,, \quad K = \{(t,0;0,0) \; : \; t \in \mathbb{R}\} \,. \tag{3.7}$$

3.2. **Regularization of $\varphi_\pm$.** We start with two general lemmas:

Lemma 3.2. *Suppose $\Gamma \subset \mathbb{R}^m$ is a closed set. For any $\epsilon > 0$ there exists $\varphi_\epsilon \in \mathcal{C}^\infty(\mathbb{R}^m)$ such that*

$$\varphi_\epsilon \geq \epsilon \,, \quad \varphi_\epsilon \sim d(\bullet, \Gamma)^2 + \epsilon \,, \quad \partial^\alpha \varphi_\epsilon = \mathcal{O}(\varphi_\epsilon^{1-|\alpha|/2}) \,,$$

uniformly on compact sets.

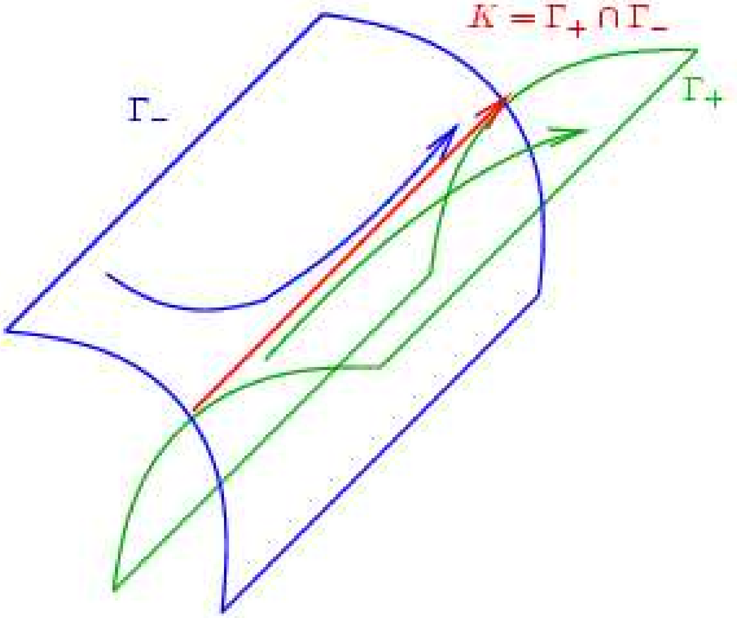

FIGURE 2. Outgoing and incoming sets in the case of one orbit in a three dimensional energy hypersurface.

Proof. We can find a sequence $x_j \in \mathbb{R}^m$ such that

$$\bigcup_j B(x_j, d(x_j, \Gamma)/8) = \mathbb{R}^m \setminus \Gamma \,,$$

every $x \in Q \setminus \Gamma$, $Q \Subset \mathbb{R}^m$, is in at most $N_0 = N_0(Q)$ balls $B(x_j, d(x_j, \Gamma)/2)$.

Let $\chi \in \mathcal{C}^\infty_{\rm c}(\mathbb{R}^m; [0, 1])$ be supported in $B(0, 1/4)$, and be identically one in $B(0, 1/8)$. We define

$$\varphi_\epsilon(x) \stackrel{\rm def}{=} \epsilon + \sum_{d(x_j,\Gamma)>\sqrt{\epsilon}} d(x_j, \Gamma)^2 \chi\left(\frac{x - x_j}{d(x_j, \Gamma) + \sqrt{\epsilon}}\right)$$

We first note that the number non-zero terms in the sum is uniformly bounded by N_0. In fact, $d(x_j, \Gamma) + \sqrt{\epsilon} < 2d(x_j, \Gamma)$, and hence if $\chi((x - x_j)/(d(x_j, \Gamma) + \sqrt{\epsilon})) \neq 0$ then

$$1/4 \geq |x - x_j|/(d(x_j, \Gamma) + \sqrt{\epsilon}) \geq (1/2)|x - x_j|/d(x_j, \Gamma) \,,$$

and $x \in B(x_j, d(x_j, d(x_j, \Gamma))/2)$. This shows that $\varphi_\epsilon(x) \leq 2N_0(\epsilon + d(x, \Gamma)^2)$, and

$$\partial^\alpha \varphi_\epsilon(x) = \mathcal{O}((d(x, \Gamma)^2 + \epsilon)^{1-|\alpha|/2}) \,,$$

uniformly on compact sets.

To see the lower bound on φ_ϵ we first consider the case when $d(x,\Gamma) \leq C\sqrt{\epsilon}$.

$$\varphi_\epsilon(x) \geq \epsilon \geq (\epsilon + d(x,\Gamma)^2)/C' .$$

If $d(x,\Gamma) > C\sqrt{\epsilon}$ then for at least one j, $\chi((x-x_j)/(d(x_j,\Gamma)+\sqrt{\epsilon})) = 1$ (since the balls $B(x_j, d(x_j,\Gamma)/8)$ cover the complement of Γ, and $\chi(t) = 1$ if $|t| \leq 1/8$). Thus

$$\varphi_\epsilon(x) \geq \epsilon + d(x_j,\Gamma)^2 \geq (\epsilon + d(x,\Gamma)^2)/C ,$$

which concludes the proof. □

For future use we also record the following

Lemma 3.3. *Suppose $\varphi \in \mathcal{C}^{1,1}(\mathbb{R}^m)$, $\varphi \geq 0$, and for a vectorfield $V \in \mathcal{C}^\infty(\mathbb{R}^m;\mathbb{R}^m)$, $V^k\varphi = \mathcal{O}(\varphi)$, $V^k\phi \in \mathcal{C}^{1,1}(\mathbb{R}^m)$, $k \in \mathbb{N}$. Then, uniformly on compact sets,*

$$dV^k\varphi = \mathcal{O}(\varphi^{\frac{1}{2}}) , \quad k \in \mathbb{N} .$$

Proof. For some $C > 0$ the $\mathcal{C}^{1,1}$ function $C\varphi - V^k\varphi$ is non-negative. Hence using the standard estimate based on Taylor's formula,

$$|d\varphi|^2 = \mathcal{O}(\varphi) , \quad |d(C\varphi - V^k\varphi)|^2 = \mathcal{O}(C\varphi - V^k\varphi) = \mathcal{O}(\varphi) .$$

