BREIT-WIGNER APPROXIMATION AND THE DISTRIBUTION OF RESONANCES

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1. Introduction

The origins of the theory of resonances lie in the work of Breit and Wigner who studied the behaviour of unstable particles. They postulated that if an ustable particle at energy E_0 decays according to the law $\exp(-\Gamma t)$ then the energy density should be approximately distributed according to the Breit-Wigner distribution

$$\frac{\Gamma}{\Gamma^2/4 + (E-E_0)^2}\,. \tag{1.1}$$

Hence for Γ small, that is for long living particles, $E = E_0$ becomes the *resonant* energy. In scattering experiments we consequently see "blips". A single "blip" is supposed to correspond to a creation of a particle with rest energy E_0 and then to its subsequent decay.

The mathematical content of a scattering experiment is captured by the scattering matrix and we should consequently look for "blips" in expressions derived from it. The most basic one is the *scattering phase* which measures the averaged phase shift which a wave experiences while passing through the scatterer. If $S(\lambda)$ is the scattering matrix then we put

$$s(\lambda) \stackrel{\text{def}}{=} i \log \det S(\lambda)\,, \quad s(0) = 0\,. \tag{1.2}$$

According to the Breit-Wigner theory, $s'(\lambda)$, which is the distribution function of continuous energy, should behave according to an expression similar to (1.1) with $\lambda_0^2 = E_0 - i\Gamma$, $\lambda^2 = E$:

$$P(\lambda_0, \lambda) \stackrel{\text{def}}{=} \frac{1}{\pi} \frac{\operatorname{Im} \lambda_0}{|\lambda - \lambda_0|^2}\,. \tag{1.3}$$

This is partly justified by the Poisson formula for resonances – see (3.1) below.

Since there could be many resonances, λ_0, close to the real axis, the separation between the ones corresponding to a particular "blip" and the other ones is far from apparent. In this paper we give upper and lower bounds on the variation of the scattering phase

$$s(\lambda+\delta) - s(\lambda - \delta)\,, \tag{1.4}$$

in terms of the number of resonances close to λ. That provides further justification of the Breit-Wigner approximation, in particular at high energies, and leads to new results on the distribution of resonances. Our main result is

Theorem. *For operators satisfying assumptions* (1.7)-(1.9) *below and for any fixed* $0 < \epsilon < 1$ *we have*

$$s(\tau+\delta) - s(\tau-\delta) = 2\pi \sum_{|\lambda_j - \tau| < \epsilon} \omega_{\mathbb{C}_+}(\lambda_j, [\tau - \delta, \tau+\delta]) + \mathcal{O}_\epsilon(\delta)\tau^{n-1}\,, \quad \forall \delta, \quad 0 \leq \delta < \epsilon/2, \ \forall \tau \geq 1, \tag{1.5}$$

where λ_j are resonances included according to their multiplicities and $\omega_{\mathbb{C}_+}$ is the harmonic measure for the upper half-plane:

$$\omega_{\mathbb{C}_+}(z, E) = \int_E P(z,\lambda) d\lambda\,, \quad E \subset \mathbb{R} = \partial \mathbb{C}_+\,,$$

with the Poisson kernel $P(z,\lambda)$ given by (1.3).

We note that we can apply the Theorem with δ which depends on τ. A by-product of the proof, which also justifies the statement of the theorem, is the following estimate closely related to the assumption (1.9) below:

$$\sharp\{\lambda_j \; : \; |\lambda_j - \tau| < \epsilon\} = \mathcal{O}(\tau^{n-1}) \,, \quad 0 < \epsilon < 1 \,, \tag{1.6}$$

see Proposition 2 below. The Theorem leads to new estimates on the number of resonances near the real axis.

Previous results on the Breit-Wigner approximation were obtained in a very specific semi-classical setting by Gérard-Martinez-Robert [3] and their work motivated ours. We consider a general compactly supported perturbation of the Laplacian in $\mathbb{R}^n$, $n \geq 2$. To avoid discussion of specific aspects of obstacle or metric scattering we adopt the "black box" formalism developed by Sjöstrand and the second author [20]. It is quite possible that much of this work could be generalized to include the non-compact "black box" perturbations [19] but that would involve further development of some tools of scattering theory. We choose to remain in a somewhat less technical situation at this stage.

We start by recalling the abstract assumptions of "black box" scattering. Let $\mathcal{H}$ be a complex Hilbert space with an orthogonal decomposition

$$\mathcal{H} = \mathcal{H}_{R_0} \oplus L^2(\mathbb{R}^n \setminus B(0, R_0)), \;\; n \geq 2$$

and let L be a self-adjoint operator with domain $\mathcal{D} \subset \mathcal{H}$ such that

$$1\!\!1\!\restriction_{\mathbb{R}^n \setminus B(0,R_0)} \mathcal{D} = H^2(\mathbb{R}^n \setminus B(0, R_0)), \tag{1.7}$$

$$1\!\!1\!\restriction_{\mathbb{R}^n \setminus B(0,R_0)} L = -\Delta\!\restriction_{\mathbb{R}^n \setminus B(0,R_0)}, \tag{1.8}$$

$$\sigma_{\rm pp}(L) \cap [0, \infty) = \emptyset \,, \quad L \geq -C \,, \quad C > 0 \,.$$

We also assume that L is *elliptic* in the following spectral sense: let $L^\sharp$ be a reference operator of L obtained by considering L on $\mathcal{H}^\sharp = \mathcal{H}_{R_0} \oplus L^2((\mathbb{R}^n / R\mathbb{Z}^n) \setminus B(0, R_0))$, $R \gg R_0$, in the obvious way. We then assume that

$$\sharp\{\lambda \; : \; \lambda^2 \in \sigma(L^\sharp) \,, \; |\lambda| \leq r\} = C(L) r^n + \mathcal{O}(r^{n-1}) \,. \tag{1.9}$$

In other words, the reference operator admits Weyl asymptotics with the Hörmander remainder term. We note that (1.9) implies the trace class properties needed for the Poisson formula for resonances – see [22], [19].

A typical example is provided by a metric perturbation of $-\Delta$ on an exterior domain given by

$$L = -\sum_{i,j=1}^{n} \partial_{x_j} \Big(a_{i,j}(x) \partial_{x_i} \Big) \,, \quad \text{on } \mathbb{R}^n \setminus \mathcal{O} \,,$$

where $\partial\mathcal{O}$ is smooth, $\mathbb{R}^n \setminus \mathcal{O}$ is connected and we take Dirichlet (or Neumann) boundary conditions. The coefficients $a_{i,j}(x)$ are real valued and satisfy the assumptions:

$$a_{i,j}(x) \in \mathcal{C}^\infty(\mathbb{R}^n), \;\; a_{i,j}(x) = a_{j,i}(x), \;\; 1 \leq i, j \leq n,$$

$$\sum_{i,j}^{n} a_{i,j}(x) \xi_i \, \xi_j \geq \delta_0 |\xi|^2, \;\; \delta_0 > 0, \; \forall x \in \mathbb{R}^n, \;\; \forall \xi \in \mathbb{R}^n,$$

$$\text{there exists } R_0 > 0 \text{ such that } a_{i,j}(x) = \delta_{i,j} \text{ for } |x| \geq R_0.$$

In this case L is a self-adjoint elliptic positive operator in $L^2(\mathbb{R}^n \setminus \mathcal{O})$ with the domain given by $H^2(\mathbb{R}^n \setminus \mathcal{O}) \cap H^1_0(\mathbb{R}^n \setminus \mathcal{O})$.

As was shown in [20], the resolvent

$$R(\lambda) = (L - \lambda^2)^{-1} : \, \mathcal{H} \longrightarrow \mathcal{D} \,, \;\; \mathrm{Im}\, \lambda < 0 \,, \;\; \lambda^2 \notin \sigma_{\rm pp}(L) \,,$$

admits a meromorphic continuation to $\mathbb{C}$ as an operator $R(\lambda) : \mathcal{H}_{\text{comp}} \longrightarrow \mathcal{D}_{\text{loc}}$ when n is odd and to the logarithmic plane, Λ, when n is even. The poles, λ_j, $\operatorname{Im}\lambda_j > 0$, have finite multiplicities. Moreover, for $\lambda \in \mathbb{R}$ we define the scattering operator

$$S(\lambda) : L^2(S^{n-1}) \longrightarrow L^2(S^{n-1})$$

following the usual practice in scattering theory – see [31] and [2] for the details in the "black box" setting. The scattering matrix has the form

$$S(\lambda) = I + A(\lambda)$$

with a trace class operator $A(\lambda)$, and it continues meromorphically to $\mathbb{C}$ when n is odd and to Λ when n is even. Its poles coincide with the poles of the resolvent with multiplicities, $m_S(\lambda)$, related to the multiplicities of the poles of the resolvent, $m_R(\lambda)$, by the formula

$$m_S(\lambda) = m_R(\lambda) - m_R(-\lambda)\,,$$

see [31] and references given there for more details.

The above assumptions imply a sharp polynomial upper bound for the counting function of the scattering poles when n is odd (obtained in successive generality in [9], [30], [20], [25])

$$\sharp\{\lambda_j : |\lambda_j| \leq r\} \leq A_0 r^n,\ \ r \geq A_1\,. \tag{1.10}$$

For n even we have a similar estimate

$$\sharp\{\lambda_j : |\lambda_j| \leq r,\ \ |\arg\lambda_j| \leq \rho\} \leq\ A_\rho r^n,\ \ r \geq A_1\,, \tag{1.11}$$

– see [20] for the proof which applies to $\rho < \pi/2$ and [27], [28] for the general case and the dependence of A_ρ on ρ.

