RESONANCE WAVE EXPANSIONS: TWO HYPERBOLIC EXAMPLES

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Abstract. For scattering on the modular surface and on the hyperbolic cylinder, we show that the solutions of the wave equations can be expanded in terms of resonances, despite the presence of trapping. Expansions of this type are expected to hold in greater generality but have been understood only in *non-trapping* situations.

1. Introduction

In this note we give two examples for which we can obtain, on compact sets, an asymptotic expansion of solutions to the wave equation with smooth, compactly supported initial data, although there is trapping. The expansions are given in terms of *resonances* and they generalize the standard "separation of variables" expansions in terms of eigenvalues. The examples are the modular surface where we can use detailed information about the zeta function (see Fig.1 and Theorem 1) and the hyperbolic cylinder where the resonances are particularly simple (see Fig.2(b) and Theorem 2).

Resonances or *scattering poles* are defined as poles of the meromorphic continuation of the resolvent or the scattering matrix and they constitute a natural replacement of discrete spectral data for problems on exterior domains. That point of view was emphasized early by Lax-Phillips ([12]) – see [23] for a light-hearted overview of recent results.

Although resonances are most frequently defined in the stationary framework of scattering theory they are a dynamical concept: the real part of a resonance describes the rest energy of a state and the imaginary part its rate of decay. Consequently they should be understood in terms of long time behaviour of solutions to evolution equations, and, in particular, to the wave equation. For the Schrödinger evolution equation we refer to the recent paper by Soffer-Weinstein [17] and references given there; in that case one considers resonances which come from perturbing embedded eigenvalues.

Under a *quantum* dynamical condition that the obstacle $\mathcal{O}$ is non-trapping (that is, a condition on the behaviour of solutions of the evolution equation) Lax-Phillips [12] and Vainberg [21],[22] showed that for n odd, for some $\epsilon > 0$

$$\left.\begin{aligned} &(D_t^2 - \Delta)u(t,x) = 0\,, \quad x \in \mathbb{R}^n \setminus \mathcal{O}\,, \quad u\restriction_{\mathbb{R}\times\partial\mathcal{O}} = 0\,, \\ &u\restriction_{t=0} = f \in \mathcal{C}^\infty_{\rm c}(\mathbb{R}^n \setminus \overline{\mathcal{O}}) \\ &\tfrac{1}{i}\partial_t u\restriction_{t=0} = g \in \mathcal{C}^\infty_{\rm c}(\mathbb{R}^n \setminus \overline{\mathcal{O}}) \end{aligned}\right\} \Longrightarrow \tag{1.1}$$

$$u(t,x) = \sum_{\operatorname{Im}\lambda_l \leq C} \sum_{j=1}^{m_{\mathcal{O}}(\lambda_l)} w_{\lambda_l, j}(x) e^{it\lambda_l} t^{j-1} + \mathcal{O}(e^{-(C+\epsilon)t})\,, \; x \in K\,,$$

$K \subset \mathbb{R}^n \setminus \mathcal{O}$ compact, where $m_{\mathcal{O}}(\lambda_l)$ is the multiplicity of the resonance. Here we took the convention that $\operatorname{Im}\lambda_l \geq 0$, that is, that the resonances lie in the upper half-plane. That the quantum non-trapping condition follows from the *classical* non-trapping condition was shown in the works of Andersson, Melrose, Morawetz, Ralston, Strauss, Sjöstrand and Taylor – see the appendix to the second edition of [12] and references given there.

In the ultimate trapping situation of a compact manifold, the expansion (1.1) is simply the expansion in terms of eigenvalues. Hence a naïve "interpolation" argument suggests its validity for all perturbations. That, however, is far from clear and not much is known. Tang-Zworski [19] recently showed, using the methods of [18], that the expansion (1.1) is valid for general "black box" perturbations when we sum over resonances satisfying $\operatorname{Im}\lambda_j \leq |\lambda_j|^{-M}$, M sufficiently large, and when we replace the error by $\mathcal{O}(t^{-N})$ for any N. The result is at the moment conditional and the following generically reasonable yet unverifiable assumption has to be made:

$$|\lambda_l - \lambda_k| > \frac{1}{C}(\max\{|\lambda_l|, |\lambda_k|\})^{-L}\,, \quad \text{for some fixed } L > 0\,. \tag{1.2}$$

This motivates our unconditional results in two rather explicit examples.
Remark. After this paper was written we learned that in a recent paper [1], Beyer obtained results very close to Theorem 2 here. His work was motivated by gravitational scattering – see [15] for another discussion of resonances in that setting and the relation to the hyperbolic cylinder.

