SINGULAR PART OF THE SCATTERING MATRIX DETERMINES THE OBSTACLE

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1. Introduction

The purpose of this note is to answer a question proposed by Ikawa: does the singular part of the scattering matrix determine the obstacle? It is well known that the complete scattering matrix at any fixed energy determines the obstacle – see [5], Sect.V.6 and [3]. Here we give

Theorem. *Suppose that $\mathcal{O}_1$ and $\mathcal{O}_2$ are two obstacles in $\mathbb{R}^n$ and that $S_1(\lambda)$, $S_2(\lambda)$ are the corresponding scattering matrices. If λ_0, in $\mathbb{C}$, when n is odd and in Λ_1, when n is even, is a pole of $S_1(\lambda)$ then*

$$S_1(\lambda) - S_2(\lambda) \ \text{ is holomorphic near } \ \lambda_0 \ \Longrightarrow \ \mathcal{O}_1 = \mathcal{O}_2 \,.$$

Here Λ_1 denotes the first sheet of the logarithmic plane: $-\pi < \arg \lambda_0 < \pi$.

The proof is an observation based on the complex scaling method and on the Rellich uniqueness theorem. The method does not apply to potential or metric scattering and in fact, as pointed out to the author by Michael Livshits, the analogous theorem does not hold in that case – see a remark in Sect.3. It seems likely however, that when $S_1(\lambda) - S_2(\lambda)$ has no poles at all, then any type of scatterer is determined, and that was Ikawa's original question.

We recall the basic assumptions and definitions. By an *obstacle*, $\mathcal{O}$, we mean a compact subset of $\mathbb{R}^n$ such that $\mathbb{R}^n \setminus \mathcal{O}$ is connected and $\partial\mathcal{O}$ is smooth. We then consider the Laplacian, $\Delta = -\sum_{j=1}^n \partial_j^2$, with the Dirichlet conditions on $\partial\mathcal{O}$.

There are many equivalent definition of the scattering matrix for $\mathcal{O}$. The one most relevant here, comes from considering the radiation patterns of plane waves scattered by the obstacle: for every $\lambda \in \mathbb{R} \setminus \{0\}$ and $f \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$ there exist a unique solution to the exterior problem with prescribed incoming radiation pattern:

$$\begin{gathered} (-\Delta - \lambda^2)u = 0 \,, \quad u\restriction_{\partial\mathcal{O}} = 0 \,, \\ u(x) = |x|^{-\frac{n-1}{2}} e^{i|x|\lambda} \left(f\left(\frac{x}{|x|}\right) + \mathcal{O}\left(\frac{1}{|x|}\right)\right) + |x|^{-\frac{n-1}{2}} e^{-i|x|\lambda} \left(g\left(\frac{x}{|x|}\right) + \mathcal{O}\left(\frac{1}{|x|}\right)\right) \,, \\ |x| \longrightarrow \infty \,, \quad g \in \mathcal{C}^\infty(\mathbb{S}^{n-1}) \,. \end{gathered} \tag{1.1}$$

The scattering matrix is defined as the operator mapping the incoming radiation pattern to the outgoing one:

$$S(\lambda) \ : \ f \longmapsto i^{1-n} g(-\bullet) \,, \tag{1.2}$$

where the antipodal map and the power of i were added so that $S(\lambda) = I$ when $\mathcal{O} = \emptyset$ – see [7] for a comprehensive review.

A more concrete representation of $S(\lambda)$ comes from considering solutions of the form

$$\begin{gathered} (-\Delta - \lambda^2)u_\omega = 0 \,, \quad u_\omega\restriction_{\partial\mathcal{O}} = 0 \,, \quad \omega \in \mathbb{S}^{n-1} \\ u_\omega(x) = e^{i\lambda\langle x, \omega\rangle} + v_\omega(x) \,, \quad v_\omega(r\theta) = r^{-\frac{n-1}{2}} e^{-i\lambda r} \left(a(\lambda, \theta, \omega) + \mathcal{O}\left(\frac{1}{r}\right)\right) \,. \end{gathered} \tag{1.3}$$

