SCATTERING MATRIX IN CONFORMAL GEOMETRY

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1. Statement of the results

This paper describes the connection between scattering matrices on conformally compact asymptotically Einstein manifolds and conformally invariant objects on their boundaries at infinity. This connection is a manifestation of the general principle that the far field phenomena on a conformally compact Einstein manifold are related to conformal theories on its boundary at infinity. This relationship was proposed in [5] as a means of studying conformal geometry, and the principle forms the basis of the AdS/CFT correspondence in quantum gravity – see [16],[26],[12],[10] and references given there.

We first define the basic objects discussed here. By a conformal structure on a compact manifold M we mean an equivalence class $[h]$ determined by a metric representative h:

$$\hat{h} \in [h] \iff \hat{h} = e^{2\Upsilon} h\,, \quad \Upsilon \in \mathcal{C}^\infty(M)\,.$$

Let X be a compact $n+1$-manifold with $\partial X = M$, or in the case M does not bound a manifold, with $\partial X = M \sqcup M$. This can be achieved trivially by putting $X = [0,1] \times M$.

Let x be a defining function of ∂X in X:

$$x|_{\overset{\circ}{X}} > 0\,, \quad x|_{\partial X} = 0\,, \quad dx|_{\partial X} \neq 0\,.$$

We say that g is a conformally compact metric on X with conformal infinity $[h]$ if

$$g = \frac{\overline{g}}{x^2}\,, \quad \overline{g}|_{T\partial X} \in [h]\,, \tag{1.1}$$

where $\overline{g}$ is a smooth metric on X. Since we can choose different defining functions, the metric g determines only the conformal class $[h]$. A conformally compact metric is said to be asymptotically hyperbolic if its sectional curvatures approach -1 at ∂X; this is equivalent to $|dx|_{\overline{g}} = 1$ on ∂X. The basic example is hyperbolic space $\mathbb{H}^{n+1}$, with boundary given by $\mathbb{R}^n$ in the half-space model and by $\mathbb{S}^n$ in the ball model: the two being conformally equivalent.

One of the results of [5] is that given a conformal structure $[h]$ on M, one can construct a conformally compact metric g with conformal infinity $[h]$ which satisfies

$$\mathrm{Ric}(g) + ng = \begin{cases} \mathcal{O}(x^\infty) & \text{for } n \text{ odd} \\ \mathcal{O}(x^{n-2}) & \text{for } n \text{ even.} \end{cases} \tag{1.2}$$

When n is even, the condition (1.2) is augmented by a vanishing trace condition to the next order. We call a metric g satisfying these conditions asymptotically Einstein. When n is odd, the condition (1.2) together with an asymptotic evenness condition uniquely determine g mod $\mathcal{O}(x^\infty)$ up to diffeomorphism. We shall call a metric g which is asymptotically Einstein and which also satisfies the asymptotic evenness condition, a Poincaré metric associated to $[h]$.

Our first theorem relates the scattering matrix of a Poincaré metric g to the "conformally invariant powers of the Laplacian" on $(M, [h])$. The scattering matrix of (X, g) is a meromorphic family $S(s)$ of pseudodifferential operators on M defined in terms of the behaviour at infinity of solutions of $[\Delta_g - s(n-s)]u = 0$, which we discuss in §3. The conformally invariant powers of the Laplacian are a family P_k, $k \in \mathbb{N}$ and $k \leq n/2$ if n is even, of scalar differential operators on M constructed in [8], which we discuss in §4. These operators are *natural* in the sense that they can be written in terms of covariant derivatives and curvature of a representative metric h, and they are *invariant* in the sense that if $\hat{h} = e^{2\Upsilon}h$, then

$$\widehat{P}_k = e^{(-n/2-k)\Upsilon} P_k \, e^{(n/2-k)\Upsilon} \,. \tag{1.3}$$

The operator P_k has the same principal part as Δ^k (in our convention the Laplacian is a positive operator) and equals Δ^k if h is flat.

Theorem 1. *Let $(M^n, [h])$ be a compact manifold with a conformal structure, and let (X, g) be a Poincaré metric associated to $[h]$. Suppose that $k \in \mathbb{N}$ and $k \leq n/2$ if n is even, and that $(n/2)^2 - k^2$ is* not *an L^2-eigenvalue of Δ_g. If $S(s)$ is the scattering matrix of (X, g), and P_k the conformally invariant operator on M, then $S(s)$ has a simple pole at $s = n/2 + k$ and*

$$c_k P_k = -\mathrm{Res}_{\, s=n/2+k} \, S(s) \,, \quad c_k = (-1)^k [2^{2k} k!(k-1)!]^{-1} \,, \tag{1.4}$$

where $\mathrm{Res}_{\, s=s_0} \, S(s)$ denotes the residue at s_0 of the meromorphic family of operators $S(s)$.

We remark that the condition on the spectrum is automatically satisfied if $k \geq n/2$ and in general can be guaranteed by perturbing the metric g in the interior. Since $S(s)$ is self-adjoint for $s \in \mathbb{R}$, as an immediate consequence of Theorem 1 and of this remark we obtain

Corollary. *The conformally invariant operators, P_k, are self-adjoint.*

This was previously known only for small values of k for which the operators can be explicitly calculated. The first two of the invariant operators are the conformal Laplacian

$$P_1 = \Delta + \frac{n-2}{4(n-1)} R$$

and the Paneitz operator

$$P_2 = \Delta^2 + \delta T d + (n-4)(\Delta J + \frac{n}{2} J^2 - 2|P|^2)/2.$$

Here R denotes the scalar curvature, $J = R/(2(n-1))$, $P_{ij} = \frac{1}{n-2}(R_{ij} - Jh_{ij})$ where R_{ij} is the Ricci curvature, $T = (n-2)Jh - 4P$ acting as an endomorphism on 1-forms, $|P|^2 = P_{ij}P^{ij}$, and δ is the adjoint of d (the divergence operator).

Another important notion of conformal geometry is Branson's Q-curvature in even dimensions. It is a scalar function on M constructed from the curvature tensor and its covariant derivatives, with an invariance property generalizing that of scalar curvature in dimension two: if once again $\hat{h} = e^{2\Upsilon}h$, then

$$e^{n\Upsilon}\widehat{Q} = Q + P_{n/2}\Upsilon\,. \tag{1.5}$$

There has been great progress recently in understanding the Q-curvature and its geometric meaning in low dimensions and on conformally flat manifolds – see [4] for an example of recent work. However, in general it remains a rather mysterious quantity – its definition (given in [2] and reviewed in §4 below) in the general case is via analytic continuation in the dimension. In dimension two it is given by $Q = R/2$, and in dimension four by $6Q = \Delta R + R^2 - 3|\mathrm{Ric}|^2$.

If n is even, then the operator $P_{n/2}$ has no constant term, i.e. $P_{n/2}1 = 0$. It therefore follows from Theorem 1 that $S(s)1$ extends holomorphically across $s = n$, so $S(n)1 = \lim_{s\to n} S(s)1$ is a well-defined function on M.

Theorem 2. *With the notation of Theorem 1, for n even, we have*

$$c_{n/2}Q = S(n)1\,. \tag{1.6}$$

Theorem 2 can be used as an alternative definition of the Q-curvature. We show in §4 that the conformal transformation law (1.5) is an easy consequence of Theorems 1 and 2. It follows from (1.5), the self-adjointness of $P_{n/2}$, and the fact that $P_{n/2}1 = 0$, that $\int_M Q$ is a conformal invariant. For $(M,[h])$ conformally flat, it follows from a result of [3] that $\int_M Q$ is a multiple of the Euler characteristic $\chi(M)$.

A specific mathematical object which appeared in the study of the AdS/CFT correspondence is the renormalized volume of an asymptotically hyperbolic manifold (X,g) – see [7] for a discussion and references. It has also appeared in geometric scattering theory – see [11],[15]. As shown in [9] (Lemma 5.2 and the subsequent paragraph), a choice of metric h in the conformal class on ∂X uniquely determines a defining function x near ∂X and an identification of a neighborhood of ∂X with $\partial X \times [0,\epsilon]$ such that g takes the form

$$g = x^{-2}(h_x + dx^2), \qquad h_0 = h, \tag{1.7}$$

where h_x is a 1-parameter family of metrics on ∂X. The renormalized volume is defined as the finite part in the expansion of $\mathrm{vol}_g(\{x > \epsilon\})$ as $\epsilon \to 0$. For asymptotically Einstein metrics the expansions take a special form

$$\mathrm{vol}_g(\{x > \epsilon\}) = c_0\epsilon^{-n} + c_2\epsilon^{-n+2} + \cdots + c_{n-1}\epsilon^{-1} + V + o(1)$$

for n odd,

(1.8)

$$\mathrm{vol}_g(\{x > \epsilon\}) = c_0\epsilon^{-n} + c_2\epsilon^{-n+2} + \cdots + c_{n-2}\epsilon^{-2} + L\log(1/\epsilon) + V + o(1)$$

for n even.

It turns out that for asymptotically Einstein metrics g, V is independent of the conformal representative h on the boundary at infinity when n is odd, and L is independent of the conformal representative when n is even. The dependence of V on the choice of h for n even is the so-called holographic anomaly – see [12],[7]. Anderson [1] has recently identified V when $n = 3$. In an appendix to [23], Epstein shows that for conformally compact hyperbolic manifolds, the invariants L for n even and V for n odd are each multiples of the Euler characteristic $\chi(X)$.

Using the connection with the scattering matrix, we are able to identify L in terms of the Q-curvature:

Theorem 3. *Let n be even and let L be defined by* (1.8)*. Then*

$$L = 2c_{n/2}\int_M Q\,, \tag{1.9}$$

where $c_{n/2}$ is defined in (1.4)*.*

We should stress that despite the fact that the scattering matrix is a global object, in some sense our results are all formal Taylor series statements at the boundary of X. In fact, in [6] it will be shown that, using a variant of the ideas introduced here, a direct definition of Q and proofs of Theorem 3 and the self-adjointness of the P_k's can be given based purely on formal asymptotics, avoiding the analytic continuation via the scattering matrix. It is nevertheless worthwhile to proceed with the full scattering theory: as a byproduct, this allows us to clarify certain confusing issues about the infinite rank poles of the scattering matrix at $s = n/2 + l/2$, $l \in \mathbb{N}$.

The paper is organized as follows. In §2 we present a simple one dimensional introduction to the relevant aspects of scattering theory. The more involved theory for asymptotically hyperbolic manifolds is then discussed in §3: using results of Mazzeo-Melrose [19] we give direct arguments for the existence and properties of the Poisson operator and the scattering matrix, focussing particularly on their behaviour for s near $n/2+\mathbb{N}/2$. Although it is not relevant for our main results, at the end of §3 we outline a method for the study of the structure of the scattering matrix complementing the treatment in [14]. In §4 we show how the invariant operators P_k may be constructed from a Poincaré metric and discuss Branson's Q-curvature. Finally, §5 combines scattering theory with §4, providing proofs of the main results.