Since $A(\lambda)$ is a trace class opearator, the definition of the scattering phase, (1.2), makes sense. Because of the assumption (1.9), the scattering phase satisfies a Weyl law:

$$|s(\lambda) - c_0\lambda^n| \leq C_0\lambda^{n-1},\ \ \lambda \longrightarrow +\infty\,, \tag{1.12}$$

for some constants $c_0 > 0$, $C_0 > 0$ independent of λ. This was shown in by Christiansen [2] following earlier work on obstacle and metric scattering [10], [14], [15], [11] (some of those papers are concerned with non-compact perturbations not included in the presently discussed set-up).

In spite of (1.12), the derivative $s'(\tau)$ may increase exponentially for $\tau = \operatorname{Re}\lambda_j$ if the poles λ_j converge exponentially to the real axis (see [3], [11]). This of course means that the scattering phase difference

$$s(\tau+\delta) - s(\tau-\delta),\ \ \delta > 0 \tag{1.13}$$

varies quickly as $\delta \longrightarrow 0$ and τ is equal to the real part of a scattering pole. This is related to our opening discussion and to the Breit-Wigner approximation. Thus it is interesting and useful to find a link between the behaviour of the scattering phase difference and the existence of poles in small "boxes"

$$\{z \in \mathbb{C} : |\operatorname{Re} z - \tau| \leq B\delta,\ \operatorname{Im} z \leq \delta\epsilon\},\ B > 0. \tag{1.14}$$

To motivate our results let us recall the case studied in [3] (see also [15] for the long range case). Let $s_h(\lambda)$ be the relative scattering phase associated to two operators $L_j(h) = -h^2\Delta + V_j(x)$, $j = 1, 2$, $h > 0$, where $V_1(x)$ is a non-trapping potential for the energy level $0 < \lambda_0 = |\xi|^2 + V_1(x)$, while $V_2(x)$ is a "well in the island" trapping potential for the level $\lambda_0 = |\xi|^2 + V_2(x)$. Let $\Gamma(h)$ be the set of poles of the scattering operator related to $L_j(h)$, $j = 1, 2$, and let $\lambda(h) \in \Gamma(h)$ be a simple pole. Next let $\delta(h) > 0$ be such that

$$\lim_{h\searrow 0} h^{-n}\delta(h) = 0,\ \ \lim_{h\searrow 0}\frac{\operatorname{Im}\lambda(h)}{\delta(h)} = 0. \tag{1.15}$$

Assume that $\Omega(h)$ is a complex neighbourhood of λ_0 such that

$$\bigcap_{h>0}\Omega(h) = \{\lambda_0\},\ \ \operatorname{Re}\lambda(h) \pm \delta(h) \in \Omega(h) \cap \mathbb{R},\ \ 0 < h \leq h_0,$$

$$\operatorname{dist}\Big(\Gamma(h),\, \partial\Omega(h)\Big) \geq c_\epsilon e^{-\epsilon/h},\ \forall \epsilon > 0$$

and, moreover, suppose that $\lambda(h)$ is the unique pole in $\Omega(h)$. Under these assumptions Gérard, Martinez and Robert [3] (see also [15] for long range potentials) proved that

$$\lim_{h\searrow 0}\Big[s(\operatorname{Re}\lambda(h) \pm \delta(h)) - s(\operatorname{Re}\lambda(h))\Big] = \pm\pi. \tag{1.16}$$

We start by presenting some improved estimates on the scattering determinant in Sect.2. In Sect.3 we obtain a lower bound for the difference (1.13) taking $B < 1$, $0 < \epsilon \leq 1$ in (1.14). As a consequence we obtain an upper bound (1.6) for the variation of the counting function of scattering poles near the real axis. This shows that the distribution near the real axis has to be to some extent regular. Sect.4 is devoted to the proof of the main theorem and in Sect.5 we obtain upper bounds for the variation of the scattering phase (1.13). The estimate involves the number of the poles in a "box"

$$\{z \in \mathbb{C} : |\operatorname{Re} z - \tau| \leq \delta^{1-2\gamma}\,,\ \operatorname{Im} z \leq \delta^{1-\gamma}\}\,,$$

with an error $\mathcal{O}(\delta^\gamma)\tau^{n-1}$. Finally, in Sect.6, we combine the bounds for the scattering differences with the Weyl asymptotics for the scattering phase for metric perturbations [11] to establish the existence of poles λ_j with $\operatorname{Im}\lambda_j \longrightarrow 0$.

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2. Estimates on the scattering determinant

We start by recalling results on the factorization of the scattering determinant. For n odd

$$\begin{gathered}\det S(\lambda) = e^{g(\lambda)}\frac{P(-\lambda)}{P(\lambda)}\,,\quad |g(\lambda)| \leq C|\lambda|^n + C\,,\quad \lambda \in \mathbb{C}\,,\\ P(\lambda) = \prod_j E(\lambda/\lambda_j, n)\,,\quad E(z,p) = (1-z)e^{z+\frac{z^2}{2}+\cdots+\frac{z^p}{p}}\,.\end{gathered} \tag{2.1}$$

Here, as elsewhere in this paper, the resonances can have multiplicities, that is, we can have $\lambda_i = \lambda_j$ for $i \neq j$.

In the study of the relation between the scattering phase and resonances it is therefore natural to use the factorization formula rather than (3.1). In even dimensions we have a weaker factorization given in [32] (see (2.3) and (2.7) there):

$$\begin{gathered}\det S(\lambda) = e^{g_\rho(\lambda)}\frac{P_\rho(-\lambda)}{P_\rho(\lambda)}\,,\quad |\partial_\lambda^k g_\rho(\lambda)| \leq C_{\rho',k}|\lambda|^{m-k}\,,\quad \lambda \in \Lambda'_\rho\,,\ m \geq n\,, \rho' < \rho\,,\\ P_\rho(\lambda) = \prod_{\lambda_j\in\Lambda_\rho} E(\lambda/\lambda_j, n)\,,\quad \Lambda_\rho = \{\lambda \in \mathbb{C}\ :\ |\arg \pm\lambda| < \rho < \pi/2\,,\ \pm\operatorname{Re}\lambda > 1\}\,.\end{gathered} \tag{2.2}$$

For λ in the physical half plane we will give an estimate which generalizes and improves the estimate given in Proposition 2 and (14) of [30]:

Lemma 1. *Let L be an operator satisfying the "black box" assumption of Sect.1 but with* (1.9) *replaced by a weaker assumption*

$$\sharp\{\lambda\ :\ \lambda^2 \in \sigma(L^\sharp)\,,\ \lambda \leq r\} \leq C(L)r^m\,,\quad m \geq n\,. \tag{2.3}$$

Then, for $\operatorname{Im}\lambda \geq 0$, $\operatorname{Re}\lambda \geq 0$, $|\lambda - \tilde{\lambda}| > \epsilon$, $\tilde{\lambda}^2 \in \sigma_{\rm pp}(L) \cup \{0\}$ *we have*

$$|\det S(-\lambda)| \leq C_\epsilon \exp(C_\epsilon \operatorname{Im}\lambda |\lambda|^{n-1}) .$$

Here we identified the first quadrant in $\mathbb{C}$ *with the correspoding subset of* Λ *when* n *is even.*

Proof. Let $\operatorname{Im}\lambda > C_0 \gg 1$. We denote below by C different constants which are independent of λ. We can then use the representation of $S(-\lambda) = I + A(-\lambda)$ given in the proof of Theorem 4 in [32] (we could also use the representation given in [31], see (2.4) in [32]):

$$A(-\lambda) = -c_n \lambda^{n-2} \mathbb{E}^{\phi_1}(\lambda)[\Delta, \chi_2] R(-\lambda)[\Delta, \chi]^t \mathbb{E}^{\phi_2}(\lambda) ,$$

where $\mathbb{E}^\phi(\lambda) : L^2(\mathbb{R}^n) \to L^2(\mathbb{S}^{n-1})$ has the kernel given by $K(\theta, x) = \exp(i\lambda\langle x, \theta\rangle)\phi(x)$, $\phi_j, \chi_2, \chi \in \mathcal{C}^\infty(\mathbb{R}^n)$, identically 1 near $B(0, R_0)$ are specially chosen. We have for $\operatorname{Im}\lambda > C_0$

$$\|R(-\lambda)\|_{\mathcal{H}\to\mathcal{H}} \leq 1/|\operatorname{Im}(\lambda^2)|$$

and

$$\|R(-\lambda)\|_{\mathcal{H}\to\mathcal{H}} \leq \frac{C}{|\lambda|},$$

provided $\operatorname{Im}\lambda > C_0$. Using the equation, we obtain

$$\|[\Delta, \chi_2] R(-\lambda)[\Delta, \chi]\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \leq C|\lambda| , \quad \operatorname{Im}\lambda > C_0 ,$$

which immediately gives

$$|A(-\lambda)(\theta, \omega)| \leq C|\lambda|^{n-1} e^{C \operatorname{Im}\lambda} , \quad \operatorname{Im}\lambda > C_0 ,$$

and the analyticity of the kernel $A(-\lambda)(\theta, \omega)$ of $A(-\lambda)$ in θ. The Cauchy estimates then yield

$$|\Delta_\theta^m A(\theta, \omega)| \leq C^m (2m)! e^{C|\lambda|} , \quad \operatorname{Im}\lambda > C_0 ,$$

and hence following [9],[30] we can estimate characteristic values of $A(-\lambda)$:

$$\begin{aligned} \mu_j(A(-\lambda)) &\leq \mu_j((-\Delta_\theta + I)^{-m}) \|(-\Delta_\theta + I)^m A(-\lambda)\| \\ &\leq C^m (2m)! j^{-\frac{2m}{n-1}} e^{C|\lambda|} \\ &\leq C e^{C|\lambda| - j^{\frac{1}{n-1}}/C} , \end{aligned}$$

where the last inequality is achieved through an optimization in m. We now observe that

$$\prod_{j \geq C A^{n-1}} \left(1 + e^{A - j^{\frac{1}{n-1}}/C} \right) \leq \widetilde{C} e^A , \quad A \geq 0 ,$$

and consequently, via the standard use of Weyl inequalities (see [30]),

$$\begin{aligned} |\det S(-\lambda)| &\leq \prod_{j=1}^\infty (1 + \mu_j(A(-\lambda))) \leq C e^{C \|A(-\lambda)\| |\lambda|^{n-1}} e^{C|\lambda|} \\ &\leq C e^{C(\operatorname{Im}\lambda + (n-1)\log|\lambda|)|\lambda|^{n-1}} , \quad \operatorname{Im}\lambda > C_0 \end{aligned}$$

since

$$\|A(-\lambda)\| \leq C e^{C \operatorname{Im}\lambda + (n-1)\log|\lambda|} , \quad \operatorname{Im}\lambda > C_0.$$