The lemma follows. □

We now have

Proposition 3.4. *Let $\widehat{\Gamma}_\pm$ be given by (3.4). For any small $\epsilon > 0$ there exist functions $\widehat{\varphi}_\pm \in \mathcal{C}^\infty(T^*\mathbb{R}^n;[0,\infty))$ such that in a neighbourhood of $\widehat{K}$,*

$$\begin{aligned} \widehat{\varphi}_\pm(\rho) &\sim d(\rho,\widehat{\Gamma}_\pm)^2 + C\epsilon , \\ \mp H_p\widehat{\varphi}_\pm(\rho) + C\epsilon &\sim \widehat{\varphi}_\pm(\rho) , \\ \partial^\alpha H_p^k \widehat{\varphi}_\pm(\rho) &= \mathcal{O}(\widehat{\varphi}_\pm(\rho)^{1-|\alpha|/2}) , \quad k \in \mathbb{N} , \\ \widehat{\varphi}_+(\rho) + \widehat{\varphi}_-(\rho) &\sim d(\rho,\widehat{K})^2 + C\epsilon . \end{aligned} \tag{3.8}$$

Proof. We modify the arguments of [16, §5], roughly speaking, adding an $\mathcal{O}(\epsilon)$ error to all the estimates. Let $\varphi_\pm$ be the functions obtained using Lemma 3.2 with $\Gamma = \Gamma_\pm$. We now put

$$\begin{gathered} \widehat{\varphi}_\pm(\rho) \stackrel{\text{def}}{=} \int_{\mathbb{R}} g_T(t)\varphi_\pm(\exp tH_p(\rho))dt , \\ g_T \in \mathcal{C}^\infty_{\rm c}((-1,T+1)) , \quad \operatorname{supp} g_T' \subset [-1,1]\cup[T-1,T+1] , \\ g_T'\restriction_{[-1,1]} \geq 0 , \quad g_T'\restriction_{[T-1,T+1]} \leq 0 , \quad g_T'(0) = 1 , \quad g_T'(T) = -1 . \end{gathered}$$

To check (3.8) we note that, by definition, $\varphi_\pm(\rho) \sim d(\rho, \widehat\Gamma_\pm)^2 + C\epsilon$. The assumptions (3.6) imply (see [16, Lemma 5.2]) that

$$\exists\, C\,,\ \forall\, T \geq 0\,,\ \exists\, \Omega_T \supset K\,, \text{an open set}, \quad d(\exp(\pm T H_p)(\rho), \widehat\Gamma_\pm) \leq C e^{-T/C} d(\rho, \widehat\Gamma_\pm)\,.$$

Hence, with constants depending on T,

$$\begin{aligned} \widehat\varphi_+(\rho) &\sim \varphi_+(\exp(TH_p)(\rho)) \sim \varphi_+(\rho) \sim d(\rho, \Gamma_+)^2 + C\epsilon\,, \\ \widehat\varphi_-(\rho) &\sim \varphi_-(\rho) \sim d(\rho, \Gamma_-)^2 + C\epsilon\,. \end{aligned}$$

This shows the first statement in (3.8).

The assumptions on g_T also show that

$$H_p \widehat\varphi_\pm(\rho) \sim \varphi_\pm(\exp TH_p(\rho)) - \varphi_\pm(\rho) \sim d(\exp TH_p(\rho), \widehat\Gamma_\pm)^2 - d(\rho, \widehat\Gamma_\pm)^2 + \mathcal{O}(\epsilon)\,.$$

so that for T large enough and for ρ in a small neighbourhood of K, (again with T depenendent constants)

$$\mp H_p \widehat\varphi_\pm(\rho) + C\epsilon \sim d(\rho, \widehat\Gamma_\pm)^2 + C'\epsilon \sim \widehat\varphi_\pm(\rho)\,.$$

This proves the second part of (3.8). The third part is proved using Lemma 3.3 for $|\alpha| = 1$ and the estimates on $\varphi_\pm$ in general.

To prove the last statement in (3.8) we first see that the transversality, $E^+_{\rho_0} + E^-_{\rho_0} = T_{\rho_0}(T^*\mathbb{R}^n)$, and the continuity, $\rho \mapsto E^\pm_\rho$, assumed in (3.6) imply that for $\rho\,, \rho_1\,, \rho_2\,,$ near a point $\rho_0 \in K$,

$$d(\rho, \widehat\Gamma_{+,\rho_1} \cap \widehat\Gamma_{-,\rho_2}) \sim d(\rho, \widehat\Gamma_{+,\rho_1}) + d(\rho, \widehat\Gamma_{-,\rho_2})\,.$$

Hence

$$\begin{aligned} \widehat\varphi_+(\rho) + \widehat\varphi_-(\rho) + \mathcal{O}(\epsilon) &\sim d(\rho, \widehat\Gamma_+)^2 + d(\rho, \widehat\Gamma_-)^2 + C\epsilon \\ &\leq d(\rho, \widehat\Gamma_{+,\rho'})^2 + d(\rho, \widehat\Gamma_{-,\rho'})^2 + C\epsilon \\ &\sim d(\rho, \widehat\Gamma_{+,\rho'} \cap \widehat\Gamma_{-,\rho'})^2 + C\epsilon\,. \end{aligned}$$

