A REMARK ON A PAPER OF E.B. DAVIES

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In a recent paper [1], Davies proves the following interesting result: if $P(h) = (hD_x)^2 + V(x)$, $D_x = (1/i)d/dx$, $V \in \mathcal{C}^\infty(\mathbb{R})$, then

$$\begin{gathered} \operatorname{Im} V'(x_0) \neq 0\,, \quad E = \xi_0^2 + V(x_0)\,,\ \xi_0 \in \mathbb{R} \setminus \{0\} \implies \\ \exists\ u(h) \in L^2(\mathbb{R})\,,\ \|u(h)\|_{L^2(\mathbb{R})} = 1\,,\ (P(h) - E)u(h) = \mathcal{O}(h^\infty)\,, \end{gathered} \tag{1}$$

where $\mathcal{O}(h^\infty)$ is a bound in any norm, for instance $\mathcal{C}^k$ for any k.

The purpose of this note is to point out that a more general statement than (1) follows immediately from the now standard results in microlocal analysis described in Chapter 26 of [5].

Let X be a manifold and $S^{m,k}(T^*X)$ be the space of semi-classical symbols on T^*X: $a \in \mathcal{C}^\infty(T^*X \times (0,1])$,

$$\partial_x^\alpha \partial_\xi^\beta a(x,\xi;h) \leq C_{\alpha,\beta} h^{-m}(1+|\xi|)^{k-|\beta|}\,, \quad (x,\xi) \in T^*X\,,\ h \in (0,1]\,. \tag{2}$$

The local quantization formula,

$$a(x, hD_x; h)u = \frac{1}{(2\pi h)^n} \int\!\!\int a(x,\xi;h) e^{i\langle x-y,\xi\rangle} u(y)\,dy\,d\xi\,,$$

defines a class of semi-classical pseudo-differential operators on X, $\Psi_h^{m,k}(X)$, with a symbol map, σ, and a short exact sequence,

$$0 \longrightarrow \Psi_h^{m-1,k-1}(X) \longrightarrow \Psi_h^{m,k}(X) \xrightarrow{\sigma} S^{m,k}(T^*X)/S^{m-1,k-1}(T^*X) \longrightarrow 0\,.$$

If we strengthen the condition (2) to

$$(hD_h)^l \partial_x^\alpha \partial_\xi^\beta a(x,\xi;h) \leq C_{l,\alpha,\beta} h^{-m}(1+|\xi|)^{k-|\beta|}\,, \quad (x,\xi) \in T^*X\,,\ h \in (0,1]\,, \tag{3}$$

we can then use, without any modifications, the $\mathcal{C}^\infty$ theory of pseudo-differential operators when working in compact subsets of T^*X. We can simply introduce a new variable $t \in \mathbb{R}$ and consider $h = 1/\tau$, where τ is the dual variable to t. Working microlocally in the region $|\xi/\tau| \leq C$, corresponds to working semi-classically in compact subsets of T^*X. In particular we have a correspondence between the frequency set and the wave front set:

$$(x,\xi) \notin WF_h(u(h)) \iff \begin{cases} \exists\ Q \in \Psi^0(\mathbb{R} \times X) \ \text{ elliptic at } \ (t,x,\tau,\tau\xi) \ \text{ for all } (t,\tau) \in T^*\mathbb{R} \\ \text{such that } Q\mathcal{F}^{-1}_{\tau \mapsto t}(u(1/\bullet)) \in \mathcal{C}^\infty(\mathbb{R} \times X), \end{cases}$$

where $\mathcal{F}^{-1}_{\tau \mapsto t}$ is the inverse Fourier transform – see [3] and [6, Proposition 9] for the definition and the relation to other wave front sets. The characterization given here follows from the one in [6].

This allows us to adapt directly the results of Hörmander and of Duistermaat and Sjöstrand (see [2] and [5, Sect.26.3]), to obtain the following statement: if $P(x, hD_x; h)$ is a semi-classical pseudo-differential operator, in the class satisfying (3), and with the principal symbol p, then

$$\begin{gathered} p(m) = 0\,, \quad \{\operatorname{Re} p, \operatorname{Im} p\}(m) < 0\,, \quad m \in T^*X \implies \\ \exists\ u(h) \in \mathcal{C}^\infty(X) \ \text{ such that } WF_h(u(h)) = \{m\} \text{ and } WF_h(Pu) = \emptyset. \end{gathered} \tag{4}$$

If we take $P(x, hD_x; h) = (hD_x)^2 + V(x) - E$ we immediately recover Davies's result: if $\chi \in \mathcal{C}_c^\infty(X)$ is equal to 1 near x_0, $m = (x_0, \xi_0)$ then, because of the WF_h statements, $P(\chi u) = \mathcal{O}(h^\infty)$ and $\chi u \in \mathcal{C}_c^\infty(X)$. The condition on the Poisson bracket generalizes the one dimensional condition $\operatorname{Im} V'(x_0) \neq 0$. For Schrödinger operators in higher dimensions, we need $E = \xi_0^2 + V(x_0)$, $\operatorname{Im} \xi_0 \cdot \nabla V(x_0) \neq 0$.

We conclude with a few remarks. The microlocal statement (4), is a generalization of the celebrated commutator condition of Hörmander [4]. The crucial part of his argument was in fact a geometric optics "quasi-mode" construction which, in a very special case, is repeated in [1]. As was pointed out to the author by Victor Ivrii, the microlocal commutator condition (4) has the following well known global analogue: let Q be an unbounded operator on a Hilbert space $\mathcal{H}$, with the domain $\mathcal{D}$. Let us also assume that Q^* has the same domain as Q and that $Q - z$, $Q^* - \bar{z} : \mathcal{D} \to \mathcal{H}$, are Fredholm operators for all z. Then

$$\pm[Q, Q^*] \geq C > 0 \implies \operatorname{spec} Q = \mathbb{C}.$$

In the case of $+$ sign, $Q - z$ fails to be injective for all z, and in the case of $-$ sign, surjective. In fact, we conclude that for all $z \in \mathbb{C}$ and $u \in \mathcal{D}$

$$\mp \|(Q - z)u\|^2 \pm \|(Q^* - \bar{z})u\|^2 \geq C\|u\|^2 .$$

If $Q - z_0$ were invertible at some z_0, then index $(Q - z) =$ index $(Q - z_0) = 0$. In the $-$ case $Q - z$ would then be invertible everywhere which gives a contradiction. In the $+$ case we draw the same conclusion for the adjoint.

We remark finally that, as pointed out by Davies, the existence of complex quasi-modes, implies lower bounds on the resolvent, and that has computation consequences.

Acknowledgments. I should like to thanks Fréderic Klopp and Victor Ivrii for helpful discussions and Steve Zelditch for encouraging me to publish this note.

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