For $\lambda \in \mathbb{R}$ we have $|\det S(-\lambda)| = 1$ and for $|\lambda - \tilde{\lambda}| = \epsilon$, $\tilde{\lambda}^2 \in \sigma_{\rm pp}(L) \cup \{0\}$, $|\det S(-\lambda)| \leq C_\epsilon$. Also, as in [4],[31], we can estimate $\det S(-\lambda)$ globally away from its poles:

$$|\det S(-\lambda)| \leq C e^{C|\lambda|^{mn+\epsilon}} , \quad -\lambda \notin \bigcup_{\lambda_j \neq 0} D(\lambda_j, |\lambda_j|^{-m-\epsilon}) , \ |\lambda| > 1 , \ n \text{ odd}.$$

For n even the same estimate is true in a conic neighbourhood of the positive real axis as was shown in [32]. Combined with the estimate on the real axis and the maximum principle this gives $|\det S(-\lambda)| \leq \exp(C|\lambda|^N)$ for $\operatorname{Im}\lambda \geq 0$, $|\lambda - \tilde{\lambda}| > \epsilon$, $\tilde{\lambda}^2 \in \sigma_{\rm pp}(L) \cup \{0\}$ and some fixed N.

To apply the Phragmén-Lindelöf principle consider the function

$$g(\lambda) = e^{iB\lambda^n} \det S(-\lambda)\,, \quad \operatorname{Im}\lambda \geq 0\,,\ |\operatorname{Re}\lambda| > C_2 > 0$$

with a constant $B > 0$ which we will choose below. On the curve

$$\gamma = \{\lambda \in \mathbb{C} : \operatorname{Im}\lambda = \log|\lambda| > C_0\,,\ \operatorname{Re}\lambda \geq C_2 > 0\}$$

we have

$$1 - \frac{1}{n}\sum_{j=1}^{n-1} C_{j+1}^n \Big(\frac{\operatorname{Im}\lambda}{\operatorname{Re}\lambda}\Big)^j \geq 1/2,$$

provided we take $C_2 > 0$ sufficiently large. Thus we deduce that

$$|e^{iB\lambda^n}| \leq \exp\Big(-\frac{nB}{2}\operatorname{Im}\lambda(\operatorname{Re}\lambda)^{n-1}\Big)\,, \quad \lambda \in \gamma.$$

Combining this with the estimate for $|\det S(-\lambda)|$, we obtain

$$|g(\lambda)| \leq C,\ \ \lambda \in \gamma$$

assuming $|\lambda|/\operatorname{Re}\lambda \leq C_3$ on γ and $B = 2CC_3^{n-1}$. On the other hand, $|g(\lambda)| = 1$ for $\operatorname{Im}\lambda = 0$ and an application of Phragmén-Lindelöf principle yields

$$|\det S(-\lambda)| \leq C|e^{-iB\lambda^n}| \leq C_4 e^{C_4 \operatorname{Im}\lambda|\lambda|^{n-1}} \ \text{ for } \ 0 \leq \operatorname{Im}\lambda \leq \log|\lambda|\,, \quad \operatorname{Re}\lambda \geq C_2.$$

By the maximum principle we complete the proof. $\square$

As the first application of this estimate we obtain a local factorization of the scattering determinant. The proof is inspired by recent progress in trace formulæ[1] for resonances, [4], [19], [32]:

Lemma 2. *If for $|\tau| \gg 0$ we write*

$$\det S(\lambda) = e^{g_{\rho,\tau}(\lambda)} \frac{\overline{P_{\rho,\tau}(\overline{\lambda})}}{P_{\rho,\tau}(\lambda)}\,, \ |\lambda - \tau| \leq |\tau|/A\,, \ \ P_{\rho,\tau}(\lambda) = \prod_{\substack{|\lambda_j - \tau| \leq |\tau|/2 \\ \lambda_j \in \Lambda_\rho}} (\lambda - \lambda_j)\,, \quad A > 2\,,$$

then for $|\lambda - \tau| \leq |\tau|/C_\rho$, we can choose the holomorphic function $g_{\rho,\tau}$ so that

$$|\partial_\lambda^l g_{\rho,\tau}(\lambda)| \leq C_l |\lambda|^{n-l},\ \ l \geq 0\,,$$

where $C_\rho \geq A > 2$ and C_l are independent of τ.

Proof. Let us take $\tau > 0$. We first notice that $g_{\rho,\tau}(\overline{\lambda}) = -\overline{g_{\rho,\tau}(\lambda)}$. Next we choose $B_\rho > A$ large enough to arrange

$$\{\lambda : |\lambda - \tau| \leq \tau/B_\rho\} \subset \Lambda_\rho \cap \{\operatorname{Re}\lambda > 1\}.$$

We fix B_ρ and we observe that by Lemma 1, for $|\lambda - \tau| \leq \tau/B_\rho$, we have

$$|\det S(\lambda)| \leq Ce^{C|\lambda|^n}\,, \ \ \operatorname{Im}\lambda \leq 0\,.$$

Here and below we denote by C different constants independent on λ and τ. We note that we can replace $P_{\rho,\tau}(\lambda)$ by

$$\prod_{\substack{|\lambda_j - \tau| \leq |\tau|/2 \\ \lambda_j \in \Lambda_\rho}} \left(\frac{\lambda - \lambda_j}{\tau}\right)\,,$$

so that the bounds on the number of resonances, (1.10), (1.11), give

$$|P_{\rho,\tau}(\lambda)| \leq Ce^{C|\lambda|^n}\,, \quad |\lambda - \tau| \leq \tau/B_\rho\,.$$

[1]As was recently pointed out by Sjöstrand, the underlying feature of these arguments is already present in the classical proof of Lidskii's theorem.

On the other hand, Cartan's Lemma (see for instance [6], Lemma 6.17), implies that outside of a union of discs with radii summing up to $\epsilon\tau$, $\epsilon > 0$

$$|P_{\rho,\tau}(\lambda)| > \exp\left(-C \log \frac{1}{\epsilon} \tau^n\right) .$$

If $\epsilon \ll 1$ we can find $B > B_\rho$ such that this estimate is valid on $|\tau - \lambda| = \tau/B$. Combining this with the upper bounds on $P_{\rho,\tau}(\lambda)$ and $S(\lambda)$ above, and with the maximum principle we obtain

$$\operatorname{Re} g_{\rho,\tau}(\lambda) \leq C_1 \tau^n , \quad \operatorname{Im} \lambda \leq 0 , \quad |\lambda - \tau| \leq \tau/B .$$

Applying Harnack's inequality to the positive harmonic function $G(\lambda) = 2C_1\tau^n - \operatorname{Re} g_{\rho,\tau}(\lambda)$ in discs $D(\tilde{\lambda}, r) \subset D(\hat{\lambda}, 3r)$, $\operatorname{Im} \hat{\lambda} < -3r$ and comparing the maximum of $G(\lambda)$ on $D(\tilde{\lambda}, r)$ with the minimum of $G(\lambda)$ on $D(\tilde{\lambda}, r)$ gives

$$-3C_1\tau^n \leq \operatorname{Re} g_{\rho,\tau}(\lambda) \leq C_1\tau^n , \quad \operatorname{Im} \lambda \leq 0 , \quad |\lambda - \tau| \leq \tau/B .$$

Hence $|\operatorname{Re} g_{\rho,\tau}(\lambda)| \leq C|\lambda|^n$ for $\operatorname{Im} \lambda \leq 0$, $|\lambda - \tau| \leq \tau/B$, which, due to the symmetry of $g_{\rho,\tau}$, immediately yields

$$|\operatorname{Re} g_{\rho,\tau}(\lambda)| \leq C|\lambda|^n \quad \text{for} \quad |\lambda - \tau| \leq \tau/B .$$

From this we see that $|g_{\rho,\tau}(\lambda)| \leq C|\lambda|^n$ for $|\lambda - \tau| \leq \tau/2B$. In fact, Carathéodory's inequality (see for instance [24], 5.5) shows that

$$\max_{|\lambda-\tau|\leq\tau/2B} |g_{\rho,\tau}(\lambda)| \leq 2 \max_{|\lambda-\tau|\leq\tau/B} \operatorname{Re} g_{\rho,\tau}(\lambda) + 3|g_{\rho,\tau}(\tau)| .$$

We now note that $g_{\rho,\tau}$ is determined up to an integral multiple of $2\pi i$, and hence we can choose $g_{\rho,\tau}$ so that $|g_{\rho,\tau}(\tau)| \leq 2\pi$. The Cauchy inequalities then give the symbolic estimates. $\square$

3. Lower bounds on the variation of the scattering phase

Under the assumptions (1.7)-(1.9) (or in fact under a much weaker assumption than (1.9)) we can define a distribution

$$u(t) = 2 \operatorname{tr} \left(\cos(t\sqrt{L}) - \mathbb{1}\restriction_{\mathbb{R}^n \setminus B(0,R_0)} \cos(t\sqrt{\Delta}) \mathbb{1}\restriction_{\mathbb{R}^n \setminus B(0,R_0)} \right) \in \mathcal{D}'(\mathbb{R}) .$$

The basic trace formula of scattering theory, the *Birman-Krein* formula, relates the wave trace, $u(t)$, and the scattering phase:

$$u(t) = \frac{1}{2\pi} \widehat{\frac{ds}{d\lambda}}(t) + 2 \sum_{\substack{\lambda_j^2 \in \sigma_{\rm pp}(L) \\ \operatorname{Im} \lambda_j < 0}} \cos(\lambda_j t) , \quad t \neq 0 ,$$

see [2] and references given there.