If we choose $\rho' \in K$ so that $d(\rho, \widehat K) = d(\rho, \rho')$ then

$$d(\rho, \widehat\Gamma_{+,\rho'} \cap \widehat\Gamma_{-,\rho'})^2 \leq d(\rho, \rho')^2 = d(\rho, \widehat K)^2\,,$$

proving that

$$\widehat\varphi_+(\rho) + \widehat\varphi_-(\rho) \leq d(\rho, \widehat K)^2 + \mathcal{O}(\epsilon)\,.$$

The opposite inequality is obtained by choosing $\rho_\pm \in \widehat\Gamma_\pm$ such that $d(\rho, \rho_\pm) = d(\rho, \widehat\Gamma_\pm)$. Then using the transversality of $\widehat\Gamma_+$, $\widehat\Gamma_-$

$$\begin{aligned} d(\rho, \widehat K)^2 &\leq d(\rho, \widehat\Gamma_{+,\rho_+} \cap \widehat\Gamma_{-,\rho_-})^2 \sim d(\rho, \widehat\Gamma_{+,\rho_+})^2 + d(\rho, \widehat\Gamma_{-,\rho_-})^2 \\ &\leq d(\rho, \rho_+)^2 + d(\rho, \rho_-)^2 = d(\rho, \widehat\Gamma_+)^2 + d(\rho, \widehat\Gamma_-)^2 \\ &\leq \widehat\varphi_+(\rho) + \widehat\varphi_-(\rho) + \mathcal{O}(\epsilon)\,. \end{aligned}$$

□

3.3. **Regularized escape function.** We now use the functions constructed in Proposition 3.4 to obtain an escape function near K. We first need the following

Lemma 3.5. *Then for $|\alpha| + k \geq 1$ we have*

$$\partial_\rho^\alpha H_p^k \log(\widehat{\varphi}_\pm) = \mathcal{O}(\widehat{\varphi}_\pm^{-\frac{|\alpha|}{2}})$$

Proof. Let $f(t) = \log(t)$. Then

$$f^{(k)}(\widehat{\varphi}_\pm) = \mathcal{O}\left(\frac{1}{\widehat{\varphi}_\pm^k}\right), \quad k \geq 1,$$

and for $|\alpha| + k \geq 1$, $\partial_\rho^\alpha H_p^k f(\widehat{\varphi}_\pm)$ is a finite linear combination of terms

$$f^{(l)}(\widehat{\varphi}_\pm)\left(\partial_\rho^{\alpha_1} H_p^{k_1} \widehat{\varphi}_\pm\right)\cdots\left(\partial_\rho^{\alpha_\ell} H_p^{k_\ell} \widehat{\varphi}_\pm\right) = \mathcal{O}(1)\prod_{j=1}^{\ell} \frac{\partial_\rho^{\alpha_j} H_p^{k_j} \widehat{\varphi}_\pm}{\widehat{\varphi}_\pm},$$

with

$$|\alpha_j| + k_j \geq 1, \quad \alpha_1 + \cdots + \alpha_\ell = \alpha, \quad k_1 + \cdots + k_\ell = k.$$

The estimates in (3.8) show that $\partial_\rho^{\alpha_j} H_p^{k_j} \widehat{\varphi}_\pm / \widehat{\varphi}_\pm = \mathcal{O}(\widehat{\varphi}_\pm^{-|\alpha_j|/2})$, and hence

$$\partial_\rho^\alpha H_p^k f(\widehat{\varphi}_\pm) = \mathcal{O}(\widehat{\varphi}_\pm^{-\frac{|\alpha|}{2}}),$$

proving the lemma. □

We are now ready for the main results of this section.

Lemma 3.6. *Let $\widehat{\varphi}_\pm$ be given in Proposition 3.4 and*

$$\widehat{G} \stackrel{\text{def}}{=} \left(\log(M\epsilon + \widehat{\varphi}_-) - \log(M\epsilon + \widehat{\varphi}_+)\right). \tag{3.9}$$

Then in a neighbourhood of K we have

$$\begin{gathered} \partial_\rho^\alpha H_p^k \widehat{G} = \mathcal{O}_M(\min(\widehat{\varphi}_+, \widehat{\varphi}_-)^{-\frac{|\alpha|}{2}}) = \mathcal{O}_M(\epsilon^{-\frac{|\alpha|}{2}}), \quad |\alpha| + k \geq 1, \\ d(\rho, \widehat{K})^2 \geq C\epsilon \Longrightarrow \ H_p \widehat{G} \geq 1/C, \end{gathered} \tag{3.10}$$

where, for the second estimate, M has to be chosen large enough, independently of ϵ, and C is a large constant.

Proof. We observe that, with constants depending on M, $\widehat{\varphi}_\pm + M\epsilon$ has the same properties as $\widehat{\varphi}_\pm$. Hence the estimates on $\partial_\rho^\alpha H_p^k \widehat{G}$ follow directly from the definition (3.9) and from Lemma 3.5. To check the second part of (3.10) we compute, using Proposition 3.4,

$$H_p \widehat{G} = \left(\frac{H_p \widehat{\varphi}_-}{\widehat{\varphi}_- + M\epsilon} - \frac{H_p \widehat{\varphi}_+}{\widehat{\varphi}_+ + M\epsilon}\right) \geq \frac{1}{C_1}\left(\frac{\widehat{\varphi}_- - C_2\epsilon}{\widehat{\varphi}_- + M\epsilon} + \frac{\widehat{\varphi}_+ - C_2\epsilon}{\widehat{\varphi}_+ + M\epsilon}\right).$$