In the paper, $\mathbb{N} = \{1, 2, \cdots\}$ denotes the natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Linear operators are identified with their distributional (integral) kernels.

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2. A simple example

As a simple-minded introduction to scattering and as an illustration of the connection between residues of the scattering matrix and asymptotic expansions we present a one-dimensional example.

Thus we consider scattering on a half line, $X = [0, \infty)$ with a compactly supported real valued potential $V \in L^\infty_{\mathrm{comp}}(X; \mathbb{R})$. The quantum Hamiltonian is given by $H = -\partial_y^2 + V(y)$ and we impose, say, the Dirichlet boundary condition at $y = 0$. We are interested in the properties of generalized eigenfunctions at energies $-s^2$:

$$(H + s^2)u(y) = 0\,, \quad u(0) = 0\,,$$

which for large values of y and for $s \neq 0$ satisfy

$$u(y) = A(s)e^{sy} + B(s)e^{-sy}. \tag{2.1}$$

For $s \in i\mathbb{R} \setminus \{0\}$, consideration of the Wronskian of u and $\bar{u}$ shows that $|A(s)|^2 = |B(s)|^2$. Normalizing, we define the scattering matrix, $S(s)$ (which in this case is a one-by-one matrix!) by

$$S(s) = \frac{B(s)}{A(s)}\,.$$

This defines a meromorphic function of $s \in \mathbb{C}$, regular for $s \in i\mathbb{R}$. The definition shows that $S(s) = S(-s)^{-1}$ and that for $s \in i\mathbb{R}$, $S(s)^* = S(s)^{-1}$. Hence for all s we have

$$S(-\bar{s})^* = S(s)^{-1}\,, \quad S(\bar{s})^* = S(s)\,, \tag{2.2}$$

so that, in particular, $S(s)$ is self-adjoint for s real. When V is compactly supported then L^2-eigenvalues of H, $E \leq 0$, correspond to the poles of $S(s)$, $E = -s^2$, $\operatorname{Re} s > 0$. This follows easily from (2.1) and self-adjointness of H. The poles of S for $\operatorname{Re} s < 0$ correspond to *resonances* – see [27].

When V is not compactly supported, then under relatively mild assumptions we still have the scattering matrix for $s \in i\mathbb{R}$ but its meromorphic continuation becomes very sensitive to the behaviour of V at infinity. The first physical case in which these difficulties[1] occurred was that of Yukawa potential, $V(y) = e^{-y}$.

We can study the Yukawa potential scattering using simple regular singular point analysis, not unlike what one encounters in the study of the free hyperbolic space. We start by making a change of variables:

$$x = e^{-y}\,, \quad H + s^2 = -(x\partial_x)^2 + x + s^2\,, \quad X = (0,1]\,.$$

Note that the definition of the scattering matrix in the new variables is

$$\begin{gathered}(H+s^2)u(x) = 0\,, \quad u(1) = 0\,, \quad s \in i\mathbb{R}\setminus\{0\}\,,\\ u(x) = x^{-s} + S(s)x^s + \mathcal{O}(x)\,, \quad x \to 0\,.\end{gathered} \tag{2.3}$$

We obtain $u(x)$ from the two solutions of $(H+s^2)G = 0$:

$$G_\pm(x,s) = x^{\pm s}\sum_{j=0}^{\infty} b_j^\pm(s)x^j\,, \quad b_j^\pm(s) = [j!\Gamma(\pm 2s + j + 1)]^{-1}\,. \tag{2.4}$$

These are independent provided $2s \notin \mathbb{Z}$. The scattering matrix is obtained by finding a combination matching the boundary conditions:

$$S(s) = -\frac{G_-(1,s)b_0^+(s)}{G_+(1,s)b_0^-(s)}\,.$$

For $l \in \mathbb{N}$, $G_+(x,l/2) = G_-(x,l/2)$ and $b_0^+(l/2) \neq 0$, $b_0^-(l/2) = 0$. Hence $S(s)$ has a simple pole at $s = l/2$, $l \in \mathbb{N}$.

Another independent solution for $s = l/2$ is obtained by taking

$$\begin{aligned}&\partial_s(G_-(x,s) - G_+(x,s))|_{s=l/2}\\ &\quad = x^{-\frac{l}{2}}\left(\sum_{j=0}^{l-1}\partial_s b_j^-(l/2)x^j - 2b_l^-(l/2)x^l\log x + \mathcal{O}(x^l)\right)\,,\end{aligned}$$

and dividing by $\partial_s b_0^-(l/2)$ gives us a solution with the prescribed leading part, as in (2.3):

$$u_{l/2}(x) = x^{-l/2} + \cdots - 2p_l x^{l/2}\log x + \mathcal{O}(x^{l/2})\,, \tag{2.5}$$

where $p_l = b_l^-(l/2)/\partial_s b_0^-(l/2)$. Comparison of the definition of the scattering matrix with the expansions gives

$$p_l = -\mathrm{Res}_{\,s=l/2}\, S(s)\,. \tag{2.6}$$

Comparison of this elementary discussion with the way in which the invariant operators P_k are formulated in §4 provides motivation for Theorem 1.

[1]This is potentially rather confusing. The "false" poles of the scattering matrix discussed below almost led Heisenberg to abandoning his S-matrix formalism, until a clear explanation was provided by Jost – see [22].

3. Scattering on conformally compact manifolds

We recall from §1 that the class of asymptotically hyperbolic manifolds, (X, g), is given by conformally compact manifolds (1.1) for which all sectional curvatures tend to -1 as the boundary is approached. That is equivalent to demanding that

$$|dx|_{\bar{g}} = 1 \quad \text{at} \quad \partial X \,, \tag{3.1}$$

which depends only on g – see [17]. This class of manifolds comes with a well developed scattering theory, which originated in the study of infinite volume hyperbolic quotients $\Gamma\backslash\mathbb{H}^{n+1}$ by Patterson, Lax-Phillips, Agmon, Guillopé, Perry and others – see [24], and references given there.

The basic facts about the spectrum $\sigma(\Delta_g)$ of the Laplacian of an asymptotically hyperbolic metric were established by Mazzeo and Mazzeo-Melrose [17], [19], [18]. It is given by

$$\sigma(\Delta_g) = [(n/2)^2, \infty) \cup \sigma_{\rm pp}(\Delta_g) \,, \quad \sigma_{\rm pp}(\Delta_g) \subset (0, (n/2)^2) \,,$$

where the pure point spectrum, $\sigma_{\rm pp}(\Delta_g)$ (the set of L^2 eigenvalues), is finite.

More refined statements follow from the main result of [19], which is the existence of the meromorphic continuation of the resolvent

$$R(s) = (\Delta_g - s(n-s))^{-1} \,.$$

This traditional choice of the spectral parameter s corresponds to

$$\begin{gathered} \operatorname{Re} s \neq \frac{n}{2} \iff s(n-s) \notin [(n/2)^2, \infty) \\ \operatorname{Re} s > n \implies s(n-s) \notin [0, \infty) \,. \end{gathered}$$

Since $\Delta_g \geq 0$, the second implication and the spectral theorem show that for $\operatorname{Re} s > n$,

$$R(s) : L^2(X) \longrightarrow L^2(X)$$

is a holomorphic family of operators. In fact $R(s)$ is meromorphic on $L^2(X)$ for $\operatorname{Re} s > n/2$, and on a smaller space continues meromorphically to $\mathbb{C}$:

Proposition 3.1. *(Mazzeo-Melrose* [19, Theorem 7.1, Lemma 6.13]*) Let $\dot{\mathcal{C}}^\infty(X)$ denote smooth functions vanishing to infinite order at ∂X. The resolvent*

$$R(s) \; : \; \dot{\mathcal{C}}^\infty(X) \longrightarrow x^s \mathcal{C}^\infty(X)$$

is meromorphic in $\{\operatorname{Re} s > n/2\}$, with poles having finite rank residues exactly for s such that $s(n-s) \in \sigma_{\rm pp}(\Delta_g)$. It extends meromorphically to $\mathbb{C}$, with poles having finite rank residues, and is regular for $\operatorname{Re} s = n/2$, $s \neq n/2$.

In [19], the structure of the kernel of $R(s)$ is given in Theorem 7.1. The mapping property stated above is proved in Lemma 6.13 only for hyperbolic space, but the proof is valid more generally using only the properties of $R(s)$ established in Theorem 7.1. The regularity for $\operatorname{Re} s = n/2$, $s \neq n/2$ follows from the spectral theorem and the absence of embedded eigenvalues – see [18].

The line $\operatorname{Re} s = n/2$ corresponds to physical scattering. As in the one dimensional case, for such s other than $n/2$, we will consider generalized eigenfunctions u of the form

$$
\begin{aligned}
&(\Delta_g - s(n-s))u = 0 \\
&u = F x^{n-s} + G x^s \,, \quad F, G \in \mathcal{C}^\infty(X) \,.
\end{aligned}
\tag{3.2}
$$

Note that $n = 0$ in the example in §2, and as there we think of $F|_{\partial X}$ and $G|_{\partial X}$ as the incoming and outgoing scattering data respectively. The scattering matrix is the operator which maps $F|_{\partial X}$ to $G|_{\partial X}$. For $\operatorname{Re} s > n/2$, it is more appropriate to think of (3.2) as a boundary value problem. Then $F|_{\partial X}$ represents "Dirichlet data" and $G|_{\partial X}$ "Neumann data", so that the analytically continued scattering matrix can be regarded as a generalized Dirichlet to Neumann map. As we shall see, the expansion for u must in general be modified by the inclusion of a log term if $2s - n \in \mathbb{N}$.

There are two consequences of Green's identity which are relevant for us. In the case $\operatorname{Re} s = n/2$, the following identity is known as the *pairing formula*:

Proposition 3.2. *For* $\operatorname{Re} s = n/2$ *and* u_1, u_2 *satisfying*

$$
\begin{gathered}
(\Delta_g - s(n-s))u_i = r_i \in \dot{\mathcal{C}}^\infty(X), \\
u_i = x^{n-s} f_i + x^s g_i + \mathcal{O}(x^{n/2+1}) \,, \quad f_i, g_i \in \mathcal{C}^\infty(\partial X),
\end{gathered}
$$

we have

$$
\int_X (u_1 \bar{r}_2 - r_1 \bar{u}_2) \, dv_g = (2s - n) \int_{\partial X} \left(f_1 \bar{f}_2 - g_1 \bar{g}_2 \right) dv_h,
\tag{3.3}
$$

where, since we chose the defining function x of ∂X, we have a natural choice in the conformal class, $h = x^2 g|_{T\partial X}$.