The other trace formula relates the wave trace to resonances and is a non-trivial analogue of the easy spectral trace formula for operators on compact manifolds. For n odd we have

$$u(t) = \sum_j e^{i\lambda_j |t|}, \quad t \neq 0, \tag{3.1}$$

in the sense of distributions on $\mathbb{R} \setminus \{0\}$ and where the sum is taken over all poles λ_j included according to their multiplicities. This was established in successive generality by Bardos-Guillot-Ralston, Melrose and Sjöstrand-Zworski – see [19] and the references given there. As was pointed out in [4],[31],[32], the Poisson formula is equivalent to the Birman-Krein formula and the factorization results for the scattering determinant recalled in the beginning of Sect.2.

Motivated by the Poisson formula (3.1) we now apply a modification of the idea of Melrose [10] for regularizing $s(\lambda)$. It allows the best available general justification of the Breit-Wigner approximation at high energies.

Let us take $\chi(t) \in \mathcal{C}^\infty(\mathbb{R})$ such that $0 \leq \chi(t) \leq 1$, $\chi(t) = 1$ for $5/6 \leq t \leq 7/6$, $\chi(t) = 0$ for $t \geq 4/3$ and for $t \leq 2/3$. We then put

$$\frac{d}{d\lambda} s_{\mathrm{norm}}(\lambda) = 2\pi \sum_{\lambda_j \in \Lambda_\rho} \chi\left(\frac{|\lambda_j|}{\lambda}\right) P(\lambda_j, \lambda)\,, \quad s_{\mathrm{norm}}(0) = 0\,,$$

where $P(z, \lambda)$ is given by (1.3). Next, we define

$$s_{\mathrm{reg}}(\lambda) = s(\lambda) - s_{\mathrm{norm}}(\lambda)\,. \tag{3.2}$$

The essential part of Melrose's argument was showing that $s_{\mathrm{reg}}(\lambda) \in S^n((0,\infty))$ is a symbol of order n. We prove the same result in this greater generality:[2]

Lemma 3. *Let s_{reg} be given by (3.2). Then*

$$s_{\mathrm{reg}}(\lambda) \in S^n((0,\infty)). \tag{3.3}$$

Proof. First we will show that for $|\lambda - \tau| \leq |\tau|/C_\rho$, we have

$$\left|\left(\frac{d}{d\lambda}\right)^k s'_{\mathrm{reg}}(\lambda)\right| \leq C_k(1 + |\lambda|)^{n-k-1}, \ \ |\lambda| \geq 1, \ \ k \geq 0\,, \tag{3.4}$$

with constants $C_k > 0$ independent of τ. In the notation of Lemma 2 we have

$$s'(\lambda) = ig'_{\rho,\tau}(\lambda) + \sum_{\substack{|\lambda_j - \tau| \leq |\tau|/2 \\ \lambda_j \in \Lambda_\rho}} \frac{2\,\mathrm{Im}\,\lambda_j}{|\lambda - \lambda_j|^2}\,,$$

so that

$$\begin{aligned} s'_{\mathrm{reg}}(\lambda) = ig'_{\rho,\tau}(\lambda) &+ \sum_{\substack{|\lambda_j - \tau| \leq |\tau|/2 \\ \lambda_j \in \Lambda_\rho}} \left(1 - \chi\left(\frac{|\lambda_j|}{\lambda}\right)\right) \frac{2\,\mathrm{Im}\,\lambda_j}{|\lambda - \lambda_j|^2} \\ &- \sum_{\substack{|\lambda_j - \tau| > |\tau|/2 \\ \lambda_j \in \Lambda_\rho}} \chi\left(\frac{|\lambda_j|}{\lambda}\right) \frac{2\,\mathrm{Im}\,\lambda_j}{|\lambda - \lambda_j|^2}\,. \end{aligned} \tag{3.5}$$

We already have symbolic estimates for $g'_{\tau,\rho}(\lambda)$ and it remains to study the remaining terms. In the second term on the right hand side of (3.5) we have $|\lambda_j - \lambda| \geq |\lambda|/C$ and the estimate (3.4) for this term follows from the upper bound on the number of resonances in the sum, which by (1.10), (1.11) is $C|\tau|^n$. The third term is treated in the same way, once we note that in the sum, $|\lambda - \lambda_j| \geq |\tau|/C$, if C_ρ is large enough. Since the constants C_k are independent of τ, we conclude that $s'_{\mathrm{reg}}(\lambda) \in S^{n-1}((0,\infty))$ and by integration we complete the proof. ☐

We should remark that we could now give a new proof of (1.12) in even dimensions following the now standard argument of Hörmander, as applied in [10].

The normalized part of $s(\lambda)$, $s_{\mathrm{norm}}(\lambda)$, involves contributions of resonances with moduli comparable with $|\lambda|$ and behaves in the way giving "blips". The estimates (3.4) show that for the variation of $s_{\mathrm{reg}}(\lambda)$ is well behaved, $s_{\mathrm{reg}}(\tau + \delta) - s_{\mathrm{reg}}(\tau - \delta) = \mathcal{O}(\delta)\tau^{n-1}$ for $\delta > 0$.

Proposition 1. *Let $C_1 > 1$, $C = 2\left(1 - {C_1}^{-1}\right)^{-1}$ be fixed. For each $0 < \epsilon \leq 1$, each $\delta > 0$ and $\tau \geq 1$ we have*

$$s(\tau + \delta) - s(\tau) \geq (\pi - C\epsilon)\,\sharp\,\{\lambda_j : \tau \leq \mathrm{Re}\,\lambda_j \leq \tau + \frac{\delta}{C_1},\ \mathrm{Im}\,\lambda_j \leq \epsilon\delta\} + \mathcal{O}(\delta\tau^{n-1})\,, \tag{3.6}$$

$$s(\tau) - s(\tau - \delta) \geq (\pi - C\epsilon)\,\sharp\,\{\lambda_j : \tau - \frac{\delta}{C_1} \leq \mathrm{Re}\,\lambda_j \leq \tau,\ \mathrm{Im}\,\lambda_j \leq \epsilon\delta\} + \mathcal{O}(\delta\tau^{n-1})\,, \tag{3.7}$$

[2]The argument is different from the strictly odd dimensional argument of [10] based on the exact Poisson formula.

where $\mathcal{O}(\delta\tau^{n-1})$ *is independent of* ϵ.

Proof. We shall only treat the inequality (3.6) since the proof of (3.7) is similar. As we already remarked the integration of the derivative of $s_{\mathrm{reg}}(\lambda)$ over $(\tau,\ \ \tau+\delta)$ yields a term $\mathcal{O}(\delta\tau^{n-1})$, it is only necessary to examine the integral

$$\int_\tau^{\tau+\delta}\frac{d}{d\lambda}s_{\mathrm{norm}}(\lambda)d\lambda \geq 2\pi \sum_{\substack{|\operatorname{Re}\lambda_j-\tau|\leq\gamma\\ \operatorname{Im}\lambda_j\leq\epsilon\delta}}\int_\tau^{\tau+\delta}P(\lambda_j,\lambda)d\lambda$$
$$+\ 2\pi\sum_{\substack{|\operatorname{Re}\lambda_j-\tau|\leq\gamma\,,\ \operatorname{Im}\lambda_j\leq\epsilon\delta\\ 7\tau/6\leq|\lambda_j|\leq 4(\tau+\delta)/3\\ 2\tau/3\leq|\lambda_j|\leq 5(\tau+\delta)/6}}\int_\tau^{\tau+\delta}\left(\chi\left(\frac{|\lambda_j|}{\lambda}\right)-1\right)P(\lambda_j,\lambda)d\lambda\,,$$

where we take $\gamma<\delta$. In the second sum we have

$$|\lambda_j-\lambda|\geq\lambda/6\geq\tau/6$$

and

$$\frac{2\operatorname{Im}\lambda_j}{|\lambda_j-\lambda|^2}\leq 72\epsilon\delta\tau^{-2}.$$

Combining this with the upper bound of the number of poles in the disk $|z|\leq A\tau$, $\mathcal{O}(\tau^n)$, we obtain for this term the bound $\mathcal{O}(\epsilon\delta^2\tau^{n-2})$.