Since, in the sense of distributions on $\mathbb{S}^{n-1}_\theta$, we have

$$e^{i\lambda r\langle\theta,\omega\rangle}=\left(\frac{2\pi}{i\lambda}\right)^{\frac{n-1}{2}}r^{-\frac{n-1}{2}}\left[e^{i\lambda r}\left(\delta_\omega(\theta)+\mathcal{O}\left(\frac{1}{r}\right)\right)+i^{n-1}e^{-i\lambda r}\left(\delta_{-\omega}(\theta)+\mathcal{O}\left(\frac{1}{r}\right)\right)\right],$$

the scattering matrix is represented by the kernel

$$S(\lambda)(\omega,\theta)=\delta_\omega(\theta)+c_n\lambda^{\frac{n-1}{2}}a(\lambda,-\theta,\omega).$$

For $\lambda\in\mathbb{R}$, $S(\lambda)$ is a unitary operator on $L^2(\mathbb{S}^{n-1})$. As was shown in [5] (see also [11] and references given there) it continues meromorphically to $\mathbb{C}$ when in n is odd and to Λ, the logarithmic place, when n is even. The poles coincide with the poles of the meromorphic continuation of the resolvent, $(-\Delta-\lambda^2)^{-1}: L^2_{\text{comp}}(\mathbb{R}^n\setminus\mathcal{O})\to L^2_{\text{loc}}(\mathbb{R}^n\setminus\mathcal{O})$, from the physical half-plane, $\operatorname{Im}\lambda<0$, to $\mathbb{C}$ and Λ, when n is odd or even respectively.

The multiplicity of a pole can be defined in any sensible way and all of such definitions coincide (we never have the troublesome pole at $\lambda=0$ in obstacle scattering). For instance, we can put

$$m_S(\lambda_0)=\dim\sum_{j=1}^M A_j(L^2(\mathbb{S}^{n-1})),\quad S(\lambda)=\sum_{j=1}^M\frac{A_j}{(\lambda-\lambda_0)^j}+A_{\lambda_0}(\lambda),\quad A_{\lambda_0}\text{ holomorphic near }\lambda_0\ . \tag{1.4}$$

Although the issues of the multiplicity will be implicit in Sect.2, in this note, only an existence of a pole will matter.

2. Structure of the singular part

We will describe the relation between the singular part of the resolvent and the singular part of the scattering matrix. In the case of potential scattering, this is well known – see [9], [2], [1]. For future reference, and to avoid specific aspects of obstacle scattering, we will now procceed in the generality of "black box" scattering introduced by Sjöstrand and the author in [10]. We briefly recall the assumptions. Let $\mathcal{H}$ be a complex Hilbert space which is a complexification of a real Hilbert space, so that the complex conjugate is well defined: $\mathcal{H}\ni g\mapsto\bar g\in\mathcal{H}$. We assume the existence of an orthogonal decomposition

$$\mathcal{H}=\mathcal{H}_{R_0}\oplus L^2(\mathbb{R}^n\setminus B(0,R_0)),\ \ n\geq 2$$

and let P be a real self-adjoint operator ($\overline{Pu}=P\bar u$, $(Pu,v)_\mathcal{H}=(u,Pv)_\mathcal{H}$) with domain $\mathcal{D}\subset\mathcal{H}$ such that

$$1\!\!1\restriction_{\mathbb{R}^n\setminus B(0,R_0)}\mathcal{D}=H^2(\mathbb{R}^n\setminus B(0,R_0)), \tag{2.1}$$

$$1\!\!1\restriction_{\mathbb{R}^n\setminus B(0,R_0)}P=-\Delta\restriction_{\mathbb{R}^n\setminus B(0,R_0)}, \tag{2.2}$$

$$1\!\!1\restriction_{B(0,R_0)}(P-i)^{-1}\text{ is compact.}$$

The assumption that P is real is not necessary but, as in [4], it make the structure of the singular part of the resolvent particularly nice.

Theorem 1 of [10] gives the meromorphic continuation of $R(\lambda)=(P-\lambda^2)^{-1}$ as an operator

$$R(\lambda)=\mathcal{H}_{\text{comp}}\longrightarrow\mathcal{D}_{\text{loc}}\,.$$

In fact, for any $\chi\in\mathcal{C}^\infty_{\text{comp}}(\mathbb{R})$, $\chi=1$ near $B(0,R_0+a_0)$, $a_0>0$ we have

$$\begin{gathered}R(\lambda)\chi=(Q_0(\lambda)+Q_1(\mu))\chi(I+K(\lambda,\mu)\chi)^{-1}\,,\\ Q_0(\lambda)=(1-\chi_0)R_0(\lambda)(1-\chi_1)\,,\quad Q_1(\mu)=\chi_2R(\mu)\chi_1\,,\end{gathered} \tag{2.3}$$

where χ_i's are functions with the same property as χ, $\chi\chi_i=\chi_i$ (and some other properties), and $K(\lambda,\mu)\chi$ is a compact operator on $\mathcal{H}$. $R_0(\lambda)$ denotes the free resolvent and $\operatorname{Im}\mu\ll 0$ (the convention is the same as in Sect.1: $R(\mu)$ is bounded on $\mathcal{H}$ for $\operatorname{Im}\mu<0$).