Proof. This is a standard application of Green's identity – see [20] and the proof of Proposition 3.3 below. □

The following analogous identity for $s \in \mathbb{R}$ will be crucial in the proof of Theorem 3.

Proposition 3.3. *Suppose $s \in \mathbb{R}$, $s > n/2$, $2s - n \notin \mathbb{N}$. Let u_1, u_2 be real-valued and satisfy (3.2) with $F_i, G_i \in \mathcal{C}^\infty(X)$. Then*

$$
\operatorname{pf} \int_{x > \epsilon} [\langle du_1, du_2 \rangle - s(n-s) u_1 u_2] dv_g = -n \int_{\partial X} F_1 G_2 dv_h = -n \int_{\partial X} G_1 F_2 dv_h \,,
\tag{3.4}
$$

where pf *denotes the finite part of the divergent integral.*

A special case of (3.4) was discussed by Witten [26] in a physical context, for $X = \mathbb{H}^{n+1}$, using explicit formulæ for u and $G|_{\partial X}$ in terms of $F|_{\partial X}$ given by the Poisson operator and the scattering matrix. Our project originated from an attempt to understand that discussion in the general setting.

Proof. With g in the form (1.7), Green's identity gives

$$\int_{x>\epsilon} [\langle du_1, du_2\rangle - s(n-s)u_1u_2]dv_g = -\epsilon^{1-n}\int_{x=\epsilon} u_1\partial_x u_2 dv_{h_\epsilon}.$$

Substituting (3.2) and calculating the finite part gives

$$\mathrm{pf}\int_{x>\epsilon} [\langle du_1, du_2\rangle - s(n-s)u_1u_2]dv_g = -\int_{\partial X} [sF_1G_2 + (n-s)F_2G_1]dv_h.$$

By symmetry of the left hand side, we deduce that $\int_{\partial X} F_1G_2dv_h = \int_{\partial X} F_2G_1dv_h$, so the result follows. □

Remark. If we assume that u_i's have expansions (3.2) and that

$$(\Delta_g - s(n-s))u_2 = 0\,, \quad (\Delta_g - s(n-s))u_1 \in \dot{\mathcal{C}}^\infty(X)\,, \quad F_2 = 0\,,$$

then the same argument gives

$$\int_X u_2(\Delta_g - s(n-s))u_1 dv_g = (n-2s)\int_{\partial X} G_2F_1dv_h\,, \quad s\in\mathbb{R}\,, \quad s>n/2\,, \tag{3.5}$$

which will be useful later.

To have the incoming and outgoing data f_i, g_i (or $F_i|_{\partial X}$, $G_i|_{\partial X}$ in (3.2)) invariantly defined we introduce density bundles of [21], $|N^*\partial X|^s$, which keep track of changes of the defining function to first order, or equivalently of the choice of the metric in the conformal class[2]: somewhat informally,

$$f\in\mathcal{C}^\infty(\partial X, |N^*\partial X|^s) \iff f = a|dx|^s\,,\ a\in\mathcal{C}^\infty(\partial X)\,.$$

We will often identify $\mathcal{C}^\infty(\partial X, |N^*\partial X|^s)$ with $\mathcal{C}^\infty(\partial X)$ in the presence of a chosen defining function.

Using Proposition 3.1 we will construct a family of *Poisson operators* $\mathcal{P}(s)$ for $\mathrm{Re}\, s\geq n/2$, $s\neq n/2$, which is meromorphic for $\mathrm{Re}\, s>n/2$ with poles only for s such that $s(n-s)\in\sigma_{\mathrm{pp}}(\Delta_g)$, and continuous up to $\mathrm{Re}\, s = n/2$, $s\neq n/2$, with the properties:

$$\begin{gathered}\mathcal{P}(s)\,:\,\mathcal{C}^\infty(\partial X, |N^*\partial X|^{n-s}) \longrightarrow \mathcal{C}^\infty(\mathring{X})\,,\\ (\Delta_g - s(n-s))\mathcal{P}(s)\equiv 0\,,\\ \mathcal{P}(s)f = x^{n-s}f + o(x^{n-s})\,, \quad \text{if } \mathrm{Re}\, s>\frac{n}{2},\\ \mathcal{P}(s)f = x^{n-s}f + x^s g + \mathcal{O}(x^{n/2+1})\,, \quad \text{if } \mathrm{Re}\, s = \frac{n}{2}, \quad s\neq\frac{n}{2},\end{gathered} \tag{3.6}$$

where $g\in\mathcal{C}^\infty(\partial X)$ and we chose trivializations of the density bundles as indicated above. If in addition $s\notin n/2 + \mathbb{N}_0/2$, then we will have

$$\mathcal{P}(s)f\in x^{n-s}\mathcal{C}^\infty(X) + x^s\mathcal{C}^\infty(X)\,. \tag{3.7}$$

[2] These bundles are equivalent to the density bundles of conformal geometry – see §4.

As in the example in §2, special care will be needed at $s \in n/2 + \mathbb{N}/2$. To construct $\mathcal{P}(s)$ for $\operatorname{Re} s \geq n/2$ and $s \notin n/2 + \mathbb{N}_0/2$, we first construct a non-linear *formal solution* operator:

$$
\begin{aligned}
&\Phi(s) \, : \, \mathcal{C}^\infty(\partial X) \longrightarrow x^{n-s}\mathcal{C}^\infty(X)\,, \\
&\Gamma(n+1-2s)^{-1}\Phi(s) \text{ is entire in } s, \\
&(\Delta_g - s(n-s))\Phi(s)f \in \dot{\mathcal{C}}^\infty(X)\,, \\
&\Phi(s)f = x^{n-s}f + \mathcal{O}(x^{n-s+1})\,.
\end{aligned} \tag{3.8}
$$

This is done by the usual asymptotic construction which we briefly recall. Choose a metric h in the conformal class on the boundary; this determines a defining function x and product identification near ∂X so that g is given by (1.7). It is easy to see that the Laplacian, Δ_g, can be written as

$$
\Delta_g = -(x\partial_x)^2 + nx\partial_x + x^2\Delta_h + xE\,, \quad E \in \operatorname{Diff}_0^2(X)\,,
$$

that is E can be locally written as a polynomial of degree 2 in $x\partial_x$ and $x\partial_{y_i}$ with coefficients in $\mathcal{C}^\infty(X)$, where y_i are local coordinates on ∂X.

From this we see that for $f_j \in \mathcal{C}^\infty(\partial X)$,

$$
(\Delta_g - s(n-s))(x^{n-s+j}f_j) = j(2s-n-j)x^{n-s+j}f_j + (x\Delta_h + \tilde{E})(x^{n-s+j+1}f_j)\,, \tag{3.9}
$$

with $\tilde{E} \in \operatorname{Diff}_0^2(X)$. By an iterative procedure we obtain a formal power series solution (see the proof of Theorem 4.2 for further details), which then gives a solution modulo $\dot{\mathcal{C}}^\infty(X)$ by applying Borel's lemma (see for instance [13, Theorem 1.2.6]). It is here that our operator becomes non-linear: the Borel construction depends on the sequence of functions f_j.[3] This gives $\Phi(s)$ satisfying (3.8):

$$
\begin{aligned}
\Phi(s)f = {} & fx^{n-s} + p_{1,s}fx^{n-s+1} + \cdots \\
& + \; p_{j,s}fx^{n-s+j} + \mathcal{O}(x^{n-s+j+1})\,,
\end{aligned} \tag{3.10}
$$

where $p_{j,s}$ is a differential operator on ∂X of order at most $2[j/2]$, such that

$$
\Gamma(n+1+j-2s)\Gamma(n+1-2s)^{-1}p_{j,s} \tag{3.11}
$$

is polynomial in s for each j.

We can now easily prove

Proposition 3.4. *For* $\operatorname{Re} s \geq n/2$, $s \notin n/2 + \mathbb{N}_0/2$, $s(n-s) \notin \sigma_{\rm pp}(\Delta_g)$, *there exists a unique linear operator* $\mathcal{P}(s)$ *satisfying* (3.6), (3.7).

[3] The use of Borel's Lemma could be avoided by truncating the expansions at a large finite order. Either direct estimation or a Banach-Baire type of argument imply that in Proposition 3.1, $\dot{\mathcal{C}}^\infty(X)$ can be replaced by $x^M\mathcal{C}^\infty(X)$ for some $M = M(s)$, which is locally constant. This approach has the advantage that linearity and continuity are transparent. However, the use of Borel's Lemma suffices for our purposes since we are primarily interested in the dependence on s with f fixed.

Proof. We define $\mathcal{P}(s)$ by starting with $\Phi(s)$ and using the resolvent to correct the error in the eigen-equation:

$$\mathcal{P}(s) = (I - R(s)(\Delta_g - s(n-s)))\Phi(s) .$$

For $f \in \mathcal{C}^\infty(\partial X)$ and $\operatorname{Re} s \geq n/2$, $s \notin n/2 + \mathbb{N}_0/2$, it follows from (3.8) that $\Phi(s)f \in x^{n-s}\mathcal{C}^\infty(X)$ and $(\Delta_g - s(n-s))\Phi(s)f \in \dot{\mathcal{C}}^\infty(X)$. So by Proposition 3.1, $\mathcal{P}(s)f$ is defined if also $s(n-s) \notin \sigma_{\rm pp}(\Delta_g)$ and has the form (3.7). Since $(\Delta_g - s(n-s))R(s) = I$ on $\dot{\mathcal{C}}^\infty(X)$, we obtain $(\Delta_g - s(n-s))\mathcal{P}(s)f = 0$. Therefore $\mathcal{P}(s)$ is a (possibly nonlinear) operator on $\mathcal{C}^\infty(\partial X)$ satisfying (3.6), (3.7).