To deal with the first sum, we write

$$2\pi\int_\tau^{\tau+\delta}P(z,\lambda)d\lambda = 2\int_{(\tau-\operatorname{Re}z)/\operatorname{Im}z}^{(\tau-\operatorname{Re}z+\delta)/\operatorname{Im}z}\frac{1}{1+t^2}\,dt\,.$$

We then set

$$0<\frac{1}{y}=\frac{\operatorname{Im}z}{\delta}\leq\epsilon\,,\ \ x=\frac{\tau-\operatorname{Re}z}{\operatorname{Im}z}\leq 0$$

and assume that $|x|\leq\frac{1}{C_1}y$ which corresponds to assuming that $|\tau-\operatorname{Re}z|\leq\delta/C_1$. We obtain

$$2\int_x^{x+y}\frac{1}{1+t^2}\,dt\geq 2\int_0^{y\left(1+\frac{x}{y}\right)}\frac{1}{1+t^2}\,dt\geq 2\int_0^{y\left(1-\frac{1}{C_1}\right)}\frac{1}{1+t^2}\,dt$$
$$=\pi-2\int_{y\left(1-\frac{1}{C_1}\right)}^{\infty}\frac{1}{1+t^2}\,dt\geq\pi-\frac{C}{y}$$
$$\geq\pi-C\epsilon\,,\ \ \frac{2}{C}=\left(1-\frac{1}{C_1}\right)>0\,.$$

The estimate (3.6) now follows by choosing $\gamma=\delta/C_1$ and summing over all poles, λ_j, which satisfy

$$\tau\leq\operatorname{Re}\lambda_j\leq\tau+\frac{\delta}{C_1},\ \operatorname{Im}\lambda_j\leq\epsilon\delta.$$

$\square$

Remark. The constant $\pi-C\epsilon$ on the right hand side of (3.6) corresponds to the constant π in the semi-classical result (1.16). In fact, assume the existence of scattering poles λ_m such that

$$\operatorname{Im}\lambda_m\leq C(\operatorname{Re}\lambda_m)^{-n},\ \ 0<\operatorname{Re}\lambda_m\longrightarrow+\infty\,,\ \ m\longrightarrow\infty\,,\ \ C>1.$$

Choose

$$0<\nu<1,\ \delta_m=C(\operatorname{Re}\lambda_m)^{-n+\nu},\ \ \epsilon_m=(\operatorname{Re}\lambda_m)^{-\nu},\ \ \tau_m=\operatorname{Re}\lambda_m.$$

Then we deduce from Proposition 1

$$\lim_{m\to\infty}\Big(s(\tau_m+\delta_m)-s(\tau_m)\Big)\geq\pi. \tag{3.8}$$

We may choose δ_m , ϵ_m suitably in order that λ_m is the unique pole in the "box"

$$\{z\in\mathbb{C}:|\operatorname{Re}z-\tau_m|\leq\frac{\delta_m}{C},\ \operatorname{Im}z\leq\delta_m\epsilon_m\}.$$

We will see in Sect.5 that if this pole is simple we can get equality in (3.8).

We shall now obtain a bound for the number of poles

$$\sharp\,\{\lambda_j:\tau\leq\operatorname{Re}\lambda_j\leq\tau+1,\ \operatorname{Im}\lambda_j\leq 2\epsilon\}.$$

We take $C_1=2$, $C=4$, $\delta=2$ and $0<\epsilon\leq(2\pi-1)/8$ in Proposition 1 and we conclude that

$$\sharp\,\{\lambda_j:\tau\leq\operatorname{Re}\lambda_j\leq\tau+1,\ \operatorname{Im}\lambda_j\leq 2\epsilon\}\leq 2\Big[s(\tau+1)-s(\tau)\Big]+A_1\tau^{n-1}.$$

The difference $s(\tau+1)-s(\tau)$ can be estimated by (1.12). Thus we get

$$s(\tau+1)-s(\tau)\leq c_0((\tau+1)^n-\tau^n)+2C_0\tau^{n-1}\leq A_2\tau^{n-1},\ \ \tau\geq 1,$$

where $c_0>0$, $C_0>0$ and $A_2>0$ are independent of τ. Finally, with $A=2A_2+A_1$ we obtain the following

Proposition 2. *For operators satisfying* (1.7)-(1.9) *and for* $0<\epsilon\leq(2\pi-1)/8$, *there exists a constant* $A>0$ *independent of* τ *such that*

$$\sharp\,\{\lambda_j:\tau\leq\operatorname{Re}\lambda_j\leq\tau+1,\ \operatorname{Im}\lambda_j\leq 2\epsilon\}\leq A\tau^{n-1},\ \ \tau\geq 1. \tag{3.9}$$

In particular we have (1.6).

4. Proof of the Main Theorem

To prove the Theorem stated in Sect.1 we will use the representation of the scattering phase given in Sect.3 and the local bound given in Proposition 2. Thus all we need to prove is

$$\sum_{\epsilon<|\lambda_j-\tau|<\frac{1}{2}\tau}P(\lambda_j,\lambda)=\mathcal{O}_\epsilon(\tau^{n-1}),\ \ |\lambda-\tau|\ll 1, \tag{4.1}$$

as then integration over $[\tau-\delta,\tau+\delta]$ will furnish the needed δ factor. The convenient constant, $\frac{1}{2}$, came from the choice of the cut-off function χ in the definition of s_{norm} combined with the choice of a small ρ.

The sums of the Poisson kernels, $P(\lambda_j,\lambda)$, appear naturally in the Carleman estimate, see [24], 3.7 and 3.71: let $h(z)$ be holomorphic in $\operatorname{Im}z\geq 0$ with zeros λ_j, none of which lie on semi-circles $|z-\lambda_j|=\rho$ and $|z-\lambda_j|=2r$. Then for $\lambda\in\mathbb{R}$

$$\begin{aligned}\sum_{\rho<|\lambda-\lambda_j|<r}\frac{\operatorname{Im}\lambda_j}{|\lambda-\lambda_j|^2}\leq\frac{4}{3}\Bigg(&\frac{1}{2\pi r}\int_0^\pi\log|h(\lambda+2re^{i\theta})|\sin\theta d\theta-\frac{1}{\pi\rho}\int_0^\pi\log|h(\lambda+\rho e^{i\theta})|\sin\theta d\theta\\&+\frac{1}{2\pi}\int_\rho^{2r}\left(\frac{1}{y^2}-\frac{1}{4r^2}\right)\log|h(\lambda+y)h(\lambda-y)|dy\Bigg).\end{aligned} \tag{4.2}$$

The natural function for our problem comes from the determinant of the scattering matrix:

$$h(\lambda)\stackrel{\mathrm{def}}{=}\prod_{\substack{\lambda_j^2\in\sigma_{\mathrm{pp}}(L)\\ \operatorname{Im}\lambda_j<0}}(\lambda+\lambda_j)\det S(-\lambda). \tag{4.3}$$

We recall that for $\operatorname{Im}\lambda\leq 0$, $\lambda\neq 0$, $S(\lambda)$ is meromorphic with finitely many poles at $\lambda_j\neq 0$ satisfying $\lambda_j^2\in\sigma_{\mathrm{pp}}(L)$ and zeros at $-\lambda_j$ where λ_j are resonances. The multiplicity of the resonance λ_j is given by the order of the zero of $\det S(\lambda)$ at $-\lambda_j$ (except possibly at the finitely many eigenvalues) – see [31]. We also recall that due to unitarity of $S(\lambda)$ for λ real we have $|\det S(\lambda)|=1$ for $\lambda\in\mathbb{R}$.

To apply (4.2) we need some estimates on $\det S(-\lambda)$. The first one was already given in Lemma 1. The difficulty in the application of the Carleman estimate (4.2) lies in estimating the integral over the ρ-semicirle, uniformly in $\lambda \sim \tau$. We need to estimate $\log |h(\lambda + \rho e^{i\theta})|$ from below. That is done in

Lemma 4. *For any fixed $0 < \rho < 1/2$ and for $\tau \gg 1$ there exists $\rho/2 < \rho(\tau) < \rho$ such that*

$$\log | \det S(-\tau - \rho(\tau) e^{i\theta})| \geq -A_\rho \tau^{n-1} , \quad 0 \leq \theta \leq \pi ,$$

where $A_\rho > 0$ is independent of τ.

Proof. Let λ_j, $1 \leq j \leq J(\tau) = \mathcal{O}(\tau^{n-1})$, be the resonances in $|\lambda + \tau| < 1$, with the bound on $J(\tau)$ given by Proposition 2. Then we can write

$$H(\lambda) \stackrel{\text{def}}{=} \det S(-\lambda) = e^{g_\tau(\lambda)} \frac{\overline{P_\tau(\bar\lambda)}}{P_\tau(\lambda)}, \quad |\lambda - \tau| < 1 ,$$

$$P_\tau(\lambda) \stackrel{\text{def}}{=} \prod_{j=1}^{J(\tau)} (\lambda + \lambda_j) ,$$

where g_τ is holomorphic in $|\lambda - \tau| \leq 3/4$ and satisfies $g_\tau(\bar\lambda) = -\overline{g_\tau(\lambda)}$ so that in particular $\operatorname{Re} g_\tau(\lambda) = 0$ for λ real.