The assumption that P is real shows that $R(\lambda)$ is formally symmetric with respect to the form $(\bullet, \bar{\bullet})_{\mathcal{H}})$ (that is $1\!\!1_{B(0,R_0+a)} R(\lambda) 1\!\!1_{B(0,R_0+a)}$ is symmetric). Proceeding as in Sect.3 of [10], the structure of the singular part of the resolvent can be described as follows: for $\lambda_0 \neq 0$ we have

$$\begin{gathered} R(\lambda) = \sum_{j=1}^{M} \frac{B_j}{(\lambda - \lambda_0)^j} + B_{\lambda_0}(\lambda) \,, \quad B_{\lambda_0} \text{ holomorphic near } \lambda_0 \,, \\ (P - \lambda_0^2)^M B_1 = 0 \,, \quad \text{rank } B_1 < \infty \,, \quad B_k = (P - \lambda_0^2)^{k-1} B_1 \end{gathered} \tag{2.4}$$

The symmetry of $R(\lambda)$ with respect $(\bullet, \bar{\bullet})_{\mathcal{H}}$, shows that the operator $B_M \neq 0$ is of the form

$$B_M = \sum_{j=1}^{L} \phi_j \otimes \phi_j \,, \quad (P - \lambda_0^2)\phi_j = 0 \,, \quad \text{span } \{\phi_j\}_{j=1}^L = B_M(\mathcal{H}_{\text{comp}}) \,, \tag{2.5}$$

and ϕ_j's are independent. Here $\phi \otimes \psi$ denotes a rank one operator on $\mathcal{H}_{\text{comp}}$ given by

$$\phi \otimes \psi(g) = (\psi, \bar{g})_{\mathcal{H}} \phi \,, \quad g \in \mathcal{H}_{\text{comp}} \,.$$

Combined with (2.3) this gives

Lemma 1. *For every ϕ_j in (2.5), there exists $h_j \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ such that*

$$1\!\!1_{\mathbb{R}^n \setminus B(0,R_0+a_0)} \phi_j = R_0(\lambda) h_j \restriction_{\mathbb{R}^n \setminus B(0,R_0+a_0)} \,,$$

unless, possibly, $\lambda_0^2 < 0$, $M = 1$, and there exists $u \in \mathcal{H}_{\text{comp}}$, such that $(P - \lambda_0^2)u = 0$.

Proof. The structure of the resolvent, (2.3), (2.4), shows that for any $g \in \mathcal{H}_{\text{comp}}$ we have that

$$1\!\!1_{\mathbb{R}^n \setminus B(0,R_0+a_0)} \sum_{j=1}^{N} \phi_j (\phi_j, \bar{g})_{\mathcal{H}} \restriction_{\lambda=\lambda_0} = 1\!\!1_{\mathbb{R}^n \setminus B(0,R_0+a_0)} R_0(\lambda_0) h_g \,, \quad h_g \in \mathcal{C}_c^\infty(\mathbb{R}^n) \,.$$

Using the complex scaling method (see Sect.3 of [10]), we see that we can choose $g = g_i$ so that $(\alpha_{ij})_{1 \leq i,j \leq N} = ((\phi_j, \bar{g}_i)_{\mathcal{H}})_{1 \leq i,j \leq N}$ is invertible. In fact, let $\psi_j = \phi_j \restriction_{B(0,R_0+a_2) \setminus B(0,R_0+a_1)}$, $a_2 > a_1$, $B(0, R_0 + a_2) \subset \Gamma_\theta \cap \mathbb{R}^n$. Then ψ_j's are independent in $L^2(\mathbb{R}^n)$ as otherwise $u = \sum_{j=1}^N \alpha_j \phi_j$ would vanish in $B(0, R_0+a_2) \setminus B(0, R_0+a_1)$. As it solves $(P - \lambda_0^2)u = 0$, it would then vanish on $\mathbb{R}^n \setminus B(0, R_0 + a_1)$ (by analytic continuation). But then, $u \in \mathcal{H}_{\text{comp}}$ is an eigenfunction which, by self-adjointness of P, shows that $u \equiv 0$, unless $\lambda_0^2 < 0$. The case $M > 1$ is impossible, as then $u = (P - \lambda_0^2)\psi$, $\psi \in \mathcal{H}_{\text{loc}}$ and