From (3.8) we also have

$$d(\rho, \widehat{K})^2 \geq C\epsilon \implies \max(\widehat{\varphi}_+, \widehat{\varphi}_-) \geq (C/2 - \mathcal{O}(1))\epsilon > C_3\epsilon,$$

where C_3 can be as large as we like depending on the choice of C. Hence, since $x \mapsto (x - C_2)/(x + M)$ is increasing,

$$H_p \widehat{G} \geq \frac{1}{C_1} \left(\frac{C_3 - C_2}{C_3 + M} - \frac{C_2}{M} \right) \geq \frac{1}{C} ,$$

if we choose $C_3 \gg M \gg C_2$. □

We now modify $\widehat{G}$ using G_0 given in Proposition 3.1:

Proposition 3.7. *Let us fix $\delta_0 > 0$. Then there exist $\widehat{\chi}, \chi_0 \in \mathcal{C}^\infty_{\rm c}(T^*\mathbb{R}^n)$, $C_0 > 0$, and a neighbourhoood V of K, such that*

$$G \stackrel{\rm def}{=} \widehat{\chi}\widehat{G} + C_0 \left(\log \frac{1}{\epsilon} \right) \chi_0 G_0 ,$$

satisfies

$$\begin{gathered} \partial^\alpha H_p^k G = \left\{ \begin{array}{ll} \mathcal{O}(\log(1/\epsilon)) & \alpha = 0 \\ \mathcal{O}(\epsilon^{-|\alpha|/2}) & \textit{otherwise} \end{array} \right. , \\ d(\rho, \widehat{K})^2 \geq C\epsilon , \ \rho \in V \implies H_p G(\rho) \geq 1/C , \\ \rho \in p^{-1}([-\delta, \delta]) \setminus V , \quad |x(\rho)| \leq 3R_0 \implies H_p G(\rho) \geq \log(1/\epsilon) , \\ H_p G(\rho) \geq -\delta_0 \log(1/\epsilon) , \quad \rho \in T^*\mathbb{R}^n . \end{gathered} \tag{3.11}$$

In addition we have

$$\frac{\exp G(\rho)}{\exp G(\mu)} \leq C_0 \left\langle \frac{\rho - \mu}{\sqrt{\epsilon}} \right\rangle^{N_0} , \tag{3.12}$$

for some constants C_0 and N_0.

Proof. We obtain G_0 from Proposition 3.1 taking for V a neighbourhood of $\widehat{K}$ in which the estimates of Lemma 3.6 hold. We have $\partial^\alpha H_p^k G_0 = \mathcal{O}_{k,|\alpha|}(1)$, and consequently for any $\chi_0 \in \mathcal{C}^\infty_{\rm c}(T^*\mathbb{R}^n)$,

$$\partial^\alpha H_p^k \left(\log(1/\epsilon)\chi_0 G_0\right) = \mathcal{O}_{k,|\alpha|}(\log(1/\epsilon)) = \left\{ \begin{array}{ll} \mathcal{O}(\log(1/\epsilon)) & \alpha = 0 \\ \mathcal{O}(\epsilon^{-|\alpha|/2}) & \text{otherwise} \end{array} \right. .$$

From Lemma 3.6 we obtain, again for any $\widehat{\chi} \in \mathcal{C}^\infty_{\rm c}(T^*\mathbb{R}^n)$,

$$\partial^\alpha H_p^k (\widehat{\chi}\widehat{G}) = \left\{ \begin{array}{ll} \mathcal{O}(\log(1/\epsilon)) & \alpha = 0 \\ \mathcal{O}(\epsilon^{-|\alpha|/2}) & \text{otherwise} \end{array} \right. .$$

The loss compared to (3.10) is due to the presence of the cut-off function.

We take $\chi_0 \in \mathcal{C}^\infty_{\rm c}(T^*\mathbb{R}^n; [0,1])$ to be identically equal to 1 in

$$p^{-1}([-\delta, \delta]) \cap \{(x, \xi) \ : \ |x| \leq 3R_0\} .$$

For $\widehat{\chi} \in \mathcal{C}^\infty_{\rm c}(T^*\mathbb{R}^n)$ we take a function which is supported in a neighbhourhood of $\widehat{K}$ where (3.10) holds, and identically 1 in V. Hence for $\rho \in p^{-1}([-\delta,\delta]) \setminus V$, $|x(\rho)| \leq 3R_0$,

$$H_pG(\rho) = C_0 \log(1/\epsilon) H_p G_0(\rho) + H_p(\widehat{\chi}\widehat{G})(\rho) \geq C_0\log(1/\epsilon) - \mathcal{O}(1)\log(1/\epsilon) \geq \log(1/\epsilon)\,,$$

if C_0 is taken large enough. For $\rho \in V$, $\widehat{\chi}(\rho) = 1$, and

$$H_pG(\rho) = C_0\log(1/\epsilon)H_pG_0(\rho) + H_p\widehat{G}(\rho) \geq H_p\widehat{G}(\rho)\,,$$

and if $d(\rho, \widehat{K}) \geq C\epsilon$, $H_pG(\rho) \geq 1/C$. To complete the proof of (3.11) we need to define χ_0 for $|x| \geq R_0$. Let T and R be large positive constants to be fixed later and let $\chi(t)$ satisfy

$$\chi(t) = \begin{cases} 0 & |t| > T \\ t & |t| < \alpha T \end{cases} \,, \quad \chi'(t) \geq -2\alpha\,,$$

where α can be chosen anywhere in $(0, 1/2)$. It can be easily obtained by regularizing the piecewise linear function

$$\chi_\#(t) = \begin{cases} 0 & |t| > T \\ t & |t| < \alpha T \\ \pm\alpha(T-t)/(1-\alpha) & \alpha T \leq \pm t \leq T \end{cases} \,.$$