In the paper, C will stand for a constant whose value may change from line to line. The notation $\langle s\rangle$ means $(1+|s|^2)^{\frac{1}{2}}$.

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2. The Quotient by the Modular Group

We begin by working on a general surface with one constant curvature cusp end, which we shall identify with $(a,\infty)_y \times S^1_\theta$ for some $a > 0$. We will use z to denote a variable in the surface and, as is common for scattering on hyperbolic surfaces, we use the spectral variable $s(1-s)$. We refer to [14] for the spectral and scattering theories of such surfaces.

Let $E(z,s)$ be the generalized eigenfunction of the positive Laplacian Δ, so that $(\Delta - s(1-s))E(z,s) = 0$, such that on the cusp end

$$E(z,s) = y^s + S(s)y^{1-s} + \mathcal{O}(y^{-\epsilon})$$

for some $\epsilon > 0$. With this convention, the scattering matrix $S(s)$ is holomorphic when $\operatorname{Re} s > 1/2$, except for a finite number of values of s which correspond to the eigenvalues of Δ which lie below $1/4$.

The function $u(t) = \frac{\sin(t\sqrt{\Delta - 1/4})}{\sqrt{\Delta - 1/4}} f$ satisfies the equation

$$\begin{aligned} (D_t^2 - (\Delta - 1/4))u(t) &= 0 \\ u(0) &= 0 \\ u_t(0) &= f \end{aligned}$$

when f is sufficiently well-behaved. Using the spectral representation, we have

$$\begin{aligned} \frac{\sin(t\sqrt{\Delta - 1/4})}{\sqrt{\Delta - 1/4}} &= \frac{1}{2i}\sum_{\lambda_j \in \sigma_p(\Delta)} \frac{(e^{i\sqrt{\lambda_j - 1/4}t} - e^{-i\sqrt{\lambda_j - 1/4}t})}{\sqrt{\lambda_j - 1/4}}\phi_j(z)\overline{\phi}_j(z') \\ &\quad + \frac{1}{8\pi}\int_{-\infty}^{\infty} \frac{(e^{i\tau t} - e^{-i\tau t})}{i\tau} E(z, 1/2 + i\tau)E(z', 1/2 - i\tau)d\tau \end{aligned} \tag{2.1}$$

in the sense of distributions, where ϕ_j is a normalized eigenfunction of Δ, associated to λ_j. Here the λ_j's can take the same value depending on the multiplicity and the ϕ_j's form an orthonormal set.

Let $X_0 = \mathbb{H}^2/PSL(2;\mathbb{Z})$ be the quotient of the hyperbolic upper half plane by $PSL(2;\mathbb{Z})$, which has scattering matrix

$$S(s) = \sqrt{\pi}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\frac{\zeta(2s-1)}{\zeta(2s)} \tag{2.2}$$

where Γ is the Euler Γ-function and ζ is the Riemann ζ-function. One consequence of this is that the poles of the scattering matrix other than $s = 1$ correspond to the non-trivial zeros of $\zeta(2s)$. We recall that if $N(T)$ is the number of zeros of the function $\zeta(s)$ in $0 \leq \operatorname{Re} s \leq 1$, $0 \leq \operatorname{Im} s \leq T$, then

$$N(T) = \frac{1}{2\pi}T\log T - \frac{1+2\pi}{2\pi}T + \mathcal{O}(\log T)$$

as $T \to \infty$ ([20, Theorem 9.4]). That is, the scattering matrix for the Laplacian on X_0 has far fewer than the maximum number of poles in a ball of radius T centered at the origin, which is $\mathcal{O}(T^2)$ for general surfaces with cusps.