To see the uniqueness we first observe that if v_j, $j = 1, 2$, are two solutions in $x^{n-s}\mathcal{C}^\infty(X) + x^s\mathcal{C}^\infty(X)$ with the same x^{n-s} leading terms, $f_1 = f_2$, then the formal expansion analysis above shows that

$$v_1 - v_2 \in x^s\mathcal{C}^\infty(X) .$$

For $\operatorname{Re} s > n/2$ that would imply that $v_1 - v_2 \in L^2(X)$, contradicting the assumption that $s(n-s) \notin \sigma_{\rm pp}(\Delta_g)$. For $\operatorname{Re} s = n/2$, $s \neq n/2$, we choose $\psi \in \dot{\mathcal{C}}^\infty(X)$ arbitrarily and apply (3.3) with $u_1 = v_1 - v_2$ and $u_2 = R(n-s)\psi = x^{n-s}\Psi$:

$$\int_X (v_1 - v_2)\bar{\psi}\, dv_g = (2s-n)\int_{\partial X} (f_1 - f_2)\bar{\Psi}|_{\partial X}\, dv_h = 0 .$$

That $\mathcal{P}(s)$ is linear and well-defined on densities follows from uniqueness. □

Consider the decomposition given above:

$$\mathcal{P}(s)f = \Phi(s)f - R(s)(\Delta_g - s(n-s)))\Phi(s)f , \tag{3.12}$$

with s as in Proposition 3.4. We have $\Phi(s)f \in x^{n-s}\mathcal{C}^\infty(X)$ and $R(s)(\Delta_g - s(n-s)))\Phi(s)f \in x^s\mathcal{C}^\infty(X)$. It is evident from the construction of $\Phi(s)$ that $\Phi(s)f$ may have poles for $s \in n/2 + \mathbb{N}/2$. We shall show that $\mathcal{P}(s)f$ extends holomorphically across $s \in n/2 + \mathbb{N}/2$ so long as $s(n-s) \notin \sigma_{\rm pp}(\Delta_g)$, which implies that the contributions from these poles in the two terms cancel one another.

Fix $l \in \mathbb{N}$. To analyze $\mathcal{P}(s)$ for s near $n/2 + l/2$, we construct a modified formal solution operator $\Phi_l(s)$ which extends holomorphically across $s = n/2 + l/2$. Observe first that $\Phi(n-s)$ satisfies:

$$\begin{aligned}
&\Phi(n-s) \ : \ \mathcal{C}^\infty(\partial X) \longrightarrow x^s\mathcal{C}^\infty(X) ,\\
&\Phi(n-s) \text{ is holomorphic in } \operatorname{Re} s > n/2,\\
&(\Delta_g - s(n-s))\Phi(n-s)g \in \dot{\mathcal{C}}^\infty(X) ,\\
&\Phi(n-s)g = x^s g + \mathcal{O}(x^{s+1}) .
\end{aligned}$$

Fix a metric in the conformal class on ∂X and therefore a defining function as in (1.7). Define the modified formal solution operator

$$\Phi_l(s) \stackrel{\rm def}{=} \Phi(s) - \Phi(n-s)p_{l,s} ,$$

where $p_{l,s}$ is as in (3.10) and (3.11). By the properties of $\Phi(s)$, $\Phi(n-s)$ and $p_{l,s}$, we have that $\Phi_l(s)$ is holomorphic for s near $n/2+l/2$, $s \neq n/2+l/2$, and

$$\begin{aligned}
&\Phi_l(s) \; : \; \mathcal{C}^\infty(\partial X) \longrightarrow x^{n-s}\mathcal{C}^\infty(X) + x^s\mathcal{C}^\infty(X)\,, \\
&(\Delta_g - s(n-s))\Phi_l(s)f \in \dot{\mathcal{C}}^\infty(X)\,, \\
&\Phi_l(s)f = x^{n-s}f + o(x^{n-s})\,.
\end{aligned}$$

We claim that $\Phi_l(s)$ extends holomorphically across $s = n/2+l/2$ as a map into $\mathcal{C}^\infty(\mathring{X})$, which amounts to showing that the poles arising from $\Phi(s)$ and $\Phi(n-s)p_{l,s}$ cancel one another. From (3.11) it follows that in the expansion (3.10), each of the coefficients $p_{j,s}f$ for $j < l$ is regular at $s = n/2+l/2$. The term $p_{l,s}fx^{n-s+l}$ may have a simple pole. However, since the leading term in $\Phi(n-s)p_{l,s}f$ is $p_{l,s}fx^s$, the residues cancel. Since all of the poles involved are most simple, certainly

$$\lim_{s\to n/2+l/2}(2s-n-l)\Phi_l(s)f$$

exists. This limit therefore corresponds to a formal solution of $(\Delta_g - s(n-s))u = 0$ for $s = n/2+l/2$, which is of the form $x^{n/2+l/2+1}\mathcal{C}^\infty(X)$. Hence by the iterative formula (3.9), it vanishes identically. Thus $\Phi_l(s)$ extends holomorphically across $s = n/2+l/2$.

Using

$$\lim_{t\to 0}\frac{x^t-1}{t} = \log x,$$

evaluation of the limit as $s \to n/2+l/2$ shows that

$$\Phi_l(n/2+l/2)f = x^{n/2-l/2}F + x^{n/2+l/2}\log x \; G$$

with $F, G \in \mathcal{C}^\infty(X)$, and $F|_{\partial X} = f$, $G|_{\partial X} = -2\,\mathrm{Res}_{s=n/2+l/2}\,p_{l,s}f$.

For s near $n/2+l/2$, we define the corresponding Poisson operator by:

$$\mathcal{P}_l(s) \stackrel{\mathrm{def}}{=} (I - R(s)(\Delta_g - s(n-s)))\Phi_l(s)\,. \tag{3.13}$$

Since $\Phi_l(s)$ is holomorphic across $s = n/2+l/2$, we see that if

$$(n/2)^2 - (l/2)^2 = s(n-s)|_{s=n/2+l/2} \notin \sigma_{\mathrm{pp}}(\Delta_g)\,,$$

then $\mathcal{P}_l(s)$ is also holomorphic across $s = n/2+l/2$. From the properties of $\Phi_l(s)$ and Proposition 3.1, it follows that we have

$$\begin{gathered}
\mathcal{P}_l(n/2+l/2) \; : \; \mathcal{C}^\infty(\partial X) \;\longrightarrow\; x^{n/2-l/2}\mathcal{C}^\infty(X) + x^{n/2+l/2}\log x\,\mathcal{C}^\infty(X) \\
(\Delta_g - s(n-s))\mathcal{P}_l(n/2+l/2) = 0\,.
\end{gathered}$$

For $s \neq n/2+l/2$

$$\mathcal{P}_l(s) \; : \; \mathcal{C}^\infty(\partial X) \;\longrightarrow\; \big(x^s\mathcal{C}^\infty(X) + x^{n-s}\mathcal{C}^\infty(X)\big) \cap \ker\,(\Delta_g - s(n-s))\,,$$

so from the uniqueness part of Proposition 3.4 we see that

$$\mathcal{P}_l(s) = \mathcal{P}(s)\,, \quad \text{for } s \text{ near } n/2 + l/2, \quad s \neq n/2 + l/2.$$

We summarize the discussion of Poisson operators in the following

Proposition 3.5. *There is a unique family of Poisson operators*

$$\mathcal{P}(s) : \mathcal{C}^\infty(\partial X, |N^*\partial X|^{n-s}) \longrightarrow \mathcal{C}^\infty(\overset{\circ}{X})$$

for $\operatorname{Re} s \geq n/2$, $s \neq n/2$, *which is meromorphic in* $\{\operatorname{Re} s > n/2\}$ *with poles only for* s *such that* $s(n-s) \in \sigma_{\mathrm{pp}}(\Delta_g)$, *and continuous up to* $\{\operatorname{Re} s = n/2\} \setminus \{n/2\}$, *such that*

$$(\Delta_g - s(n-s))\mathcal{P}(s) = 0,$$

with expansions

$$\begin{array}{ll} \mathcal{P}(s)f = x^{n-s}F + x^s G & \text{if } s \notin n/2 + \mathbb{N}_0/2 \\ \mathcal{P}(s)f = x^{n/2-l/2}F + Gx^{n/2+l/2}\log x & \text{if } s = n/2 + l/2,\ l \in \mathbb{N}\,, \end{array}$$

for $F, G \in \mathcal{C}^\infty(X)$ *such that* $F|_{\partial X} = f$.

If $s = n/2 + l/2$, *then* $G|_{\partial X} = -2p_l f$, *where*

$$p_l = \operatorname{Res}_{s=n/2+l/2} p_{l,s}\,,$$

with the differential operator $p_{l,s}$ *defined in* (3.10). *For* $l = 2k$,

$$\sigma_{2k}(p_{2k}) = c_k \sigma_{2k}(\Delta_h^k)\,,$$

with c_k *as in* (1.4).

The principal symbol of p_{2k} is easily calculated from the inductive construction of the operators $p_{j,s}$; this will be described in more detail in the proof of Proposition 4.2.

Remark. A more precise notion of a meromorphic family of asymptotic expansions can be given using a Mellin transform in x. This gives a holomorphic family of meromorphic functions with poles corresponding to the exponents in the expansions and logarithmic terms to double poles. We do not need this precise description here as we will work with the explicit formula for $\mathcal{P}(s)$.

Next we define the scattering matrix

$$S(s) : \mathcal{C}^\infty(\partial X, |N^*\partial X|^{n-s}) \to \mathcal{C}^\infty(\partial X, |N^*\partial X|^{s})$$

for $\operatorname{Re} s \geq n/2$, $2s - n \notin \mathbb{N}_0$, $s(n-s) \notin \sigma_{\mathrm{pp}}(\Delta_g)$. According to Proposition 3.5, for such s and for $f \in \mathcal{C}^\infty(\partial X, |N^*\partial X|^{n-s})$ we have $\mathcal{P}(s)f = x^{n-s}F + x^s G$, so we define

$$S(s)f = G|_{\partial X}\,.$$

The construction of $\mathcal{P}(s)$ in Proposition 3.4 gives an explicit realization of the scattering matrix:

$$S(s)f = -\left(x^{-s}R(s)(\Delta_g - s(n-s))\Phi(s)f\right)|_{\partial X}\,, \quad 2s - n \notin \mathbb{N}_0\,. \tag{3.14}$$

Using (3.8) and Proposition 3.1, it follows that $S(s)$ is defined for s as indicated above and is meromorphic in $\{\operatorname{Re} s > n/2\}$.

Applying complex conjugation to (3.2) shows that for $\operatorname{Re} s = n/2$, $s \neq n/2$, we have the *functional equation*

$$S(s) = S(n-s)^{-1}\,.$$

An application of the pairing formula (3.3) with $u_j = \mathcal{P}(s) f_j$ shows the unitarity relation:

$$\int_{\partial X} \left(S(s) f_1 \overline{S(s) f_2} - f_1 \bar{f}_2 = 0 \right) dv_h\,, \quad \operatorname{Re} s = \frac{n}{2}\,, \ s \neq n/2\,.$$

By using the Schwartz reflection principle and then unitarity to establish regularity at $s = n/2$, we find a meromorphic extension of $S(s)$ to the entire complex plane, regular for $\operatorname{Re} s = n/2$, and we have the following symmetries:

$$S(n-\bar{s})^* = S(s)^{-1}\,, \quad S(n-s) = S(s)^{-1}\,.$$

For $s \in \mathbb{R}$, the functional equation and unitarity show that

$$S(s) = S(s)^*\,,$$

which also follows directly from Proposition 3.3 since $S(s)$ is a real operator for s real.