Finally, let $\psi \in \mathcal{C}^\infty_{\rm c}(\mathbb{R};[0,1])$ be equal to 1 for $|t| \leq 1$, and to 0 for $|t| \geq 2$. We define

$$\chi_0(\rho) \stackrel{\rm def}{=} \frac{\chi(G_0(\rho))}{G_0(\rho)} \psi\left(\frac{p(\rho)}{\delta}\right)\psi\left(\frac{|x(\rho)|}{R}\right)\,.$$

Then

$$\begin{aligned} H_p(\chi_0 G_0)(\rho) = {} & \chi'(G_0(\rho))H_pG_0(\rho)\psi\left(\frac{p(\rho)}{\delta}\right)\psi\left(\frac{|x(\rho)|}{R}\right) \\ & + \frac{1}{R}\chi(G_0(\rho))\psi\left(\frac{p(\rho)}{\delta}\right)\psi'\left(\frac{|x(\rho)|}{R}\right)H_p(|x|)(\rho)\,, \end{aligned}$$

and

$$H_p(\chi_0G_0)(\rho) \geq -C_1\left(\alpha + \frac{T}{R}\right)\,,$$

where C_1 is independent of T and R: we note that (3.3) guarantees the boundedness of H_pG_0, and the assumptions on p imply that $H_p(|x|)$ is uniformly bounded for $|p| \leq 2\delta$. For any $\alpha > 0$ we can choose $T = T(\alpha)$ such that $|G_0(\rho)| \leq \alpha T$ for $|x(\rho)| \leq 3R_0$, $|p(\rho)| \leq 2\delta$. We then choose α and R so that

$$C_0C_1(\alpha + T(\alpha)/R) < \delta_0\,.$$

Hence for $|x(\rho)| \geq R_0$

$$H_pG = C_0\log(1/\epsilon)H_p(\chi_0G_0) \geq -\delta_0\log(1/\epsilon)\,,$$

which is the last statement in (3.11).

It remains to show (3.12) and for simplicity of presentation we replace $T^*\mathbb{R}^n$ with $\mathbb{R}^{2n}$. We first prove that

$$\frac{\widehat{\varphi}_\pm(\rho) + M\epsilon}{\widehat{\varphi}_\pm(\mu) + M\epsilon} \leq C_1 \left\langle \frac{\rho - \mu}{\sqrt{\epsilon}} \right\rangle^2 , \quad M \geq 0 , \tag{3.13}$$

with constants depending on M. We can replace $\widehat{\varphi}_\pm + M\epsilon$ with $\widehat{\varphi}_\pm$, as $\widehat{\varphi}_\pm + M\epsilon \sim_M \widehat{\varphi}_\pm$. Thus we claim that,

$$\frac{\widehat{\varphi}_\pm(\rho)}{\widehat{\varphi}_\pm(\mu)} \leq C_1 \left\langle \frac{\rho - \mu}{\sqrt{\epsilon}} \right\rangle^2 .$$

Since $\widehat{\varphi}_\pm \sim d(\bullet, \Gamma_\pm)^2 + \epsilon$, $\widehat{\varphi}_\pm \geq \epsilon$, we have

$$\begin{aligned}\widehat{\varphi}_\pm(\rho) &\leq C(d(\rho, \Gamma_\pm)^2 + \epsilon) \leq C(d(\mu, \Gamma_\pm)^2 + |\mu - \rho|^2 + \epsilon) \\ &\leq C'(\widehat{\varphi}_\pm(\mu) + |\mu - \rho|^2) = C'(\widehat{\varphi}_\pm(\mu) + \epsilon \langle (\rho - \mu)/\sqrt{\epsilon} \rangle^2) \\ &\leq 2C' \widehat{\varphi}_\pm(\mu) \langle (\rho - \mu)/\sqrt{\epsilon} \rangle^2 .\end{aligned}$$

In the notation of Lemma 3.6, (3.13) gives

$$|\widehat{G}(\rho) - \widehat{G}(\mu)| \leq C + 2 \log \langle (\rho - \mu)/\sqrt{\epsilon} \rangle ,$$

and with $\widehat{\chi} \in \mathcal{C}^\infty_{\rm c}$,

$$|\widehat{\chi}(\rho)\widehat{G}(\rho) - \widehat{\chi}(\mu)\widehat{G}(\mu)| \leq C|\rho - \mu| \log(1/\epsilon) + C \log \langle (\rho - \mu)/\sqrt{\epsilon} \rangle .$$

Clearly,

$$|\chi_0(\rho) G_0(\rho) - \chi_0(\mu) G_0(\mu)| \leq C|\rho - \mu| \log(1/\epsilon) ,$$

and hence to obtain (3.12) we need

$$|\rho - \mu| \log(1/\epsilon) \leq C \log \langle (\rho - \mu)/\sqrt{\epsilon} \rangle + C , \quad \rho, \mu \in Q \Subset \mathbb{R}^{2n} .$$

If we put $\delta = \sqrt{\epsilon}$, $t = |\rho - \mu|/(C\delta)$ this becomes

$$\delta \log \frac{1}{\delta} \leq \frac{\log\langle t \rangle + 1}{t} , \quad 0 \leq t \leq \frac{1}{\delta} ,$$

and that is clear as $t \mapsto (\log\langle t \rangle + 1)/t$ is decreasing. □

4. Proof of the main result

Let G be the escape function given in Proposition 3.7, $\epsilon = h/\tilde{h}$ and let G^w be its Weyl quantization,

$$G^w = \mathcal{O}(\log(\tilde{h}/h)) \; : \; L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n) .$$