$$(u, u)_{\mathcal{H}} = (u, (P - \lambda_0^2)\psi)_{\mathcal{H}} = (u, (P - \lambda_0^2)\chi\psi)_{\mathcal{H}} = ((P - \lambda_0^2)u, \chi\psi)_{\mathcal{H}} = 0 \,,$$

where $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, $B(0, R_0 + a_0) \cup \text{supp } u \subset \{\chi = 1\}$. Hence, again, $u \equiv 0$.

Writing $h_i = h_{g_i}$, we obtain

$$1\!\!1_{\mathbb{R}^n \setminus B(0,R_0+a_0)} \sum_{j=1}^{N} \alpha_{ij} \phi_j = 1\!\!1_{\mathbb{R}^n \setminus B(0,R_0+a_0)} \sum_{j=1}^{N} R_0(\lambda) h_i \,,$$

and the lemma follows. □

Remark. The exceptional case in the lemma can be eliminated by adding a unique continuation assumption holding in most interesting situations:

$$(P - \lambda^2)u = 0 \,, \quad u \in \mathcal{D}_{\text{comp}} \implies u \equiv 0 \,. \tag{2.6}$$

It is in fact, possible that the exceptional case never occurs for a general perturbation.

To connect this to the structure of the scattering matrix we recall the following simple

Lemma 2. *The scattering matrix for a "black box" perturbation is given by*

$$S(\lambda) = I + A(\lambda)\,, \quad \lambda A(\lambda) = c_n {}^t\mathbb{E}(-\lambda)[\Delta, \chi_2]R(\lambda)[\Delta,\chi]\mathbb{E}(\lambda)\,, \textit{where}$$

$$\mathbb{E}(\lambda) : L^2(\mathbb{S}^{n-1}) \longrightarrow \mathcal{C}^\infty(\mathbb{R}^n)\,, \quad \mathbb{E}(\lambda)u(x) = c_n \lambda^{\frac{n-1}{2}} \int_{\mathbb{S}^{n-1}} u(\omega) e^{i\lambda\langle x,\omega\rangle} d\omega\,,$$

and $\chi_j \in \mathcal{C}^\infty_{\rm c}(\mathbb{R}^n)$ *are the same as in* (2.3)

Proof. As recalled in Sect.1, $A(\lambda)$, for λ real, comes from the radiation pattern of $R(\lambda)(-[\Delta,\chi]e^{i\langle\bullet,\omega\rangle})$. To obtain a formula for $A(\lambda)$, we write

$$(1-\chi_2)R(\lambda)\chi_1 = R_0(\lambda)(-\Delta - \lambda^2)(1-\chi_2)R(\lambda)\chi_1 = R_0(\lambda)(-[\Delta,\chi_2]R(\lambda)\chi_1)\,,$$

since $(1-\chi_2)(-\Delta-\lambda^2) = (1-\chi_2)(P-\lambda^2)$ and $(1-\chi_2)\chi_1 = 0$. The basic asymptotic formula,

$$|x|^{\frac{n-1}{2}} e^{i\lambda|x|} R_0(\lambda)(x,y) \longrightarrow \alpha_n \lambda^{\frac{n-3}{2}} e^{i\langle y,\omega\rangle}\,, \quad |x|\to\infty\,, \quad x/|x| = \omega\,, \quad y/|x| \to 0\,. \tag{2.7}$$

and the definition of $S(\lambda)$ give the lemma. ☐

Lemma 3. *If* A_M *is the operator given in* (1.4)*, then*

$$A_1 = a_n \lambda^{-1} \sum_{j=1}^N {}^t\mathbb{E}(-\lambda_0)h_j \otimes {}^t\mathbb{E}(\lambda_0)h_j\,,$$

where h_j *are given in Lemma 1, and* $\mathbb{E}(\lambda_0)$ *are as in Lemma 2.*