The construction of the Poisson kernels and Proposition 3.5 give the following:

Proposition 3.6. *The scattering matrix is meromorphic in* $\operatorname{Re} s > n/2$ *and at a pole* s_0 *we have*

$$\operatorname{Res}_{\,s=s_0} S(s) = \begin{cases} \Pi_{s_0} & s_0 \notin n/2 + \mathbb{N}/2 \\ \Pi_{n/2+l/2} - p_l & s_0 = n/2 + l/2\,, \end{cases}$$

where p_l *is as in Proposition 3.5 and*

$$\Pi_{s_0} = (2s_0 - n) \left(x^{-s_0} \frac{1}{2\pi i} \int_{\gamma_{s_0}} R(s) ds \ x^{-s_0} \right) |_{\partial X \times \partial X}$$

is non-trivial only if $s_0(n-s_0) \in \sigma_{\rm pp}(\Delta_g)$*. Here* γ_{s_0} *is a small circle in* $\mathbb{C}$ *about* s_0*, traversed counterclockwise.*

Proof. Suppose first that the resolvent is holomorphic near $n/2 + l/2$. Then (3.12), (3.14), and the holomorphy of $\mathcal{P}(s)$ near $n/2 + l/2$ show that

$$\begin{aligned} \operatorname{Res}_{\,s=n/2+l/2} S(s) &= - \left(\operatorname{Res}_{\,s=n/2+l/2} x^{-s} \Phi(s) \right) |_{\partial X} \\ &= - \operatorname{Res}_{\,s=n/2+l/2} p_{l,s} = -p_l\,, \end{aligned}$$

where $p_{l,s}$ are as in (3.10).

When s_0 is a pole of the resolvent then

$$R(s) = \frac{\sum_{k=1}^K \phi_k \otimes \phi_k}{s_0(n-s_0) - s(n-s)} + \widetilde{T}(s) = \frac{1}{2s_0 - n} \frac{\sum_{k=1}^K \phi_k \otimes \phi_k}{s - s_0} + T(s)\,, \tag{3.15}$$

where T and $\widetilde{T}$ are holomorphic near s_0, and $\phi_k \in x^{s_0}\mathcal{C}^\infty(X) \subset L^2(X)$ are the normalized eigenfunctions, $(\Delta_g - s_0(n-s_0))\phi_k = 0$. Let us first consider the case $s_0 \notin n/2 + \mathbb{N}/2$. We use (3.5) to evaluate

$$\int_X \phi_k(\Delta_g - s_0(n-s_0))\Phi(s_0) f dv_g = (n - 2s_0)\int_{\partial X}(x^{-s_0}\phi_k)|_{\partial X} f dv_h \,.$$

Using this and (3.15) in (3.14) shows that the residue of $S(s)$ at s_0 is the operator with kernel $\sum_{k=1}^K (x^{-s_0}\phi_k)|_{\partial X} \otimes (x^{-s_0}\phi_k)|_{\partial X}$, which by (3.15) is Π_{s_0}.

If s_0 is a pole of the resolvent and $s_0 = n/2 + l/2$ then we replace (3.12) by (3.13) and find the residue of $\mathcal{P}(s)$ by the same method as above:

$$\operatorname{Res}_{\, s=n/2+l/2}\mathcal{P}(s)f = \sum_{k=1}^K \phi_k \int_{\partial X}(x^{-s_0}\phi_k)|_{\partial X} f dv_h \,.$$

Now using this instead of holomorphy of $\mathcal{P}(s)$ in the argument of the first paragraph of this proof gives the desired formula for $\operatorname{Res}_{\, s=n/2+l/2} S(s)$. □

This shows that the scattering matrix *always* has poles of infinite rank at $n/2+k$, $k \in \mathbb{N}$, and possibly at $n/2+k-1/2$, $k \in \mathbb{N}$, with the residues given by the operators appearing as the log term coefficients in the expansions of the Poisson operators.

Uniqueness of $\mathcal{P}(s)$ shows that $\mathcal{P}(n)1 = 1$. It then follows from Proposition 3.5 that $p_n 1 = 0$ so that $p_{n,s}1$ extends holomorphically across $s = n$. Since we always have $0 \notin \sigma_{\rm pp}(\Delta_g)$, Proposition 3.6 shows that $S(s)1$ extends holomorphically across $s = n$. Set $S(n)1 = \lim_{s\to n} S(s)1$. Then we have:

Proposition 3.7. *If $p_{n,s}$ are defined in (3.10), then*

$$S(n)1 = -\lim_{s\to n} p_{n,s}1 \,.$$

Proof. We have $\mathcal{P}(s)1 = \Phi(s)1 + x^s G(s)$, where the expansion of $\Phi(s)1$ is as in (3.10) and $G(s) \in \mathcal{C}^\infty(X)$ satisfies $G(s)|_{\partial X} = S(s)1$. Letting $s \to n$ and recalling $\mathcal{P}(n)1 = 1$ shows that $\lim_{s\to n}(\Phi(s)1 + x^s G(s)) = 1$. It follows that $\lim_{s\to n} p_{l,s}1 = 0$ for $1 \le l < n$, and $\lim_{s\to n}(p_{n,s}1 + S(s)1) = 0$ as desired. □

We next discuss the distributional kernels of the operators $R(s)$, $\mathcal{P}(s)$, and $S(s)$. The structure of the kernel of $R(s)$ is best understood on a resolved product space. Since it is not central to our discussion, we will only sketch the construction. We first recall the blow-down map of [19, Sect.3]:

$$\beta \,:\, X \times_0 X \longrightarrow X \times X \,,$$

where $X \times_0 X$ is the blow-up of $X \times X$ along the boundary diagonal, illustrated in Fig.1. The restriction to the boundary gives us a blow-up of the diagonal in $\partial X \times \partial X$ with the corresponding blow-down map:

$$\beta_\partial \,:\, T \cap B \longrightarrow \partial X \times \partial X \,,$$

where T and B are the top and bottom faces in the blow-up, as shown in Fig.1. Let ρ, ρ' be the defining functions of the top and bottom faces respectively. Then we have

Proposition 3.8. *(Mazzeo-Melrose* [19, Proposition 6.2, Theorem 7.1]*) The kernel of the resolvent on the resolved space $X \times_0 X$, $\beta^* R(s)$, and away from its poles, can be written as*

$$\beta^* R(s) = R'(s) + R''(s)\,,$$

where $R'(s)$ is supported away *from $T \cup B$, and*

$$R''(s) = (\rho)^s (\rho')^s K(s)\,, \quad K(s) \in \mathcal{C}^\infty(X \times_0 X)\,.$$

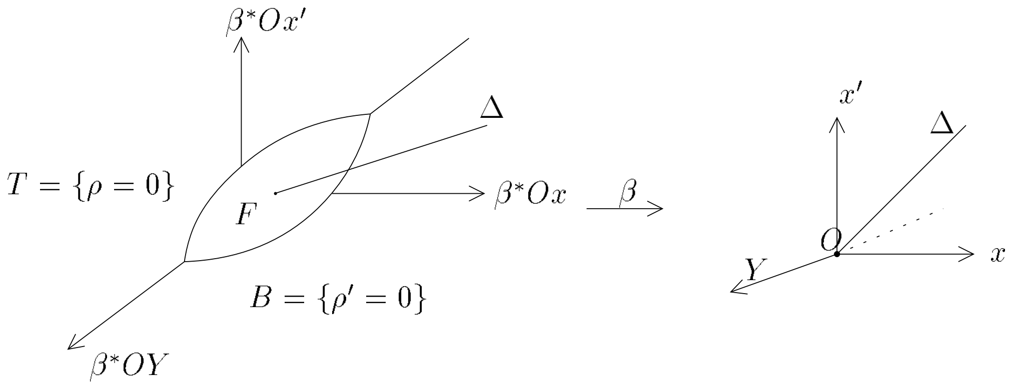

Figure 1. The boundary faces of the blown-up space $X \times_0 X$. The variable Y stands for the defining function of the diagonal in $\partial X \times \partial X$, $Y = y - y'$, in local coordinates.

The pairing formula (3.3) gives an expression for the kernel of $\mathcal{P}(s)$ in terms of that of the resolvent, much in the same spirit as in the classical derivation of the Poisson kernel from the Green function:

Proposition 3.9. *For $f \in \mathcal{C}^\infty(\partial X)$ and $\operatorname{Re} s \geq n/2$, $s \neq n/2$, we have a formal expression*

$$\mathcal{P}(s) f = (2s - n)(R(s) x^{-s})(f \otimes \delta_0(x))\,, \tag{3.16}$$

where the right hand side is understood as the restriction to the boundary of the distributional kernel of $R(s)x^{-s}$ paired with f.

Proof. First let $\operatorname{Re} s = n/2$. For any $g \in \dot{\mathcal{C}}^\infty(X)$ and $f \in \mathcal{C}^\infty(\partial X)$ apply (3.3) with $u_1 = \mathcal{P}(s) f$ and $u_2 = R(n-s) g$. We also note that for $\operatorname{Re} s = n/2$ we have

$$R(n-s)^* = R(s)\,.$$

That gives

$$\begin{aligned}\int_X \mathcal{P}(s) f \bar{g} &= (2s-n)\int_{\partial X} \overline{(x^{-n+s}R(n-s)g)}|_{\partial X} f\\ &= (2s-n)\int_X \left((R(s)x^{-s})(f\otimes\delta_0(x))\right)\bar{g}\,,\end{aligned}$$

proving the claim. That the restriction of $R(s)x^{-s}$ is well defined as a distribution on the boundary follows from Proposition 3.8. The result extends to $\operatorname{Re} s > n/2$ by analytic continuation. □

We now continue $\mathcal{P}(s)$ meromorphically to $\mathbb{C}$ so that (3.16) holds for $\operatorname{Re} s < n/2$ as well.