We use the notation of the previous section and write

$$p(x, \xi) = \xi^2 + V(x) ,$$

the real part of the symbol of P. We define a family of conjugated operators:

$$P_t \stackrel{\rm def}{=} e^{-tG^w} P e^{tG^w} . \tag{4.1}$$

It is easy to see that, in the notation of §2.2,

$$\exp(tG^w) \in \Psi^{|t|C,0,0}_{\frac{1}{2}}(\mathbb{R}^n)\,, \tag{4.2}$$

that is $\exp(tG^w) = B_t^w$, $\partial^\alpha B_t = \mathcal{O}(h^{-|t|C-|\alpha|/2}\tilde{h}^{|\alpha|/2})$. Finer estimates are however possible thanks to the results of Bony-Chemin [2]. The first of these is given in

Lemma 4.1. *Suppose that* $Q \in \Psi^{0,0,0}_{\frac{1}{2}}(\mathbb{R}^n)$*. Then*

$$\exp(-tG^w)Q\exp(tG^w) \in \Psi^{0,0,0}_{\frac{1}{2}}(\mathbb{R}^n)\,. \tag{4.3}$$

Proof. We follow §2.2 and change to the variables

$$\begin{gathered}(\tilde{x},\tilde{\xi}) = (\tilde{h}/h)^{\frac{1}{2}}(x,\xi)\,,\\ \widetilde{G}(\tilde{x},\tilde{\xi}) = G(x,\xi)\,, \quad \widetilde{Q}_t(\tilde{x},\tilde{\xi}) = Q_t(x,\xi)\,,\\ U^{-1}G^w(x,hD)U = \widetilde{G}^w(\tilde{x},\tilde{h}D_{\tilde{x}})\,, \quad U^{-1}Q_t^w(x,hD)U = \widetilde{Q}_t^w(\tilde{x},\tilde{h}D_{\tilde{x}})\,,\\ Uv(\tilde{x}) = (\tilde{h}/h)^{\frac{n}{4}}v((h/\tilde{h})^{\frac{1}{2}}\tilde{x})\,.\end{gathered}$$

We also note that

$$R \in \Psi^{0,0,0}_{\frac{1}{2}}(\mathbb{R}^n) \iff U^{-1}RU \in \Psi^{0,0}(\mathbb{R}^n)\,,$$

where on the right, $\tilde{h}$ is the small parameter – see the proof of Lemma 2.4. The estimate (3.12) shows that, in $(\tilde{x},\tilde{\xi})$ coordinates, $\widetilde{G}$ satisfies the hypothesis of Proposition 2.7 and that proves (4.3). □

The basic properties of P_t are given in

Proposition 4.2. *Let* P_t *be given by* (4.1) *and let* $\Sigma \Subset T^*\mathbb{R}^n$ *be a compact surface coinciding with* $p^{-1}(0)$ *in a neighbourhood of the support of* G*. Then for* $|t| \leq C$*,* $P_t \in \Psi^{0,0,2}_{\frac{1}{2}}$*, and more precisely*

$$P_t = P - ith\mathrm{Op}_h^w(H_pG) + E_t\,, \quad E_t \in \Psi^{-1,-1,0}_{\frac{1}{2}}(\mathbb{R}^n)\,, \tag{4.4}$$

$E_t = \mathcal{O}(h\tilde{h}) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$*, uniformly in* h *and* $\tilde{h}$*.*

Proof. Let V_1, V_2 be open neighbourhoods of $\operatorname{supp} G$,

$$\operatorname{supp} G \subset V_1 \Subset \overline{V}_2 \Subset T^*\mathbb{R}^n\,.$$

We first observe that if $\Psi \in \Psi^{0,-\infty}(\mathbb{R}^n)$ satisfies

$$\mathrm{WF}_h(\Psi) \subset V_2\,, \quad \mathrm{WF}_h(I-\Psi) \subset \complement V_1\,,$$

then

$$[\exp(tG^w),\Psi] \in \Psi^{-\infty,-\infty}(\mathbb{R}^n)\,, \quad (I-\Psi)(\exp(tG^w)-I) \in \Psi^{-\infty,-\infty}(\mathbb{R}^n)\,, \quad |t| \leq 1\,. \tag{4.5}$$

In fact, using the calculus in §2.2 we see that $[G^w, \Psi] \in \Psi^{-\infty,-\infty}(\mathbb{R}^n)$, Hence, using (4.2)

$$\frac{d}{dt}[\exp(tG^w), \Psi] = G^w[\exp(tG^w), \Psi] + [G^w, \Psi]\exp(tG^w)$$
$$= G^w[\exp(tG^w), \Psi] + A_t\,, \quad A_t \in \Psi^{-\infty,-\infty}(\mathbb{R}^n)\,.$$

Thus

$$[\exp(tG^w), \Psi] = \int_0^t \exp((t-s)G^w)A_s ds \in \Psi^{-\infty,-\infty}(\mathbb{R}^n)\,,$$

which is the first statement in (4.5). We also compute

$$\frac{d}{dt}(I-\Psi)(\exp(tG^w) - I) = (I-\Psi)G^w\exp(tG^w) \in \Psi^{-\infty,-\infty}(\mathbb{R}^n)\,,$$

and the second statement in (4.5) follows. Treating the equivalence of $(I-\Psi)Pe^{tG^w}$ and $(I-\Psi)P$ similarly we conclude that

$$P_t - e^{-tG^w}\Psi P e^{tG^w} - (I-\Psi)P \in \Psi^{-\infty,-\infty}(\mathbb{R}^n)\,.$$

We now put

$$Q \stackrel{\text{def}}{=} \Psi P \in \Psi^{0,0}(\mathbb{R}^n)\,, \quad Q_t \stackrel{\text{def}}{=} e^{-tG^w}Qe^{tG^w}\,,$$

and we only need to prove (4.4) with $P_\bullet$ replaced by $Q_\bullet$.