Let $f, \chi \in C^\infty_{\mathrm{c}}(X_0)$. We wish to obtain an asymptotic expansion as $t \to \infty$ of $\sin(t\sqrt{\Delta - 1/4})/\sqrt{\Delta-1/4})$ applied to f and truncated by χ.

Theorem 1. *Let $f, \chi \in C^\infty_{\mathrm{c}}(X_0)$. Then there exist $v_{jk} \in C^\infty_{\mathrm{c}}(X_0)$ such that as $t \to \infty$,*

$$\chi\frac{\sin(t\sqrt{\Delta-1/4})}{\sqrt{\Delta-1/4}}f = \frac{1}{2i}\sum_{\lambda_j\in\sigma_p(\Delta)}\left(\frac{e^{i\sqrt{\lambda_j-1/4}t} - e^{-i\sqrt{\lambda_j-1/4}t}}{\sqrt{\lambda_j-1/4}}\right)\chi(z)\phi_j(z)(f,\phi_j)$$
$$+ \sum_{s_j \text{ poles of } S(s)} e^{(s_j-1/2)t(\operatorname{sign}(1/2-\operatorname{Re} s_j))}\sum_{k\leq \operatorname{mult}(s_j)-1} v_{jk}t^k + \mathcal{O}(e^{-Nt})$$

for any N.

The sum over the poles should be understood as follows: for $n \in \mathbb{Z}$, there exist $\tau_n \in \mathbb{R}$, $n \leq 2\tau_n < n+1$, such that if

$$b_n(z,t) = \sum_{s_j:\tau_n<\operatorname{Im} s_j<\tau_{n+1}} e^{(s_j-1/2)t}\sum_{k\leq\operatorname{mult}(s_j)-1} v_{jk}t^k,$$

then $\sum_{n=-\infty}^{\infty} b_n(z,t)$ is absolutely convergent when $t \geq 0$. We remark that to obtain an expected expansion in the case of the modular surface, that is an expansion in which there is no need to cluster terms to obtain absolute convergence, would require more precise information about the zeta function: simplicity of zeros, lower bounds on the derivatives.

The proof of the theorem involves a contour deformation. In order to justify the deformation, we will need some bounds on $S(s)$ and on $E(z,s)$.

Let $\chi_1 \in C^\infty_{\mathrm{c}}(X_0)$ be such that the support of $(1-\chi_1)$ is contained in the cusp end of X_0. Then

$$E(z,s) = (1-\chi_1)y^s + (\Delta - s(1-s))^{-1}[\Delta,\chi_1]y^s. \tag{2.3}$$

If $\chi \in C^\infty_{\mathrm{c}}(X_0)$, then the basic spectral estimate on the resolvent

$$(\Delta - s(1-s))^{-1} = \mathcal{O}(d(s(1-s),\sigma(\Delta))^{-1}) \;:\; L^2(X_0) \longrightarrow L^2(X_0)\,,$$

the Sobolev embedding theorem, and interpolation ($\Delta^k([\Delta,\chi_1]y^s) = \mathcal{O}(|s|^{2k+1}e^{|\operatorname{Re} s|}) \in L^2(X_0)$) immediately give

$$|\chi E(z,s)| \leq C\frac{\langle s\rangle^{2+\epsilon}e^{\operatorname{Re} s}}{\operatorname{Re} s - 1/2} \quad\text{and}\quad |S(s)| \leq Ce^{\operatorname{Re} s} \tag{2.4}$$

when $\operatorname{Re} s \geq 1/2$ and $|\operatorname{Im} s| > 1$ or $\operatorname{Re} s > 2$. (For the bound on S, see [14, (3.26)-(3.27)].).