Using the structure of the resolvent described in Proposition 3.8 we can give a precise description of the kernel of $S(s)$ as a pseudodifferential operator: away from its poles, we have

$$\begin{aligned} &S(s) \in \Psi^{2\operatorname{Re} s - n}(\partial X; |N^*\partial X|^{n-s}, |N^*\partial X|^{s})\,,\\ &\sigma(S(s)) = 2^{n-2s}\frac{\Gamma(n/2-s)}{\Gamma(s-n/2)}\sigma(\Delta_h^{s-n/2}).\end{aligned} \tag{3.17}$$

To see this we follow [11] and obtain the following formula for the distributional kernel of the scattering matrix in terms of the resolvent on the resolved product space:

$$S(s) = (2s-n)(\beta_\partial)_*\left(\beta^*\left(x^{-s}x'^{-s}R(s)\right)|_{T\cap B}\right)\,, \tag{3.18}$$

(see also [14, Proposition 4.4]). This is derived using (3.12), (3.14) and Proposition 3.9, which give

$$\begin{aligned}S(s)f &= (x^{-s}\mathcal{P}(s)f - x^{-s}\Phi(s)f)|_{\partial X}\\ &= \left((2s-n)x^{-s}(R(s)x^{-s})(f\otimes\delta_0(x)) - x^{-s}\Phi(s)f\right)|_{\partial X}\,.\end{aligned}$$

For $\operatorname{Re} s < n/2$ and s away from poles of $R(s)$ we see that

$$S(s)f = \left((2s-n)x^{-s}(R(s)x^{-s})(f\otimes\delta_0(x))\right)|_{\partial X}\,.$$

Proposition 3.8 shows that for $\operatorname{Re} s \ll 0$, and s not a pole, we have

$$\left(x^{-s}(\beta_* R'(s)x^{-s})(f\otimes\delta_0(x))\right)|_{\partial X} \equiv 0\,,$$

and

$$\left(x^{-s}(\beta_* R''(s)x^{-s})(f\otimes\delta_0(x))\right)|_{\partial X} = \left(x^{-s}R(s)x^{-s})(f\otimes\delta_0(x))\right)|_{\partial X}\,,$$

and this gives (3.18) for $\operatorname{Re} s \ll 0$. Strictly speaking we need more information about the $R'(s)$ term than stated in Proposition 3.8 but those properties are stated in [19, Theorem 7.1]. We shall see below that (3.18) continues meromorphically in s.

The fact that the scattering matrix is a pseudodifferential operator of order $2\operatorname{Re} s - n$ follows from (3.18). As shown in Fig.1, near $T \cap B$ we can use y, $|Y|$, $\omega = Y/|Y|$, and ρ, ρ', the defining functions of T and B, as coordinates, where $Y = y - y'$. In terms of these functions,

$$x = |Y|\rho\,, \quad x' = |Y|\rho'\,,$$

and hence, using Proposition 3.8,

(3.19)
$$S(s) = (2s-n)(\beta_\partial)_* \left(|Y|^{-2s} K(s, y, |Y|, \omega)\right)\,, \quad K(s, \cdot) \in \mathcal{C}^\infty([\partial X \times \partial X; \Delta_{\partial X}])\,,$$

where $[\partial X \times \partial X; \Delta_{\partial X}]$ is $\partial X \times \partial X$ blown up along the diagonal, that is the space where instead of $y, y' \in \partial X$ as coordinates, we use $y, |Y|, \omega$.

Hence for $\operatorname{Re} s \ll 0$ the distributional kernel of $S(s)$ is a distribution conormal to the diagonal and hence $S(s)$ is a pseudo-differential operator. We then continue $|Y|^{-2s}$ meromorphically in s following [13, Theorem 3.2.4], and that gives the first part of (3.17). The parametrix construction in [19] gives K, and leads to a computation of the principal symbol – see [14].

4. Conformal Geometry

In this section we show how the conformally invariant powers of the Laplacian may be derived from the formal Poincaré metric associated to the conformal structure, review the definition and basic properties of Branson's Q-curvature, and remark on the interpretation of the scattering matrix for a Poincaré metric as a family of conformally invariant pseudo-differential operators.

The invariant powers of the Laplacian act on conformal densities. The metric bundle of a conformal manifold $(M, [h])$ is the ray subbundle $\mathcal{G} \subset S^2T^*M$ of multiples of the metric: if h is a representative metric, then the fiber of $\mathcal{G}$ over $p \in M$ is $\{t^2 h(p) : t > 0\}$. The space of conformal densities of weight $w \in \mathbb{C}$ is

$$\mathcal{E}(w) = \mathcal{C}^\infty(M; \mathcal{G}^{-\frac{w}{2}})\,,$$

where by abuse of notation we have denoted by $\mathcal{G}$ also the line bundle associated to the ray bundle defined above. A choice of representative h for the conformal structure induces an identification $\mathcal{E}(w) \simeq \mathcal{C}^\infty(M)$; if $\hat{h} = e^{2\Upsilon} h$ then the corresponding elements of $\mathcal{C}^\infty(M)$ transform by $\hat{f} = e^{w\Upsilon} f$. If $\partial X = M$ as in the previous section, a conformally compact metric on X with conformal infinity $[h]$ determines an isomorphism

$$\mathcal{E}(w) \simeq \mathcal{C}^\infty(M, |N^*\partial X|^{-w})\,.$$

The invariance property (1.3) can be reformulated as the statement that P_k is an invariantly defined operator

$$P_k : \mathcal{E}(-n/2 + k) \to \mathcal{E}(-n/2 - k).$$

The main result of [8] is the following existence theorem.

Proposition 4.1. *Let $k \in \mathbb{N}$ and $k \leq n/2$ if n is even. There is a conformally invariant natural differential operator $P_k : \mathcal{E}(-n/2+k) \to \mathcal{E}(-n/2-k)$ with principal part equal to that of Δ^k.*

Such operators are not unique. The construction in [8] is in terms of the ambient metric of [5]. We shall first give here a different construction of invariant operators based on the Poincaré metric and then show that the two constructions give the same operators.

We begin by recalling from [5] the formal Poincaré metric associated to a conformal structure $(M,[h])$. Given a representative metric h, one considers metrics on $M \times [0,1]$ of the form (1.7). The Einstein equation $\mathrm{Ric}(g)+ng=0$ can be calculated directly in terms of h_x and the formal asymptotics of solutions studied; see [7]. If n is odd, then there is a unique formal smooth solution h_x to

$$\mathrm{Ric}(g)+ng=\mathcal{O}(x^\infty)$$

which is even in x. If n is even, the condition $\mathrm{Ric}(g)+ng=\mathcal{O}(x^{n-2})$ uniquely determines $h_x \mod \mathcal{O}(x^n)$, which is even in x (mod $\mathcal{O}(x^n)$). Although in general smooth solutions do not exist to higher orders, the condition

$$\mathrm{tr}_g(\mathrm{Ric}(g)+ng)=\mathcal{O}(x^{n+2})$$

can be satisfied and uniquely determines the h-trace of the x^n coefficient in h_x; this is the vanishing trace condition referred to in the introduction. The indicated Taylor coefficients of h_x are determined inductively from the equation and are given by polynomial formulae in terms of h, its inverse, and its curvature tensor and covariant derivatives thereof.

Since any asymptotically hyperbolic metric can be put uniquely in the form (1.7) upon choosing h, it follows that the equivalence class of the solution g up to diffeomorphism and up to terms vanishing to the indicated orders is uniquely determined by the conformal structure. This equivalence class is called the formal Poincaré metric associated to $[h]$. When n is even, the higher order terms in h_x are not determined; however for simplicity in statements below, we shall restrict consideration to h_x which are smooth and even in x to all orders. If X is a manifold with $\partial X = M$, any metric on X whose restriction to a collar neighborhood of M is in this equivalence class is called a Poincaré metric associated to $[h]$.

Let g be a Poincaré metric and h a representative for the conformal infinity. If x is a defining function such that $x^2 g|_{TM}=h$, and $f \in \mathcal{C}^\infty(M)$ represents a section of $\mathcal{E}(w)$, then it is a conformally invariant statement to require that a function u on X be asymptotic to $x^{-w}f$. The invariant operators P_k arise from solving

$$(\Delta_g - s(n-s))u = \mathcal{O}(x^\infty) \tag{4.1}$$

for u with such asymptotic behaviour. The characteristic exponents of $\Delta_g - s(n-s)$ are $s, n-s$, so generically solutions behave like x^s and x^{n-s}. As in the previous section, we take u asymptotic to $x^{n-s}f$, which according to the above remarks means that f is to be interpreted as a density of weight $w = s - n$.

We saw in §3 that if g is asymptotically hyperbolic and $s - n/2 \notin \mathbb{N}/2$, then for any $f \in \mathcal{C}^\infty(M)$ there is a formal solution $u \mod \mathcal{O}(x^\infty)$ to (4.1) of the form $u = x^{n-s}F$ with $F \in \mathcal{C}^\infty(X)$ and $F|_M = f$. As we shall see below, for Poincaré metrics this holds if only $s - n/2 \notin \mathbb{N}$. In §3, solutions for $s - n/2 \in \mathbb{N}/2$ were constructed as a limit of solutions for nearby s. Here we make a direct analysis at the exceptional values of s and obtain the invariant operators as obstructions to the existence of formal smooth solutions.

Proposition 4.2. *Let (X, g) be a Poincaré metric associated to $(M, [h])$ and let $f \in \mathcal{C}^\infty(M)$. If $k \in \mathbb{N}$, there is a formal solution of (4.1) for $s = n/2 + k$ of the form*

$$u = x^{n/2-k}(F + Gx^{2k}\log x) \tag{4.2}$$

with $F, G \in \mathcal{C}^\infty(X)$ and with $F|_M = f$. F is uniquely determined $\mod \mathcal{O}(x^{2k})$ and G is uniquely determined $\mod \mathcal{O}(x^\infty)$. Moreover,

$$G|_M = -2c_k P_k f, \tag{4.3}$$

where P_k is a differential operator on M with principal part Δ^k.

If n is odd and $k \in \mathbb{N}$ or if n is even and $k \leq n/2$, then P_k depends only on h and defines a conformally invariant operator : $\mathcal{E}(-n/2+k) \to \mathcal{E}(-n/2-k)$.

Proof. Choose a representative metric h and write g in the form (1.7). A straightforward calculation shows that $[\Delta_g - s(n-s)] \circ x^{n-s} = x^{n-s+1}\mathcal{D}_s$, where

$$\mathcal{D}_s = -x\partial_x^2 + (2s - n - 1 - \frac{x}{2}h^{ij}h'_{ij})\partial_x - \frac{n-s}{2}h^{ij}h'_{ij} + x\Delta_{h_x}. \tag{4.4}$$

Here h_{ij} denotes the metric h_x with x fixed, and $h'_{ij} = \partial_x h_{ij}$. Therefore for $f_j \in \mathcal{C}^\infty(M)$, one has

$$\mathcal{D}_s(f_j x^j) = j(2s - n - j)f_j x^{j-1} + \mathcal{O}(x^j). \tag{4.5}$$

If $2s - n \notin \mathbb{N}$, then we can use (4.5) to construct a smooth solution of (4.1) inductively. Beginning with $f_0 = F_0 = f$, define f_j, F_j for $j \geq 1$ by

$$\begin{aligned} j(2s-n-j)f_j &= -(x^{1-j}\mathcal{D}_s(F_{j-1}))|_{x=0} \\ F_j &= F_{j-1} + f_j x^j. \end{aligned} \tag{4.6}$$

By (4.5) we have $\mathcal{D}_s F_j = \mathcal{O}(x^j)$ so that the definition of f_j makes sense. Then $F = \sum f_j x^j$ is a formal solution of (4.1). Observe that since h_x is even in x, $\mathcal{D}_s$ maps even functions to odd and vice versa. Therefore $f_j = 0$ for j odd. For $j = 2k$ even, an easy induction shows that f_{2k} takes the form

$$f_{2k} = c_{k,s}P_{k,s}f, \qquad c_{k,s} = (-1)^k \frac{\Gamma(s - n/2 - k)}{2^{2k}\,k!\,\Gamma(s - n/2)}, \tag{4.7}$$

where $P_{k,s}$ is a differential operator on M, the principal part of which agrees with that of Δ_h^k. Since the Taylor expansion of h_x is determined (to the appropriate order for n even) in terms of h, one sees by counting derivatives that if $k \in \mathbb{N}$ and $k \leq n/2$ if n is even, then $P_{k,s}$ also depends only on h and is a natural differential operator with coefficients which are polynomial in s.