For $|\lambda - \tau| \leq 3/4$ we clearly have

$$|P_\tau(\lambda)| \leq C \exp(C |\tau|^{n-1}).$$

We now proceed as in the proof of Lemma 2: using Cartan's Lemma, we find $\rho/2 < \rho(\tau) < \rho$ and $0 < \epsilon \ll \rho$ independent of τ so that for $|\tau - \lambda| = \rho(\tau)$,

$$|P_\tau(\lambda)| > \left(\frac{\epsilon}{2e} \right)^{J(\tau)} > \exp\left(-C\tau^{n-1}\right) . \tag{4.4}$$

Combining this with the estimate provided by Lemma 1 and the maximum principle we obtain

$$\operatorname{Re} g_\tau(\lambda) \leq C_1 \tau^{n-1} , \quad \operatorname{Im} \lambda \geq 0 , \quad |\lambda - \tau| \leq 3/4 .$$

Applying Harnack's inequality to the positive harmonic function $G(\lambda) = 2C_1\tau^{n-1} - \operatorname{Re} g_\tau(\lambda)$ in discs $D(\tilde\lambda, r) \subset D(\hat\lambda, 3r)$, $\operatorname{Im} \hat\lambda > 3r$ and comparing the maximum of $G(\lambda)$ on $D(\tilde\lambda, r)$ with the minimum of $G(\lambda)$ on $D(\tilde\lambda, r)$ gives

$$-3C_1\tau^{n-1} \leq \operatorname{Re} g_\tau(\lambda) \leq C_1 \tau^{n-1} , \quad \operatorname{Im} \lambda \geq 0 , \quad |\lambda - \tau| \leq 3/4 .$$

Recalling (4.4) and with the choice of $\rho(\tau)$ which followed it, we now see that

$$\log |H(\lambda)| \geq -C_2 \tau^{n-1} , \quad |\lambda - \tau| = \rho(\tau) , \quad \operatorname{Im} \lambda \geq 0 ,$$

which is the statement of the lemma. □

The proof of (4.1) based on (4.2) is now immediate as the estimates of Lemmas 1 and 4 hold also for $h(\lambda)$ given by (4.3). The choice of χ (the constant $\frac{1}{2}$ in (4.1)) guarantees that we stay in the first quadrant outside a neighbourhood of 0 in applying Carleman's inequality. From this the main theorem follows from an integration over $[\tau - \delta, \tau + \delta]$.

We remark that the use of the Carleman estimate (4.2) in the study of resonances has a long tradition. Selberg used it to obtain a Blaschke factorization of the scattering matrix for finite volume non-compact hyperbolic surfaces (see the references in [4]). An observation of Titchmarsh concerning zeros of Fourier transforms was recalled in [29] to show that potentials in dimension one have $o(\theta)r$ poles in θ-conic neighbourhoods of $\mathbb{R}$. In higher dimensions and for more general scattering problems, Vodev [26] and Petkov-Vodev [12] used (4.2) to show that

$$\sum_{0 < |\lambda_j| \leq r} \frac{\operatorname{Im} \lambda_j}{|\lambda_j|^2} \leq C r^{n-1} .$$

Lemma 1 provides a somewhat more conceptual proof of this result since we can use the scattering determinant rather than determinants of some specially constructed operators.

As an immediate consequence of the main theorem we have the following observation which seems new even in the case a resonance free strip and obstacle scattering

Proposition 3. *Suppose there exists a non-decreasing continuous function $F(\lambda) > 0$ satisfying $F(\lambda+\delta) < CF(\lambda)$ for $0 < \delta \leq 1$ and such that there are no resonances in the region*

$$\{ \lambda \ : \ \mathrm{Re}\, \lambda > \tau_0 > 0 \, , \ \mathrm{Im}\, \lambda < C_0 (F(\mathrm{Re}\, \lambda))^{-1} \ \} \, .$$

Then

$$\left| \frac{ds}{d\lambda}(\tau) \right| \leq C_1 F(|\tau|) |\tau|^{n-1} \, , \quad |\tau| \geq 1 \, .$$

Proof. Assuming $\tau_0 > 1$, we have proved that

$$s'(\lambda) = 2\pi \sum_{|\lambda - \lambda_j| < 1} P(\lambda_j, \lambda) + \mathcal{O}(\lambda^{n-1}), \ \ \lambda > \tau_0 \, .$$

Since $P(\lambda_j, \lambda) \leq \pi^{-1} (\mathrm{Im}\, \lambda_j)^{-1}$, the assumption of the proposition and Proposition 2, immediately give the claimed estimate. □

We note that using the recent results of Burq [1] this gives the exponential bound on $|s'(\lambda)|$ whenever L is a metric Laplacian on the exterior of an obstacle. Moreover, for every fixed $A > 0$ we can apply the argument of Proposition 3 to the resonances in the region

$$\{\lambda : \mathrm{Re}\, \lambda > A, \ \mathrm{Im}\, \lambda < C_0 (F(\mathrm{Re}\, \lambda))^{-1} \}$$

and this implies the following

Corollary 1. *Let $F(\lambda)$ be the function in Proposition 3. Assume there exists a sequence $\tau_j \nearrow \infty$ such that*

$$\limsup_{j \to \infty} \frac{|s'(\tau_j)|}{F(\tau_j) \tau_j^{n-1}} = +\infty. \tag{4.5}$$

Then for every fixed $C_0 > 0$ we have

$$\sharp \{\lambda_j : \mathrm{Re}\, \lambda_j > 0, \ \mathrm{Im}\, \lambda_j < C_0 (F(\mathrm{Re}\, \lambda_j))^{-1} \} = \infty.$$

This completes the results in [3] and [11] where the existence of resonances z_j close to the real axis was related to the blow-up of $|s'(\tau_j)|$ for $\tau_j = \mathrm{Re}\, z_j$.

5. Upper bounds for the variation of the scattering phase

We have already given some lower bounds on the variation of the scattering phase in Proposition 1. To give upper bounds in terms of the number of resonances we need to understand the sum of harmonic measures

$$\sum_{|\lambda_j - \tau| < 1} \omega_{\mathbb{C}_+} (\lambda_j, [\tau - \delta, \tau + \delta]) \, ,$$

and that is done in

Proposition 4. *For any $\epsilon > 0$, $\gamma_1 > \gamma_2 > 0$, $0 < \gamma_2 \leq 1$ and $0 < \delta \leq \delta(\gamma_1)$, $\tau \geq \tau_0$ we have*

$$|s(\tau - \delta) - s(\tau + \delta)| \leq 2\pi \, \sharp \, \{\lambda_j \ : \ |\, \mathrm{Re}\, \lambda_j - \tau | \leq \min\{1, \delta^{1-\gamma_1}\} \, , \ \mathrm{Im}\, \lambda_j < \epsilon \delta^{1-\gamma_2} \} + C_\epsilon \delta^{\min(\gamma_1 - \gamma_2, \gamma_2)} \tau^{n-1} \, .$$

with a constant $C_\epsilon > 0$ independent of γ_1, γ_2.

Proof. We only have to show that

$$\sum_{\substack{\operatorname{Im}\lambda_j > \epsilon\delta^{1-\gamma_2} \\ |\operatorname{Re}\lambda_j - \tau| < 1}} \omega_{\mathbb{C}_+}(\lambda_j, [\tau-\delta, \tau+\delta]) = \mathcal{O}\left(\frac{\delta^{\gamma_2}}{\epsilon}\right)\tau^{n-1}, \tag{5.1}$$

$$\sum_{\substack{\operatorname{Im}\lambda_j \leq \epsilon\delta^{1-\gamma_2} \\ 1 > |\operatorname{Re}\lambda_j - \tau| > \delta^{1-\gamma_1}}} \omega_{\mathbb{C}_+}(\lambda_j, [\tau-\delta, \tau+\delta]) = \mathcal{O}\left(\epsilon\delta^{\gamma_1-\gamma_2}\right)\tau^{n-1}, . \tag{5.2}$$

To see (5.1) we put $x_j = \tau - \operatorname{Re}\lambda_j$, $y_j = \operatorname{Im}\lambda_j$ and estimate the sum of $\mathcal{O}(\tau^{n-1})$ terms of the form

$$\int_{\frac{x_j-\delta}{y_j}}^{\frac{x_j+\delta}{y_j}} \frac{1}{1+t^2}\,dt \leq \frac{2\delta}{y_j} \leq \frac{2\delta^{\gamma_2}}{\epsilon}.$$

For (5.2) we use the same notation and observe that now either $x_j - \delta$, $x_j + \delta < -\delta^{1-\gamma_1}/2$ or $x_j - \delta$, $x_j + \delta > \delta^{1-\gamma_1}/2$ (provided δ is small enough depending on γ_1). Hence

$$\int_{\frac{x_j-\delta}{y_j}}^{\frac{x_j+\delta}{y_j}} \frac{1}{1+t^2}\,dt \leq \int_{\delta^{1-\gamma_1}/(2\epsilon\delta^{1-\gamma_2})}^{\infty} \frac{1}{1+t^2}\,dt \leq 2\epsilon\delta^{\gamma_1-\gamma_2},$$

completing the proof of the proposition. □

6. Distribution of scattering poles close to the real axis

In this section we shall apply our results to the case of metric perturbations, that is, to the case where L has the form

$$L = -\sum_{i,j=1}^{n} \partial_{x_j}\left(a_{i,j}(x)\partial_{x_i}\right)$$

given in Sect.1. Let $l(x,\xi)$ be the principal symbol of L and let

$$\Sigma = \{(x,\xi) \in T^*(\mathbb{R}^n) : \, l(x,\xi) = 1\}.$$

Set $r(x,\xi) = \sqrt{l(x,\xi)}$ and consider the flow $\Phi^t = \exp(tH_r)$ on Σ related to the Hamiltonian field H_r of r. A point $\nu \in \Sigma$ is called periodic with period $T > 0$ if

$$\exp(TH_r)\nu = \nu. \tag{6.1}$$

By $T(\nu) > 0$ we denote the minimal $T > 0$ for which (6.1) holds. The set ot periodic point in Σ will be denoted by Π. Clearly,

$$\Pi \subset \{(x,\xi) \in T^*(\mathbb{R}^n) : |x| \leq R_0\}.$$

For fixed $\nu \in \Pi$ let

$$\gamma(\nu) = \{x(\sigma),\, \xi(\sigma) \in T^*(\mathbb{R}^n) : 0 \leq \sigma \leq T(\nu)\}$$

be the periodic trajectory passing through ν for $\sigma = 0$. Denote by $m(\nu) \in \mathbb{Z}_4$ the Maslov index (see Sect.21.6 of [7]) related to $\gamma(\nu)$ and the Lagrangian manifold

$$C = \{(t,x,y,\tau,\xi,\eta) \in T^*\left(\mathbb{R}\times\mathbb{R}^{2n}\right) : \tau + r(x,\xi) = 0,\ (x,\xi) = \exp(tH_r)(y,\eta)\}$$

and set $q(\nu) = \frac{\pi}{2}m(\nu)$. A point $\nu \in \Pi$ is called absolutely periodic with period $T(\nu) > 0$ if in any local coordinates z in a neighbourhood of ν with $z(\nu) = 0$ we have

$$\partial_z^\alpha\left(\Phi^T(z) - z\right)\restriction_{z=0} = 0,\ \forall\alpha.$$

The set of absolutely periodic points $\nu \in \Pi_a$ with period T will be denoted by $\Pi_a(T)$ and $\Pi_a = \cup_{t>0}\Pi_a(t)$ is the set of absolutely periodic points. Notice that $\mu(\Pi) = \mu(\Pi_a)$ (cf. [17]), where here and in the following we use the Liouville measure μ on Σ.