Proof. Because of Lemmas 1 and 2, we only need to show that

$${}^t\mathbb{E}(\lambda)h = -{}^t\mathbb{E}(\lambda)[\Delta,\chi]R_0(\lambda)h\,, \quad h \in \mathcal{C}^\infty_{\rm c}(\mathbb{R}^n)\,, \quad (1-\chi)h = 0\,. \tag{2.8}$$

In fact,

$$(1-\chi)R_0(\lambda_0)h = R_0(\lambda)(\Delta-\lambda^2)(1-\chi)R_0(\lambda)h = R_0(\lambda)(-[\Delta,\chi]R_0(\lambda)h)\,,$$

where the equalities are justified for $\operatorname{Im}\lambda < 0$, with analytic continuation showing the equality of the extreme terms for all λ. Applying (2.7) to both sides gives (2.8). ☐

3. Proof of Theorem

The real meaning of Lemma 3 is in relating the singular part of the scattering matrix to radiation patterns of the resonant states, ψ_j. Using the complex scaling for large angles (see [10] for the large angle scaling for compactly supported perturbations and for references to the origins of the method), we obtain

Lemma 4. *For a "black box" operator,* P*, satisfying the assumptions above, and* (2.6)*, the singular part of* $(\lambda-\lambda_0)^{M-1}S(\lambda)$ *at* $\lambda_0 \neq 0$ *determines the singular part of* $(\lambda-\lambda_0)^{M-1}R(\lambda)$ *at* λ_0*. If we do not assume* (2.6)*, then the same exception as in Lemma 1 has to be allowed.*

Proof. We apply the large angle scaling of Sect.3 of [10] which deforms P to an operator P_θ on $\mathcal{H}_{R_0} \otimes L^2(\Gamma_\theta \setminus B(0,R_0))$. For a_0 large enough $1\!\!1_{\Gamma_\theta\setminus B(0,R_0+a_0)} P_\theta = 1\!\!1_{\Gamma_\theta\setminus B(0,R_0+a_0)} e^{-2i\theta}\Delta_x$, where we parametrized

$$\Gamma_\theta\restriction_{|z|>R_0+a_1} = e^{i\theta}\mathbb{R}^n\restriction_{|z|>R_0+a_1} \subset \mathbb{C}^n\,.$$

The resonant states, ψ_j, continue holomorphically (as multivalued functions) to a neighbourhood of $\cup_{0<\theta}\Gamma_\theta$, and the restriction to Γ_θ is given by

$$1\!\!1_{\Gamma_\theta\setminus B(0,R_0+a_0)}\psi_j^\theta = 1\!\!1_{\Gamma_\theta\setminus B(0,R_0+a_0)} R(e^{i\theta}\lambda_0)h_j\,.$$

We now apply this with $\theta = -\arg\lambda_0$ (the difference in sign comes from a switch in the sign convention), so that

$$(\Delta_x - (e^{i\theta}\lambda_0)^2)\psi_j^\theta = 0\,, \quad \text{on } e^{i\theta}\mathbb{R}^n_x\restriction_{|z|>R_0+a_1}\,,$$

$$\psi_j^\theta = \alpha_n \lambda^{-1}|x|^{-\frac{n-1}{2}} e^{-i(e^{i\theta}\lambda_0)|x|}\left({}^t\mathbb{E}(\lambda_0)h_j + \mathcal{O}(|x|^{-1})\right)\,,$$

where, again, we used (2.7). Since, $e^{i\theta}\lambda_0$ is real, the Rellich uniqueness theorem (see [5], Chapter V) shows that $1\!\!1_{\Gamma_\theta \restriction |x| > R_0 + a_1} \psi_j^\theta$, is uniquely determined by ${}^t\mathbb{E}(\lambda_0) h_j$, its radiation pattern. Analytic continuation now shows that $1\!\!1_{\mathbb{R}^n \setminus B(0, R_0 + a_0)} \psi_j$ is determined by ${}^t\mathbb{E}(\lambda_0) h_j$. As in the proof of Lemma 1, we see that this determines all $\psi_j' s$. $\square$