We now establish the expansion in (4.4). Lemma 2.5 implies that

$$[Q, G^w] = (h/i)\mathrm{Op}_h^w(H_pG) + R\,,$$

where $R \in \Psi_{\frac12}^{-3/2,-3/2,0}(\mathbb{R}^n) \subset \Psi_{\frac12}^{-1,-1,0}(\mathbb{R}^n)$. It also shows that

$$[[Q, G^w], G^w] = (h/i)[\mathrm{Op}_h^w(H_pG), G^w] + [R, G^w] \in \Psi_{\frac12}^{-1,-1,0}(\mathbb{R}^n)\,.$$

Here we used the special structure of G,

$$G = \widehat{\chi}\widehat{G} + C_0\log(1/h)\chi_0 G_0\,,$$

where $\widehat{\chi}, \chi_0$ and G_0 are uniformly smooth. When derivatives fall on these terms in error estimates (2.5) the gain in h compensates for the logarithmic growth, while for $|\alpha| > 0$, $\partial^\alpha\widehat{G} \in S_{\frac12}^{|\alpha|/2,-|\alpha|/2}$.

This gives,

$$\frac{d}{dt}E_t = [Q_t, G^w] - (h/i)\mathrm{Op}_h^w(H_pG) + (h/i)\mathrm{Op}_h^w(H_{p-p}G) = [Q_t - Q, G^w] + R_t\,,$$

with

$$E_0 = (h/i)\mathrm{Op}_h^w(H_{p-p}G) \in (h\log(1/h))^2\Psi_{\frac12}^{0,0,0}(\mathbb{R}^n) \subset \Psi_{\frac12}^{-1,-1,0}(\mathbb{R}^n)\,,$$

and $R_t \in \Psi_{\frac12}^{-1,-1,0}(\mathbb{R}^n)$. We also have

$$\frac{d}{dt}[(Q_t - Q), G^w] = e^{-tG^w}[[Q, G^w], G^w]e^{tG^w} \in \Psi_{\frac12}^{-1,-1,0}(\mathbb{R}^n)\,, \quad Q_0 - Q = 0\,.$$

Hence $[Q_t - Q, G^w] \in \Psi_{\frac{1}{2}}^{-1,-1,0}(\mathbb{R}^n)$, and consequently $E_t \in \Psi_{\frac{1}{2}}^{-1,-1,0}$. $\square$

We now modify our operator to obtain global invertibility. For that we define $a \in S_{\frac{1}{2}}^{0,0,-\infty}(T^*\mathbb{R}^n)$ as follows

$$a(x,\xi) \stackrel{\text{def}}{=} \chi\left(\frac{p(x,\xi)}{\delta_0}\right)\chi(H_pG(x,\xi))\psi(x,\xi)\,, \tag{4.6}$$
$$\chi \in \mathcal{C}_{\rm c}^\infty(\mathbb{R};[0,1])\,, \quad \chi(t) \equiv 1\,, \quad |t| \leq 1\,,$$

and ψ be one in a fixed small neighbourhood of $\widehat{K}$ and zero outside of another sufficiently small neighbourhood of $\widehat{K}$. We then put

$$\widetilde{P}_t = P_t - i(h/\tilde{h})\mathrm{Op}_h(a) \in \Psi_{\frac{1}{2}}^{0,0,2}(\mathbb{R}^n)\,.$$

We first treat the region away from the trapped set:

Lemma 4.3. *Suppose that $\Psi_0 \in \Psi^{0,0}(T^*\mathbb{R}^n)$ satisfies*

$$\mathrm{WF}_h(\Psi_0) \cap \widehat{K} = \emptyset\,.$$

Then for $u \in \mathcal{C}_{\rm c}^\infty(\mathbb{R}^n)$, $z \in [E-\delta, E+\delta] - i[0, Ch]$ we have

$$\|(\widetilde{P}_t - z)\Psi_0 u\|_{L^2} \geq th\|\Psi_0 u\|_{L^2(\mathbb{R}^n)}/C - \mathcal{O}(h^\infty)\|u\|_{L^2(\mathbb{R}^n)}\,,$$
$$0 < h \leq h_0(\tilde{h})\,, \quad 0 < \tilde{h} \leq \tilde{h}_0(t)\,.$$

Proof. Let us assume that $\|u\| = 1$. Microlocally near $\mathrm{WF}_h(\Psi_0)$, $a \equiv 0$ we can replace $\widetilde{P}_t$ by P_t, with error $\mathcal{O}(\tilde{h}^\infty h)$. For $z \in D(0, Ch)$, t sufficiently large,

$$P_t - z = \mathrm{Op}_h^w(p - \mathrm{Re}\, z) - iW - iht\mathrm{Op}_h^w(H_pG) - \mathrm{Im}\, z + \mathcal{O}_t(h\tilde{h})\,,$$
$$|\,\mathrm{Re}\, p - \mathrm{Re}\, z| < \delta \implies -W(x) + htH_pG(x,\xi) + \mathrm{Im}\, z \geq th/C\,.$$

Lemma 2.3 applied with Ψ_j's such that $|\,\mathrm{Re}\, p - \mathrm{Re}\, z| > \delta$ on $\mathrm{WF}_h(\Psi_1)$ (with Ψ_j's constructed using Lemma 2.2) completes the proof. $\square$

Near the trapped set we obtain

Lemma 4.4. *Let $z \in D(0, Ch)$. For $u \in \mathcal{C}_{\rm c}^\infty(\mathbb{R}^n)$, $\|u\| = 1$, with $\mathrm{WF}_h(u)$ in a fixed small neighbourhood of $\widehat{K}$ we have*

$$\|(\widetilde{P}_t - z)u\|_{L^2(\mathbb{R}^n)} \geq th\|u\|_{L^2(\mathbb{R}^n)}/C\,, \quad 0 < h \leq h_0(\tilde{h})\,, 0 < \tilde{h} \leq \tilde{h}_0(t)\,. \tag{4.7}$$

provided that t is large enough.