Next consider the function

$$Q(\lambda) = (2\pi)^{-n} \int_{\Pi_a} [\pi - \lambda T(\nu) - q(\nu)]_{2\pi} T^{-1}(\nu) d\nu$$

introduced in [5], [16]. Here $-\pi < [z]_{2\pi} \leq \pi$ is such that $z = [z]_{2\pi} + 2k\pi,\ k \in \mathbb{Z}$. Notice that $Q(\lambda)$ is right continuous and

$$Q(\lambda + 0) - Q(\lambda - 0) = (2\pi)^{-n+1} \int_{\Omega_\lambda} T^{-1}(\nu) d\nu$$

with

$$\Omega_\lambda = \{\nu \in \Pi_a : \lambda T(\nu) + q(\nu) \equiv 0 \bmod (2\pi)\}.$$

In [11] the first author established for n odd the following estimate for the scattering phase

$$\begin{aligned} &\left[Q(\tau + \frac{\delta}{2}) - Q(\tau - \frac{\delta}{2})\right] \tau^{n-1} - C_0 \delta \tau^{n-1} - o_\delta(\tau^{n-1}) \leq s(\tau + \delta) - s(\tau - \delta) \\ &\leq \left[Q(\tau + \frac{3\delta}{2}) - Q(\tau - \frac{3\delta}{2})\right] \tau^{n-1} + C_0 \delta \tau^{n-1} + o_\delta(\tau^{n-1}) \end{aligned} \tag{6.2}$$

with $C_0 > 0$ independent of τ and δ, while $o_\delta(\tau^{n-1})$ means that for every fixed $\delta > 0$ we have

$$\frac{o_\delta(\tau^{n-1})}{\tau^{n-1}} \longrightarrow_{\tau \to \infty} 0.$$

It is easy to generalize the argument of [11] to the even dimensional case. In fact for the analysis in Sect.4 of [11] we may write $s(\lambda) = s_{\text{norm}}(\lambda) + s_{\text{reg}}(\lambda)$, where s_{norm} and s_{reg} are defined in Sect.3. Therefore the function $s_1(\lambda) = s_{\text{norm}}(\lambda)$ is increasing and a Tauberian type argument can be applied for s_1 since we have the following estimate

$$\left| \left(\frac{ds_1}{d\lambda} * \varphi \right)(\lambda) \right| \leq \left| \left(\frac{ds}{d\lambda} * \varphi \right)(\lambda) \right| + \left| \left(\frac{ds_{\text{reg}}}{d\lambda} * \varphi \right)(\lambda) \right|.$$

Here $\varphi(\lambda) \in S(\mathbb{R})$ is a function with Fourier transform $\hat{\varphi}(t)$ supported in a small neighbourhood of 0.

Now suppose there exists $T_0 > 0$ such that $\mu(\Pi_a(T_0)) > 0$. Since $m(\nu)$ takes values in $\mathbb{Z}_4$, without loss of the generality we may suppose that for some set $\Pi_0 \subset \Pi_a(T_0)$ with $\mu(\Pi_0) \geq \frac{1}{4}\mu(\Pi_a(T_0))$ we have

$$T(\nu) = T_0,\ q(\nu) = q_0,\ \forall \nu \in \Pi_0.$$

Introduce the numbers

$$\tau_m = \frac{2m\pi}{T_0} - \frac{q_0}{T_0},\ m \in \mathbb{N}$$

and observe that $\Pi_0 \subset \Omega_{\tau_m}$, hence $\mu(\Omega_{\tau_m}) \geq \mu(\Pi_0) > 0$. This implies

$$Q(\tau_m + 0) - Q(\tau_m - 0) \geq (2\pi)^{-n+1} T_0^{-1} \mu(\Pi_0) = \eta_0 > 0.$$

We claim that there exists $\epsilon_0 > 0$ such that for all $0 < \epsilon \leq \epsilon_0$ we have

$$Q(\tau_m + \epsilon) - Q(\tau_m - \epsilon) \geq \eta_0 - 2\epsilon (2\pi)^{-n} \mu(\Pi_a),\ \forall m \in \mathbb{N}. \tag{6.3}$$

To prove this we need the following representation given by Safarov ([16], Proposition 1)

$$Q(\lambda + \epsilon) - Q(\lambda - \epsilon) = -2\epsilon(2\pi)^{-n}\, \mu(\Pi_a) + (2\pi)^{-n+1} \int_{\Pi_a} T^{-1}(\nu) \sum_{k \in \mathbb{Z}} \chi^{\epsilon}_{\lambda, k}(\nu) d\nu, \tag{6.4}$$

where $\chi^{\epsilon}_{\lambda,k}$ is the characteristic function of the set

$$\Omega^{\epsilon}_{\lambda,\, k} = \{\nu \in \Pi_a :\ -\epsilon T(\nu) \leq \lambda T(\nu) + q(\nu) - 2\pi k < \epsilon T(\nu)\}.$$

Given $\lambda = \tau_m$, we obtain

$$(2\pi)^{-n+1} \int_{\Pi_a} T^{-1}(\nu) \sum_{k \in \mathbb{Z}} \chi^{\epsilon}_{\tau_m,\, k}(\nu) d\nu \geq (2\pi)^{-n+1} T_0^{-1} \int_{\Pi_0} \sum_{k \in \mathbb{Z}} \chi^{\epsilon}_{\tau_m,\, k}(\nu) d\nu \geq \eta_0,$$

since for each $\nu \in \Pi_0$ we have $\chi_{\tau_m, m}(\nu) = 1$, and the claim is proved.

Now we shall apply Proposition 4 with $\gamma_1 = 2/3$ and $\gamma_2 = 1/3$ to prove the existence of scattering poles with arbitrary small imaginary parts. Fix $\epsilon > 0$ and assume that the error term in Proposition 4 is of modulus less than $C\delta^{\frac{1}{3}}\tau^{n-1}$. Choose $0 < \delta \leq \epsilon^3$ small enough to arrange

$$\frac{1}{2\pi}\Big((2\pi)^{-n}\mu(\Pi_a)\delta + C_0\delta + C\delta^{\frac{1}{3}}\Big) \leq \frac{\eta_0}{4},$$

Next fix $\delta > 0$ and choose τ_δ so that

$$\frac{1}{2\pi}|o_\delta(\tau^{n-1})| \leq \frac{\eta_0}{4}\tau^{n-1} \text{ for } \tau \geq \tau_\delta.$$

Therefore Proposition 4 combined with (6.3) and the left hand estimate in (6.2) gives the following

Proposition 5. *Let $\Pi_0 \subset \Pi_a$ be a subset with $\mu(\Pi_0) > 0$ and let $T(\nu) = T_0,\ q(\nu) = q_0,\ \forall \nu \in \Pi_0$. Then for every fixed $\epsilon > 0$ there exists $\tau_\epsilon > 0$ such that for the sequence $\tau_m = (2\pi m - q_0)T_0^{-1}$, $m \in \mathbb{N}$, we have*

$$\sharp\,\{\lambda_j : |\operatorname{Re}\lambda_j - \tau_m| \leq \epsilon\,,\ \operatorname{Im}\lambda_j \leq \epsilon^2\} \geq \frac{(2\pi)^{-n}}{2T_0}\mu(\Pi_0)\tau_m^{n-1},\ \ \tau_m \geq \tau_\epsilon\,. \tag{6.5}$$

This result is similar to clustering properties of the eigenvalues established in [5], [16], [18] since the constant $(2\pi)^{-n}(2T_0)^{-1}\mu(\Pi_0) > 0$ is independent of $\epsilon > 0$. Proposition 5 is sharper than the results in [21], [12] and [13]. In particular, in [12] the existence of scattering poles λ_j with $\operatorname{Im}\lambda_j \longrightarrow 0$ was established under some conditions. Popov [13] used the assumption $\mu(\Pi_a(T_0)) > 0$ to prove that for every $\rho > 0$ we have

$$\sharp\{\lambda_j : \operatorname{Im}\lambda_j \leq \rho\log|\lambda_j|,\ \ |\operatorname{Re}\lambda_j| \leq r\} \geq \frac{(2\pi)^{-n}}{n}\mu(\Pi_a(T_0))r^n(1 - o_\rho(1)),\ \ r \longrightarrow +\infty.$$