The proof of the Theorem in Sect.1 now follows from the classical argument of Shiffer, presented in [5] and in a slightly corrected form in [3]. If $S_1 - S_2$ is regular near λ_0, a pole of S_1, then the proof of Lemma 4 shows that there exists u such that

$$
\begin{aligned}
&(-\Delta - \lambda_0^2) u_1 = 0 \,, \quad \text{on } \mathbb{R}^n \setminus \mathcal{O}_1 \quad u_2 \restriction_{\partial \mathcal{O}_1} = 0 \,, \\
&(-\Delta - \lambda_0^2) u_2 = 0 \,, \quad \text{on } \mathbb{R}^n \setminus \mathcal{O}_2 \quad u_2 \restriction_{\partial \mathcal{O}_2} = 0 \,, \\
&u_1 = u_2 \ \text{ on } G, \text{ the connected component of infinity in } \mathbb{R}^n \setminus (\overline{\mathcal{O}}_1 \cup \overline{\mathcal{O}}_2) \ .
\end{aligned}
$$

In particular $u_1 \restriction_{\partial G} = u_2 \restriction_{\partial G} = 0$. If, say, $\mathcal{O} = (\mathbb{R}^n \setminus \overline{G}) \setminus \overline{\mathcal{O}}_1 \neq \emptyset$, then, $\mathcal{O}$ is a bounded open set, and

$$(-\Delta - \lambda_0^2) u_1 = 0 \,, \quad u \in H^2(\mathcal{O}) \cap H_0^1(\mathcal{O}) \,.$$

Since $\lambda_0^2 \in \mathbb{C} \setminus [0, \infty)$, integration by parts shows that this is impossible.

Remark. As was pointed out by Livshits [6] the result does not hold for potentials and metrics: the knowledge of one pole and its polar part will not, in general, determine the scatterer. To outline his argument, consider a radial potential of compact support (or a rotationally invariant metric). Then, $S(\lambda)$ can be diagonalized using spherical harmonics and, if λ_0 is a resonance, it appears as a pole of finitely many diagonal terms. The singular part is then given by a finite set of numbers. It is intuitively clear that a one dimensional potential cannot be recovered from finitely many parameters. More specifically, take $V_{\alpha,\beta}(r) = \sum_{i=1}^N \alpha_i 1\!\!1_{0 \leq r < \beta_i}(r)$, $\alpha \in \mathbb{R}^N$ and $0 < \beta_1 < \beta_2 < \cdots < \beta_N$. Then, for any λ, the existence of an outgoing solution is equivalent to $H_l(\alpha, \beta; \lambda) = 0$, for some l. Here, for each spherical mode l, $H_l(\alpha, \beta, \bullet)$ is a transcendental equation obtained from matching boundary conditions at each β_i, and using the explicit solutions on each step given by Hankel functions (in one dimension we simply have exponentials – see for instance [8]). For $|\lambda| \leq C$ only finitely many $l's$ can give a solution (see for instance [12]). The finite number of parameters corresponding to the singular part is then given by equations in α and β. With a sufficiently large N, we can keep the solution and those parameters fixed, while varying α and β.

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References

[1] E. Balslev and E. Skibsted, *Asymptotic and analytic properties of resonant functions,* Ann. Inst. H. Poincaré (Phys. Theor.) **51** (1989), 129-154.

[2] A. Jensen, *Resonances in an abstract scattering theory,* Ann. Inst. H. Poincaré **33** (1980), 209-223.

[3] A. Kirsch and R. Kress, *Uniqueness in inverse obstacle scattering,* Inverse Problems **9**(1993), 285-299.

[4] F. Klopp and M. Zworski, *Generic simplity of resonances,* Helv. Phys. Acta **68** (1995), 531-538.

[5] P. Lax and R. Phillips, *Scattering Theory,* Academic Press, 1969.

[6] M. Livshits, *private communication.*

[7] R. B. Melrose, *Geometric Scattering Theory,* Cambridge University Press, 1995.

[8] H.M. Nussenzweig, *The poles of S-matrix of a rectangular potential well or barrier,* Nuclear Phys. **11**(1959), 499-521.

[9] N. Shenk and D. Thoe, *Resonant states and the poles of the scattering matrix for perturbations of* $-\Delta$, J. Math. Anal. Appl. **37**(1972), 467-491.

[10] J. Sjöstrand and M. Zworski, *Complex scaling and the distribution of scattering poles*, J. Amer. Math. Soc. **4** (1991), 729-769.

[11] B. Vainberg, *Asymptotic methods in equations of mathematical physics.* Gordon and Breach, 1988.

[12] M. Zworski, *Sharp polynomial bounds on the number of scattering poles of radial potentials,* J. Funct. Anal. **82**(1989), 370-403.

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