The bounds (2.4) are valid for other surfaces with one cusp end, but we shall need an improved bound in the special case of X_0. As a by-product of their work on L^∞ bounds on L^2-eigenfunctions, Iwaniec and Sarnak obtain the bound

$$|\chi(z)E(z,1/2+it)| \ll |t|^{\frac{5}{12}+\epsilon}\,,$$

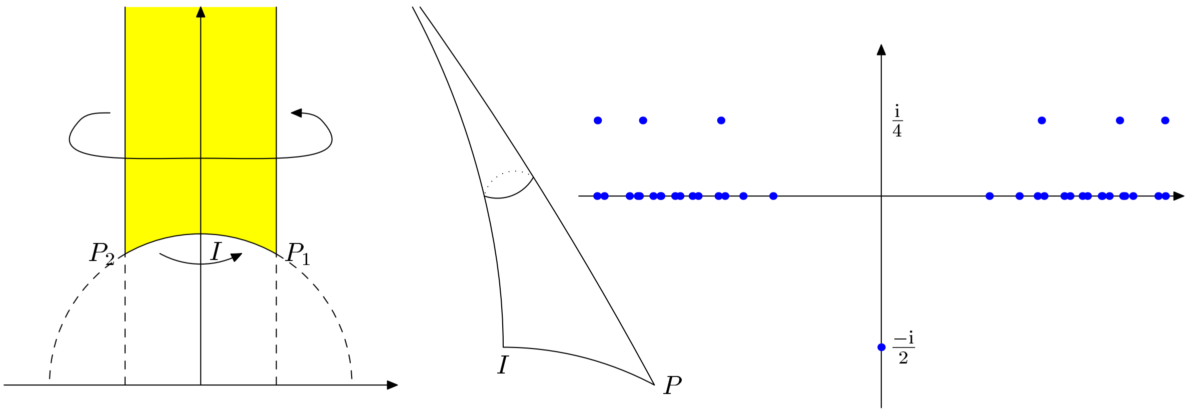

FIGURE 1. The fundamental domain of the modular group and the resonances of the modular surface in the λ-plane, $s = \frac{1}{2} - i\lambda$.

see [11, (A.12)] where now the positive contribution of the Eisenstein series should be kept (see also the discussion around [16, (2.18)]).

Here we give a direct proof of a weaker estimate, in the spirit of general scattering theory: the estimate depends only on the separation of the poles of E from the continuous spectrum.

Lemma 2.1. *For the generalized eigenfunctions $E(z,s)$ on X_0,*

$$|\chi(z)E(z,s)| \leq C\langle s\rangle^{2+\epsilon}e^{\operatorname{Re}s} \text{ if } \operatorname{Re}s \geq 1/2,\ |s-1| > \epsilon > 0.$$

Proof. We shall use (2.2) to improve our bound on $E(z,s)$ when $\operatorname{Re}s \geq 1/2$.

We start by recalling an exponential bound on the resolvent:

$$\|\chi_2(\Delta - s(1-s))^{-1}\chi_2\|_{L^2\to L^2} \leq C\exp(C|s|^{N_0}), \quad \text{if } s \notin \left(\cup_{j=1}^{\infty} D(s_j,\langle s_j\rangle^{-3}) \cup_{\sigma_j:\sigma_j(1-\sigma_j)=\lambda_j\in\sigma_p(\Delta)} D(\sigma_j,\langle\sigma_j\rangle^{-3})\right) \tag{2.5}$$

for any $\chi_2 \in C_c^\infty(X_0)$ for some N_0 (see the representation of the resolvent in [7, Sect. 5] and [8, Lemma 3.6]; the estimate also follows from the general "black box" scattering estimate, see [18, Lemma 1]).

Moreover, by Theorem 3.8 of [20] there is a constant $A > 0$ such that $\zeta(s)$ is not zero for $\operatorname{Re}s \geq 1 - A/\log|\operatorname{Im}s|$, $|\operatorname{Im}s| > t_0$. Since $(\Delta - s(1-s))^{-1}[\Delta,\chi_1]y^s$ is regular away from the poles of the scattering matrix, using (2.3), (2.5), and the maximum principle this leads to a bound on the generalized eigenfunctions:

$$|\chi(z)E(z,s)| \leq C\exp(C|s|^{N_0}) \tag{2.6}$$

when $|\operatorname{Im}s|$ is sufficiently large and $\operatorname{Re}s \geq 1/2 - A/(4\log|\operatorname{Im}s|)$. Then, just as in [18, Lemma 2], we can use the maximum principle, the existence of a pole free region, the bound in the good half plane (2.4) and the exponential bound (2.6) to obtain, for $|\operatorname{Im}s|$ sufficiently large,

$$|\chi(z)E(z,s)| \leq C\langle s\rangle^{2+\epsilon}e^{\operatorname{Re}s} \text{ if } \operatorname{Re}s \geq 1/2. \tag{2.7}$$