Proof. In a small neighbourhood of $\widehat{K}$ the operator is microlocally equal to

$$P_t^\flat \stackrel{\text{def}}{=} P - ith\mathrm{Op}_h(H_p\widehat{G}) - i(h/\tilde{h})\mathrm{Op}_h(a) + \mathcal{O}_{L^2\to L^2}(h\tilde{h})\,,$$

that is,

$$\|(\widetilde{P}_t - z)u\|_{L^2(\mathbb{R}^n)} = \|(P_t^\flat - z)u\|_{L^2(\mathbb{R}^n)} + \mathcal{O}(h^\infty)\,, \quad \|u\|_{L^2(\mathbb{R}^n)} = 1\,,$$

for u with $\mathrm{WF}_h(u)$ near $\widehat{K}$. We also note that $W = 0$ there. We now consider

$$-\operatorname{Im}\langle (P_t^\flat - z)u, u\rangle = h\langle (B_t(z)u, u\rangle\,, \quad B_t(z) \stackrel{\text{def}}{=} -(P_t^\flat - (P_t^\flat)^*)/(2hi) + \operatorname{Im} z/h\,.$$

For $z \in [E-\delta, E+\delta] - i[0, Ch]$, and (x,ξ) in a neighbourhood of $\widehat{K}$,

$$\sigma_h(B_t(z)) = tH_p G(x,\xi) + (1/\tilde{h})a + \operatorname{Im} z \geq t/C\,.$$

The sharp Gårding inequality applied in the $\Psi_{\frac{1}{2}}^{0,0,-\infty}$ calculus of §2.2 (see [4, Theorem 7.12]) gives, for $\|u\| = 1$, with $\mathrm{WF}_h(u)$ near $\widehat{K}$,

$$\langle (B_t(z)u, u\rangle \geq t/C - \mathcal{O}(\tilde{h}) - C \geq t/(2C)\,, \quad t \geq t_0(\tilde{h}, C)\,.$$

Hence

$$-\operatorname{Im}\langle (P_t^\flat - z)u, u\rangle \geq t/(2C)\,,$$

and we compte the proof by writing

$$\begin{aligned}\|(\widetilde{P}_t - z)u\|_{L^2(\mathbb{R}^n)} &= \|(P_t^\flat - z)u\|_{L^2(\mathbb{R}^n)} + \mathcal{O}(h^\infty) \\ &\geq ht/(2C)\,, \quad \mathrm{WF}_h(u) \text{ near } \widehat{K},\ \|u\|_{L^2(\mathbb{R}^n)} = 1.\end{aligned}$$

□

The two lemmas are now combined using Lemma 2.3 which gives for large t, $0 < \tilde{h} \leq \tilde{h}_0(t)$, and $0 < h < h_0(t, \tilde{h})$, the invertibility of $\widetilde{P}_t - z$, $z \in [E-\delta, E+\delta] - [0, iCh]$:

$$(\widetilde{P}_t - z)^{-1} = \mathcal{O}(1/h) \;:\; L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)\,.$$

Our main theorem will follow from showing that

$$\operatorname{Op}_h(a) = R + E\,, \quad \operatorname{rank}(R) = \mathcal{O}(h^{-\tilde{m}/2})\,, \quad E = \mathcal{O}(\tilde{h}^\infty) \;:\; L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)\,, \tag{4.8}$$

$\tilde{m} > m$, where m is the dimension of the trapped set near energy E, $\widetilde{K}$, allowing $\tilde{m} = m$ if the trapped set is of pure dimension.

That follows from Proposition 2.6 and the definition of the Minkowski dimension:

$$m = 2n - \sup\{d \;:\; \limsup_{\epsilon \to 0} \epsilon^{-d} \operatorname{vol}(\{\rho \;:\; d(\rho, \widehat{K}) < \epsilon\}) < \infty\}\,,$$

with the set being of pure dimension if

$$\limsup_{\epsilon \to 0} \epsilon^{-d} \operatorname{vol}(\{\rho \;:\; d(\rho, \widehat{K}) < \epsilon\}) < \infty\,.$$

In other words, for ϵ small

$$\operatorname{vol}(\{\rho \;:\; d(\rho, \widehat{K}) < \epsilon\}) \leq C\epsilon^{2n - \widetilde{m}}\,, \quad \widetilde{m} > m\,,$$

and $\tilde{m}$ replaceable by m when $\widehat{K}$ is of pure dimension. The definition of a in (4.6) then gives

$$\mathrm{vol}(\mathrm{supp}\, a) \leq C_{\tilde{h}} h^{(2n-\tilde{m})/2} = C_{\tilde{h}} h^{n-\widetilde{m}/2}\,, \quad \widetilde{m} > m\,,$$

with equality if $\widehat{K}$ is of pure dimension.

The standard covering arguments (see [16, Lemma 3.3]) show that the hypothesis of Proposition 2.6 are satisfied with

$$K(h) \leq C_{\tilde{h}} h^{-m/2}\,,$$

which completes the proof of Theorem in §1.

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Mathematics Department, University of California, Evans Hall, Berkeley, CA 94720

E-mail address: `zworski@math.berkeley.edu`