A direct application of Proposition 5 implies a lower bound for the counting function

$$N_\epsilon(r) = \sharp\{\lambda_j : |\operatorname{Re}\lambda_j| \leq r,\ \operatorname{Im}\lambda_j \leq \epsilon^2\}$$

and we have the following

Corollary 2. *Suppose the assumptions of Proposition 5 are fulfilled. Then for every $\epsilon > 0$ we have*

$$N_\epsilon(r) \geq \frac{(2\pi)^{-n-1}}{n}\mu(\Pi_0)r^n\left(1 - O_\epsilon(\frac{1}{r})\right)\ ,\ \ r \longrightarrow +\infty. \tag{6.6}$$

Proof. Fix $0 < \epsilon < \pi/T_0$ and let $\tau_m \geq \tau_\epsilon$ for $m \geq p+1$. Therefore the small "boxes"

$$\{\lambda_j : |\operatorname{Re}\lambda_j - \tau_m| \leq \epsilon,\ \operatorname{Im}\lambda_j \leq \epsilon^2\}$$

are disjoint and we may sum the lower bounds for all "boxes" with $p + 1 \leq m \leq M$. Thus we get

$$\sum_{m=p+1}^{M}\tau_m^{n-1} \geq \left(\frac{2\pi}{T_0}\right)^{n-1}\sum_{m=p+1}^{M}m^{n-1} - C_0\sum_{m=p+1}^{M}m^{n-2} \geq \frac{1}{n}\left(\frac{2\pi}{T_0}\right)^{n-1}M^n\left[1 - O_p(\frac{1}{M})\right]$$

with a constant $C_0 > 0$ independent of M and p. Given $r \gg 1$, let $M(r) \in \mathbb{N}$ be the biggest integer so that

$$\frac{2\pi M(r) - q_0}{T_0} \leq r - \epsilon$$

which yields

$$M(r) \geq \frac{T_0(r-\epsilon) + q_0}{2\pi} - 1 \geq \left(\frac{T_0}{2\pi}\right)r\left(1 - O_\epsilon(\frac{1}{r})\right).$$

Finally, using the symmetry of the resonances with respect to the imaginary axis, we obtain

$$N_\epsilon(r) \geq \frac{(2\pi)^{-n}}{T_0}\mu(\Pi_0)\sum_{m=p+1}^{M(r)}\tau_m^{n-1} \geq \frac{(2\pi)^{-n-1}}{n}\mu(\Pi_0)r^n\left(1 - O_\epsilon(\frac{1}{r})\right).$$

$\square$

Remark. It is important to note that the constant in (6.6) is independent of T_0 and ϵ and that this constant modulo the factor $(2\pi)^{-1}$ coincides with that in the result of Popov cited above. A similar constant appears also in the lower bounds with factor r^n obtained by Stefanov [23] who assumed existence of a suitable elliptic periodic ray.

The assumption $\mu(\Pi_a) > 0$ is necessary for the clustering results. More precisely, we have the following

Proposition 6. *Assume there exist constants $\kappa > 0$ and $B > 0$ and sequences $\epsilon_m \searrow 0$, $\tau_m \nearrow +\infty$ such that*

$$\sharp\{\lambda_j : |\operatorname{Re}\lambda_j - \tau_m| \leq B\epsilon_m,\ \operatorname{Im}\lambda_j \leq \epsilon_m^2\} \geq \kappa\tau_m^{n-1},\ \ \forall m \in \mathbb{N}.$$

Then

$$2(2\pi)^{-n}\int_{\Pi_a} T^{-1}(\nu)d\nu \geq \kappa. \tag{6.7}$$

Proof. Choose $\delta_m = 2B\epsilon_m$ and consider the term

$$\Big[Q\Big(\tau_m + \frac{3\delta_m}{2}\Big) - Q\Big(\tau_m - \frac{3\delta_m}{2}\Big)\Big]\tau_m^{n-1} + o_{\delta_m}(\tau_m^{n-1})$$

involved in the right hand side of (6.2). To overcome the difficulties related to $o_{\delta_m}(\tau_m^{n-1})$, we shall use the argument of [5], [16] replacing the sequence δ_m by a slowly decreasing sequence $\delta'_m \geq \delta_m$, $\delta'_m \longrightarrow 0$. Choose $N_1 < N_2 < \ldots < N_m < \ldots$, $N_k \geq k$ so that

$$|o_{\delta_m}(\tau_j^{n-1})| \leq \frac{1}{m}\tau_j^{n-1},\ \forall j \geq N_m.$$

Next define $\delta'_k = \delta_m$ for $N_m \leq k < N_{m+1}$, $m \in \mathbb{N}$, and note that

$$\frac{|o_{\delta'_m}(\tau_m^{n-1})|}{\tau_m^{n-1}} \longrightarrow_{m\to\infty} 0.$$

Now we apply Proposition 1 with $C_1 = 2$, $C = 4$, $\epsilon = \epsilon_m(2B)^{-1}$ and deduce

$$\lim_{m\to\infty}\Big(s(\tau_m + \delta'_m) - s(\tau_m - \delta'_m)\Big)\tau_m^{-n+1} \geq$$

$$\lim_{m\to\infty}\Big(\tau_m^{-n+1}\Big[(\pi - 2\epsilon_m B)\sharp\{\lambda_j : |\operatorname{Re}\lambda_j - \tau_m| \leq \frac{\delta'_m}{2},\ \operatorname{Im}\lambda_j \leq \delta'_m\epsilon_m(2B^{-1})\}\Big] - C_1\delta'_m\Big) \geq \pi\kappa$$

with $C_1 > 0$ independent of ϵ_m, τ_m and δ'_m. Now applying the right hand estimate in (6.2), we get

$$\lim_{m\to\infty}\Big[Q\Big(\tau_m + \frac{3\delta'_m}{2}\Big) - Q\Big(\tau_m - \frac{3\delta'_m}{2}\Big) + C_0\delta'_m + \frac{o_{\delta'_m}(\tau_m^{n-1})}{\tau_m^{n-1}}\Big] \geq \pi\kappa.$$

Setting $\epsilon'_m = \frac{3\delta'_m}{2}$ and taking into account the representation (6.4), we conclude that

$$(2\pi)^{-n+1}\lim_{\epsilon'_m\to 0}\int_{\Pi_a} T^{-1}(\nu)\sum_{k\in\mathbb{Z}}\chi^{\epsilon'_m}_{\tau_m,k}(\nu)d\nu \geq \pi\kappa. \tag{6.8}$$

It is easy to see that

$$\sum_{k\in\mathbb{Z}}\chi^{\epsilon'_m}_{\tau_m,k}(\nu) \leq \frac{\epsilon'_m}{\pi}T(\nu) + 1$$

and (6.7) follows from (6.8). The proof is complete. ∎

Remark. The above result is similar to Proposition 5 in [16] treating the necessary conditions for the clustering of eigenvalues.

The clustering of resonances is closely related to the non uniform continuity of the function $Q(\lambda)$ on $\mathbb{R}^+$. Thus we can generalize Proposition 5 in the following way

Proposition 7. *Assume there exist a constant $\kappa > 0$ and sequences $\epsilon_m \searrow 0$, $\tau_m \nearrow +\infty$ such that*

$$\lim_{m\to\infty}\Big[Q(\tau_m+\epsilon_m)-Q(\tau_m-\epsilon_m)\Big]\geq\kappa.$$

Then there exists a sequence $\eta_m \searrow 0$ such that

$$\sharp\{\lambda_j : |\operatorname{Re}\lambda_j-\tau_m|\leq\eta_m, \operatorname{Im}\lambda_j\leq\eta_m^2\}\geq\frac{\kappa}{4\pi}\tau_m^{n-1},\ \ m\geq m_0.$$

Proof. First notice that if $\epsilon'_m \geq \epsilon_m$, $\epsilon'_m \longrightarrow 0$, by using (6.4), we get

$$Q(\tau_m+\epsilon'_m)-Q(\tau_m-\epsilon'_m)\geq -2(\epsilon'_m-\epsilon_m)(2\pi)^{-n}\mu(\Pi_a)+Q(\tau_m+\epsilon_m)-Q(\tau_m-\epsilon_m),$$

hence

$$\lim_{m\to\infty}\Big[Q(\tau_m+\epsilon'_m)-Q(\tau_m-\epsilon'_m)\Big]\geq\kappa.$$

Now consider the term

$$\Big[Q(\tau_m+\epsilon_m)-Q(\tau_m-\epsilon_m)\Big]\tau_m^{n-1}-o_{2\epsilon_m}(\tau_m^{n-1})$$

and, as in the proof of Proposition 6, choose a sequence $\epsilon'_m \geq \epsilon_m$, $\epsilon'_m \longrightarrow 0$ so that

$$\frac{o_{2\epsilon'_m}(\tau_m^{n-1})}{\tau_m^{n-1}}\longrightarrow 0.$$

An application of Proposition 4 with $\gamma_1 = 2/3$, $\gamma_2 = 1/3$, $\epsilon = 1$ and $\delta = 2\epsilon'_m$ combined with the left hand estimate in (6.2) yields for m large

$$\Big[Q(\tau_m+\epsilon'_m)-Q(\tau_m-\epsilon'_m)\Big]\tau_m^{n-1}-o_{2\epsilon'_m}(\tau_m^{n-1})-2C_0\epsilon'_m\tau_m^{n-1}-C_1(2\epsilon'_m)^{1/3}\tau_m^{n-1}\leq$$

$$2\pi\sharp\{\lambda_j : |\operatorname{Re}\lambda_j-\tau_m|\leq(2\epsilon'_m)^{1/3},\ \operatorname{Im}\lambda_j\leq(2\epsilon'_m)^{2/3}\}.$$

For $m \geq m_0$ the left hand side of the above inequality can be estimated from below by $\frac{\kappa}{2}\tau_m^{n-1}$ and setting $\eta_m = (2\epsilon'_m)^{1/3}$ we complete the proof. □

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