□

Proof of Theorem 1. We use the representation (2.1) of $\sin(t\sqrt{\Delta-1/4})/\sqrt{\Delta-1/4}$. For the term of (2.1) with an $e^{it\tau}$ we deform the contour of integration into the upper half plane; the term with $e^{-it\tau}$ is deformed into the lower half-plane. We note that this is possible despite the singularity at $\tau = 0$ since

$$\lim_{\epsilon \downarrow 0} \Bigg(\Bigg(\int_{\substack{\operatorname{Im}\tau=\epsilon \\ -\delta<\operatorname{Re}\tau<\delta}} \frac{e^{i\tau t}}{\tau} E(z,1/2+i\tau)E(z',1/2-i\tau)d\tau - \int_{\substack{\operatorname{Im}\tau=-\epsilon \\ -\delta<\operatorname{Re}\tau<\delta}} \frac{e^{-i\tau t}}{\tau} E(z,1/2+i\tau)E(z',1/2-i\tau)d\tau - \int_{\substack{\operatorname{Im}\tau=0 \\ -\delta<\operatorname{Re}\tau<\delta}} \frac{(e^{i\tau t}-e^{-i\tau t})}{\tau} E(z,1/2+i\tau)E(z',1/2-i\tau)d\tau \Bigg) = 0. \tag{2.8}$$

We consider the term with $e^{it\tau}$, for which we deform the contour of integration into the upper half-plane. There $E(z,1/2-i\tau)$ is bounded as in Lemma 2.1. We will use that $E(z,1/2+i\tau) = S(z,1/2+i\tau)E(z,1/2-i\tau)$, and this, combined with some estimates on $S(s)$ and the estimates above, will allow us to justify the contour deformation.

We give some bounds on the scattering matrix, using the fact that we know it explicitly in terms of Γ and ζ. If $|\operatorname{Re}(s)| < C_1$, $|\operatorname{Im} s| > 1$, then

$$\left| \frac{\Gamma(s-1/2)}{\Gamma(s)} \right| \leq C_2 \tag{2.9}$$

by Stirling's formula. If $\operatorname{Re}(s) \geq -\delta$,

$$|\zeta(s)| = \mathcal{O}(|\operatorname{Im} s|^{3/2+\delta}) \tag{2.10}$$

([20, p. 95]). Moreover, by [20, Theorem 9.7], there is a constant A such that each interval $(T, T+1)$ contains a value τ of $\operatorname{Im} s$ such that

$$|\zeta(s)| \geq |\operatorname{Im} s|^{-A}, \ \operatorname{Im} s = \tau, \ -1 \leq \operatorname{Re} s \leq 2. \tag{2.11}$$

For each $n \in \mathbb{Z}$ we choose such a $2\tau_n \in (n, n+1)$. Using the equations (2.2), (2.7), (2.9), (2.10), (2.11), the fact that $\Delta E(z,1/2-i\tau) = (1/4+\tau^2)E(z,1/2+i\tau)$ and $\Delta^k f \in C_c^\infty(X_0)$ for any $k \in \mathbb{N}$, by integrating by parts in z' we can show that

$$|\chi(z)E(z,1/2+i\tau)\int_{z'\in X_0} E(z',1/2-i\tau)f(z')| \leq C(1+|\tau|)^{-m} \tag{2.12}$$

for any $m \in \mathbb{R}$, if $\operatorname{Im}\tau = 0$, $\operatorname{Im}\tau = 3/2$, or $\operatorname{Re}\tau = \tau_n$, $0 \leq \operatorname{Im}\tau \leq 3/2$.