If $2s-n = l \in \mathbb{N}$, then the corresponding coefficient $2s-n-j$ vanishes for $j = l$ in the first equation of (4.6). For l odd, by parity considerations it follows that the right hand side of this equation also vanishes, so f_l can be chosen arbitrarily (for example $f_l = 0$ to preserve parity) and the induction continued to infinite order.

However, if $2s - n = 2k$ is even, then the right hand side of the first equation of (4.6) need not vanish for $j = 2k$ and there is an obstruction to solving with F smooth. This is of course reflected in the pole of $c_{k,s}$ at $s = n/2 + k$. This obstruction can be incorporated into a log term and the formal solution continued to higher order as follows. Observe that

$$\begin{aligned} \mathcal{D}_s(g_j x^j \log x) =& j(2s-n-j)g_j x^{j-1}\log x + (2s-n-2j)g_j x^{j-1} \\ &+ \mathcal{O}(x^j \log x)\,. \end{aligned} \tag{4.8}$$

Therefore if we take

$$g_{2k} = (2k)^{-1}(x^{1-2k}\mathcal{D}_s(F_{2k-1}))|_{x=0}, \tag{4.9}$$

choose f_{2k} arbitrarily, and set

$$F_{2k} = F_{2k-1} + g_{2k}x^{2k}\log x + f_{2k}x^{2k}\,,$$

then we have $\mathcal{D}_s F_{2k} = \mathcal{O}(x^{2k}\log x)$. Using (4.5) and (4.8), it is easily seen that the construction can be continued to all higher orders to obtain a solution of the form (4.2) as claimed. We have $G|_{x=0} = g_{2k}$, which is given by a differential operator on M of the claimed form by the same reasoning as above. In fact, one sees easily that (4.3) holds with $P_k = P_{k,n/2+k}$ and

$$c_k = \operatorname{Res}_{s=n/2+k} c_{k,s} = (-1)^k[2^{2k}k!(k-1)!]^{-1}\,.$$

Suppose now we change the conformal representative h. We obtain a different defining function x, a different product identification for X, and a different representation (1.7). However, g and u remain unchanged, so by uniqueness we deduce that $G|_{x=0}$ must transform as a density, proving the conformal invariance of P_k. □

Remark. Equation (4.4) holds also for general asymptotically hyperbolic metrics in the form (1.7) and can be used to explicitly compute the operators $p_{j,s}$ of §3. Of course in general it no longer need be the case that h_x is even in x. Therefore log terms may also occur for $2s-n$ odd, and the scattering matrix may have poles for such s according to Proposition 3.6.

Note that if n is even, then $P_{n/2}$ is invariantly defined from $\mathcal{E}(0)$ to $\mathcal{E}(-n)$, that is, from $\mathcal{C}^\infty(M)$ to the space of volume densities. Since the constant function $1 \in \mathcal{E}(0)$ has a smooth extension annihilated by Δ_g, we have $P_{n/2}1 = 0$. Therefore $P_{n/2}$ has zero constant term.

We next recall from [8] the original construction of the invariant operators via the ambient metric. Denote by $\pi : \mathcal{G} \to M$ the natural projection of the metric bundle, and by $\mathbf{g}$ the tautological symmetric 2-tensor on $\mathcal{G}$ defined for $(p, h) \in \mathcal{G}$ and $X, Y \in T_{(p,h)}\mathcal{G}$ by $\mathbf{g}(X, Y) = h(\pi_* X, \pi_* Y)$. There are dilations $\delta_s : \mathcal{G} \to \mathcal{G}$ for $s > 0$ given by $\delta_s(p, h) = (p, s^2 h)$, and we have $\delta_s^* \mathbf{g} = s^2 \mathbf{g}$. Denote by T the infinitesimal dilation vector field $T = \frac{d}{ds}\delta_s|_{s=1}$. Define the ambient space $\tilde{\mathcal{G}} = \mathcal{G} \times (-1, 1)$. Identify $\mathcal{G}$ with its image under the inclusion $\iota : \mathcal{G} \to \tilde{\mathcal{G}}$ given by $\iota(h) = (h, 0)$ for $h \in \mathcal{G}$. The dilations δ_s and infinitesimal generator T extend naturally to $\tilde{\mathcal{G}}$.

The ambient metric $\tilde{g}$ is a Lorentzian metric on $\tilde{\mathcal{G}}$ which satisfies the initial condition $\iota^* \tilde{g} = \mathbf{g}$, is homogeneous in the sense that $\delta_s^* \tilde{g} = s^2 \tilde{g}$, and is an asymptotic solution of $\mathrm{Ric}(\tilde{g}) = 0$ along $\mathcal{G}$. For n odd, these conditions uniquely determine a formal power series expansion for $\tilde{g}$ up to diffeomorphism, but for n even and $n > 2$, a formal power series solution exists in general only to order $n/2$.

An element of $\mathcal{E}(w)$ can be regarded as a homogeneous function of degree w on $\mathcal{G}$. One of the ways that P_k is derived in [8] is as the obstruction to extending $\tilde{f} \in \mathcal{E}(-n/2 + k)$, regarded as such a homogeneous function, to a smooth function $\tilde{F}$ on $\tilde{\mathcal{G}}$, such that $\tilde{F}$ is also homogeneous of degree $-n/2 + k$ and satisfies

$$\tilde{\Delta} \tilde{F} = \mathcal{O}(\rho^\infty), \tag{4.10}$$

where $\tilde{\Delta}$ denotes the Laplacian in the metric $\tilde{h}$. Similarly to Proposition 4.2, the Taylor expansion of $\tilde{F}$ is formally determined to order $k - 1$ in ρ, but there is an obstruction at order k which defines the operator P_k.

Proposition 4.3. *Let $k \in \mathbb{N}$ with $k \leq n/2$ if n is even. The conformally invariant operator P_k defined in* [8] *in terms of the ambient metric agrees with the operator of Proposition 4.2.*

Proof. As described in [5], the formal Poincaré metric associated to a conformal structure can be constructed from the ambient metric and vice versa; the two constructions are equivalent. We review this equivalence.

In the ambient space $\tilde{\mathcal{G}}$, the equation $\tilde{g}(T, T) = -1$ defines a hypersurface $\mathring{X}$ which lies on one side of $\mathcal{G}$ and which intersects exactly once each dilation orbit on this side of $\mathcal{G}$. The Poincaré metric g is the pullback to $\mathring{X}$ of $\tilde{g}$. The equation $\mathrm{Ric}(\tilde{g}) = 0$ is equivalent to $\mathrm{Ric}(g) = -ng$. To see this, one uses the normal form from [5] for ambient metrics: in suitable coordinates on $\tilde{\mathcal{G}}$, the ambient metric takes the form

$$\tilde{g} = 2tdtd\rho + 2\rho dt^2 + t^2\overline{h}_\rho. \tag{4.11}$$

Here ρ is a defining function for $\mathcal{G} \subset \tilde{\mathcal{G}}$, t is homogeneous of degree 1 with respect to the dilations on $\tilde{\mathcal{G}}$, and $\overline{h}_\rho$ is a smooth 1-parameter family of metrics on M. In these coordinates we have $T = t\partial_t$, so $\overset{\circ}{X} = \{2\rho t^2 = -1\}$. Introduce a new variable $x = \sqrt{-2\rho}$ and set $s = xt$ so that $\overset{\circ}{X} = \{s = 1\}$. A straightforward calculation shows that (4.11) becomes

$$\tilde{g} = s^2 g - ds^2, \tag{4.12}$$

where g is given by (1.7) with $h_x = \overline{h}_\rho$. The equivalence of $\mathrm{Ric}(\tilde{g}) = 0$ and $\mathrm{Ric}(g) = -ng$ is a straightforward calculation given the relationship (4.12) (see Proposition 5.1 of [9]). Note that h_x is automatically even in x.

The equations (4.1) and (4.10) are equivalent. To see this, rewrite (4.12) as

$$\tilde{g} = s^2(g - ds^2/s^2)\,,$$

and transform under a conformal change to obtain

$$\tilde{\Delta} = s^{-2}[\Delta_g + (s\partial_s)^2 + ns\partial_s]\,.$$

If $\tilde{F}$ is homogeneous of degree w, we therefore have

$$\tilde{\Delta}\tilde{F} = s^{-2}[\Delta_g + w(w+n)]\tilde{F},$$

so (4.10) is equivalent to (4.1) for $s = n + w$ and $u = \tilde{F}|_{\overset{\circ}{X}}$. One may recover $\tilde{F}$ from u via homogeneity by $\tilde{F} = s^w u = t^w x^w u$. In order for $\tilde{F}$ to be smooth up to $\rho = 0$, we require therefore that $x^w u$ be smooth up to $x = 0$ (and be even in x). Thus the two extension problems are equivalent, so the normalized obstruction operators must agree. □

We next define the Q-curvature as in [2]. For this discussion we shall denote by P_k^n the operator P_k in dimension n. Fix $k \in \mathbb{N}$. One consequence of the construction of Proposition 4.2 is that the operator P_k^n is natural in the strong sense that $P_k^n f$ may be written as a linear combination of complete contractions of products of covariant derivatives of the curvature tensor of a representative for the conformal structure with covariant derivatives of f, with coefficients which are rational in the dimension n. Also, it follows from the fact that the zeroth order term of $\mathcal{D}_s$ in (4.4) has a factor of $n - s$, that the zeroth order term of P_k^n may be written as $(n/2 - k)Q_k^n$ for a scalar Riemannian invariant Q_k^n with coefficients which are rational in n and regular at $n = 2k$. (This is of course consistent with the fact mentioned above that $P_{n/2}^n 1 = 0$.) The Q-curvature in even dimension n is then defined as $Q = Q_{n/2}^n$.

We may also consider the Q-curvature as arising in a similar way from the zeroth order terms of the operators $P_{k,s}$ with n fixed but as s varies. For the same reason as above, the zeroth order term of $P_{k,s}$ is of the form

$$P_{k,s}1 = (n-s)Q_{k,s} \tag{4.13}$$

for a scalar Riemannian invariant $Q_{k,s}$ which is polynomial in s. Taking $s = n/2+k$ and recalling that $P_k = P_{k,n/2+k}$ shows that $Q_k^n = Q_{k,n/2+k}$. In particular,

$$Q = Q_{n/2,n}. \tag{4.14}$$

If g is a Poincaré metric with conformal infinity $[h]$, then according to (3.17), the scattering matrix $S(s)$ is a family of pseudodifferential operators on M, which is conformally invariant in the sense that it acts invariantly on conformal densities. Even though $S(s)$ depends on the choice of Poincaré metric g, it is shown in [14] that its full symbol depends only on the infinite jet of g at ∂X, which, as discussed above, is determined by $[h]$ (to the appropriate order for n even). One can therefore view the symbol as determined by the conformal structure, and the choice of g as a geometrically natural means of fixing smoothing terms to obtain a globally well-defined operator having this symbol. Peterson [25] has defined an analogous family of symbols directly by analytic continuation from the differential operators P_k.

Branson [2] derived the transformation law (1.5) for Q by analytic continuation in the dimension. It also follows easily from Theorems 1 and 2 by analytic continuation in s. The conformal invariance of $S(s)$ is equivalent to $\widehat{S(s)} = e^{-s\Upsilon}S(s)e^{(n-s)\Upsilon}$, where here $S(s)$ is realized as an operator on $\mathcal{C}^\infty(M)$ corresponding to the choice of conformal representative. Therefore,

$$e^{s\Upsilon}\widehat{S(s)}1 = S(s)1 + S(s)(e^{(n-s)\Upsilon}-1).$$

Letting $s \to n$ and applying Theorems 1 and 2 immediately yields (1.5).

5. Proofs of the main results

Theorems 1 and 2 follow from the corresponding results of §3 in the special case when g is a Poincaré metric associated to the conformal structure. It follows from (4.7) that for Poincaré metrics, the differential operators $p_{l,s}$ of (3.10) are given by

$$p_{l,s} = 0 \text{ for } l \text{ odd}, \qquad p_{2k,s} = c_{k,s}P_{k,s}, \ k \in \mathbb{N}. \tag{5.1}$$

Thus for the residues we obtain

$$p_l = 0 \text{ for } l \text{ odd}, \qquad p_{2k} = c_kP_k, \ k \in \mathbb{N}. \tag{5.2}$$

Proof of Theorem 1. This is immediate from Proposition 3.6 and (5.2). □

Observe that Proposition 3.6 and (5.2) also show that for Poincaré metrics, $S(s)$ has no pole at $s = n/2 + l/2$ if $l \in \mathbb{N}$ is odd.

Proof of Theorem 2. We apply Proposition 3.7. By (5.1) and (4.13), (4.14), we obtain

$$S(n)1 = -\lim_{s\to n} c_{n/2,s} P_{n/2,s} 1 = \mathrm{Res}_{s=n}\, c_{n/2,s}\, Q = c_{n/2} Q.$$

□

We note that Proposition 3.7 and (5.1) also show that for Poincaré metrics, $S(n)1 = 0$ if n is odd.

Proof of Theorem 3. Let n be even and let g be a Poincaré metric. We first recall (as described in [7]) the expression for L in terms of the expansion of the volume form of g. If h is a representative for the conformal infinity of g, we may write g in the form (1.7), and from the fact that g is a Poincaré metric it follows that

$$dv_g = x^{-n-1}(1 + v^{(2)}x^2 + (\text{even powers}) + v^{(n)}x^n + \ldots)dv_h dx, \tag{5.3}$$

where each $v^{(2j)}$ for $1 \le j \le n/2$ is a smooth function on M determined by h. Integration yields (1.8), and shows that

$$L = \int_M v^{(n)} dv_h. \tag{5.4}$$

Now take s near but not equal to n and apply Proposition 3.3 with $u_1 = u_2 = \mathcal{P}(s)1$, which we now denote by u_s. Here we have used h to trivialize the density bundle $|N^*\partial X|^{n-s}$. We obtain

$$\mathrm{pf} \int_{x>\epsilon} [|du_s|^2 - s(n-s)u_s^2] dv_g = -n \int_M S(s)1 dv_h. \tag{5.5}$$

We consider the limiting behaviour in this equation as $s \to n$. According to Theorem 2, $\int_M S(s)1 dv_h \to c_{n/2} \int_M Q dv_h$. We shall show that the left hand side in (5.5) converges to $-nL/2$ as $s \to n$, thereby proving Theorem 3.

Since $u_s = \mathcal{P}(s)1 \to 1$ as $s \to n$, the integrand in the left hand side of (5.5) converges to 0 pointwise on $\mathring{X}$. It follows that

$$\int_{x>x_0} [|du_s|^2 - s(n-s)u_s^2] dv_g \to 0$$

for each fixed $x_0 > 0$. So it suffices to consider

$$\int_{\epsilon<x<x_0} [|du_s|^2 - s(n-s)u_s^2] dv_g, \tag{5.6}$$

for which we may use the product identification (1.7) coming from the representative metric h, provided we choose x_0 small enough.

By (3.12), (3.10), (5.1), and (4.13), for s near n we have

$$u_s = x^{n-s}(1 + \sum_{k=1}^{n/2} c_{k,s}(n-s)Q_{k,s}x^{2k}) + x^s S(s)1 + \mathcal{O}(x^{n+3/4}), \tag{5.7}$$

where the power $n+3/4$ can be replaced by $n+a$, $a<1$, if we take s close to n. In considering (5.7), recall that $c_{k,s}$ is regular at $s=n$ for $1\le k<n/2$, but that $c_{n/2,s}$ has a simple pole at $s=n$ with residue $c_{n/2}$. Also, by Theorem 2, $S(n)1=c_{n/2}Q=c_{n/2}Q_{n/2,n}$. Since $u_n=1$, the error term vanishes identically when $s=n$ and in general is seen to be of the form $\mathcal{O}(|n-s|x^{n+3/4})$. There is a similar bound when we differentiate the expansion. Let y^i denote local coordinates on M and let ∂ denote any first coordinate derivative ∂_x or ∂_{y^i}; then we have

$$(5.8)\quad x\partial u_s = x\partial\left[x^{n-s}(1+\sum_{k=1}^{n/2}c_{k,s}(n-s)Q_{k,s}x^{2k})+x^sS(s)1\right]+\mathcal{O}(|n-s|x^{n+3/4}).$$

We begin by considering the u_s^2 term in (5.6). Upon squaring (5.7), one obtains

$$u_s^2=x^{2(n-s)}(1+\sum_{k=1}^{n/2}A_{k,s}x^{2k})+2x^nS(s)1+\mathcal{O}(x^{n+1/2}),$$

where the coefficients $A_{k,s}$ are smooth functions on M, holomorphic in s near $s=n$, and satisfying $A_{k,n}=0$ for $1\le k<n/2$ and $A_{n/2,n}=-2c_{n/2}Q$. Multiplying by dv_g and using (5.3) gives

$$u_s^2dv_g=\left[x^{n-2s-1}(1+\sum_{k=1}^{n/2}B_{k,s}x^{2k})+2x^{-1}S(s)1+\mathcal{O}(x^{-1/2})\right]dv_hdx,$$

with coefficients $B_{k,s}$ again holomorphic in s, and with $B_{n/2,n}=-2c_{n/2}Q+v^{(n)}$. In order to evaluate $\lim_{s\to n}(n-s)\text{ pf }\int_{\epsilon<x<x_0}u_s^2dv_g$, observe first that the $\mathcal{O}(x^{-1/2})$ error term is integrable, so its contribution vanishes upon letting $s\to n$. Now

$$\text{pf}\int_{\epsilon<x<x_0}x^{n-2s+2k-1}B_{k,s}dv_hdx=x_0^{n-2s+2k}(n-2s+2k)^{-1}\int_M B_{k,s}dv_h.$$

If $k<n/2$, then this approaches a finite limit as $s\to n$, so these terms also do not contribute. The same is true for the $x^{-1}S(s)1$ term. Evaluating the limit for $k=n/2$ and recalling (5.4) then gives

$$(5.9)\qquad \lim_{s\to n}(n-s)\text{ pf }\int_{\epsilon<x<x_0}u_s^2dv_g=\frac{1}{2}\int_M B_{n/2,n}dv_h=-c_{n/2}\int_M Q+L/2.$$

For the derivative term, we have $|du_s|^2=(x\partial_xu_s)^2+h_x^{ij}(x\partial_{y^i}u_s)(x\partial_{y^j}u_s)$. Consider first $(x\partial_xu_s)^2$. Expanding and squaring (5.8) gives

$$(5.10)\qquad (x\partial_xu_s)^2=x^{2(n-s)}\sum_{k=0}^{n/2}A'_{k,s}x^{2k}+2s(n-s)x^nS(s)1+\mathcal{O}(|n-s|x^{n+1/2}),$$

where $A'_{k,s}$ are smooth functions on M, holomorphic in s, satisfying

$$\begin{aligned} A'_{k,s} &= \mathcal{O}(|n-s|^2), \quad 0 \le k < n/2, \\ \lim_{s\to n}(n-s)^{-1}A'_{n/2,s} &= -2nc_{n/2}Q. \end{aligned} \tag{5.11}$$

Multiplying (5.10) by dv_g gives

$$(x\partial_x u_s)^2 dv_g = \\ \left[x^{n-2s-1}\sum_{k=0}^{n/2} B'_{k,s}x^{2k} + 2s(n-s)x^{-1}S(s)1 + \mathcal{O}(|n-s|x^{-1/2}) \right] dv_h dx,$$

where the $B'_{k,s}$ satisfy the same properties (5.11) as the $A'_{k,s}$. Integrating and evaluating the finite part and the limit as above yield

$$\lim_{s\to n} \text{ pf} \int_{\epsilon<x<x_0} (x\partial_x u_s)^2 dv_g. = -nc_{n/2}\int_M Q\ . \tag{5.12}$$

Differentiation with respect to y does not decrease the order of vanishing in x, and because of this one finds by a similar calculation that

$$\lim_{s\to n} \text{ pf} \int_{\epsilon<x<x_0} h_x^{ij}(x\partial_{y^i}u_s)(x\partial_{y^j}u_s)dv_g = 0. \tag{5.13}$$

Combining (5.9), (5.12), and (5.13) gives

$$\lim_{s\to n} \text{ pf} \int_{\epsilon<x<x_0} [|du_s|^2 - s(n-s)u_s^2]dv_g = -nL/2$$

as desired. □

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