Now we do a contour deformation, deforming the integral over $\tau_n \leq \tau \leq \tau_{n+1}$, $\operatorname{Im}\tau = 0$, to the contour consisting of the three line segments $\operatorname{Re}\tau = \tau_n$, $0 \leq \operatorname{Im}\tau \leq 3/2$; $\operatorname{Im}\tau = 3/2$, $\tau_n \leq \operatorname{Re}\tau \leq \tau_{n+1}$; and $\operatorname{Re}\tau = \tau_{n+1}$, $0 \leq \operatorname{Im}\tau \leq 3/2$. The estimate (2.12) and the residue theorem show that the sum of the residues in this region is bounded in absolute value by $C_k(1+|\tau_n|)^{-k}$ for any k; this will give us the convergence of the sum over the poles as desired. By doing a contour deformation to the line $\operatorname{Im}\tau = 3/2$ and using the estimates (2.7), (2.9), (2.10), and (2.11), along with the residue theorem, we obtain a sum over the poles, with an error term of order $e^{-3t/2}$.

To improve the error term, we will need bounds on $S(s)$ when $\operatorname{Re} s < -1$. By [20, Theorem 8.7],

$$\left| \frac{1}{\zeta(s)} \right| \leq \frac{\zeta(\operatorname{Re} s)}{\zeta(\operatorname{Re} 2s)} \quad \text{if} \quad \operatorname{Re} s > 1. \tag{2.13}$$

Since

$$\zeta(s) = 2^s \pi^{s-1} \sin\frac{\pi s}{2} \Gamma(1-s)\zeta(1-s),$$

the estimate (2.13) combined with our earlier estimates is enough to show that the contour of integration can be deformed to any line $\operatorname{Im}\tau = c_0$, resulting in an error as claimed.

The term with $e^{-it\tau}$ is treated similarly. □

Suppose the Riemann hypothesis is true. Then, using a contour deformation argument similar to that above, one can show that

$$\chi \frac{\sin(t\sqrt{\Delta - 1/4})}{\sqrt{\Delta - 1/4}} f = \frac{1}{2i} \sum_{\lambda_j \in \sigma_p(\Delta)} \left(\frac{e^{i\sqrt{\lambda_j - 1/4}t} - e^{-i\sqrt{\lambda_j-1/4}t}}{\sqrt{\lambda_j - 1/4}} \right) \chi(z)\phi_j(z)(f, \phi_j) + \mathcal{O}(e^{(-1/4+\epsilon)t})$$

for any $\epsilon > 0$ and where we have used the notation and assumptions of Theorem 1.

On the other hand, just using the *known* fact that the ζ-function has no zeros in $\{s : \operatorname{Re} s > 1 - C(\log |\operatorname{Im} s|)^{-1}, |\operatorname{Im} s| > t_0\}$ for some $C > 0$, one can show, by moving the contour of integration off the real axis, to $\operatorname{Im} \tau = C/4 \operatorname{Re} \tau$ for sufficiently large τ, that

$$\chi \frac{\sin(t\sqrt{\Delta - 1/4})}{\sqrt{\Delta - 1/4}} f = \frac{1}{2i} \sum_{\lambda_j \in \sigma_p(\Delta)} \left(\frac{e^{i\sqrt{\lambda_j - 1/4}t} - e^{-i\sqrt{\lambda_j-1/4}t}}{\sqrt{\lambda_j - 1/4}} \right) \chi(z)\phi_j(z)(f, \phi_j) + \mathcal{O}(t^{-N})$$

for any N.

3. The Hyperbolic Half-Cylinder

The second example we consider is the hyperbolic half-cylinder $Y_{0l} \simeq (\mathbb{R}_+)_r \times (\mathbb{R}/l\mathbb{Z})_\theta$ with metric $dr^2 + \cosh^2 r d\theta^2$. The analysis is equally applicable to the case of the full cylinder $Y_l \simeq (\mathbb{R})_r \times (\mathbb{R}/l\mathbb{Z})_\theta$ with the same metric. In both cases the trapped set consists of one closed hyperbolic orbit which is well known to generate resonances on a lattice (as was pointed out by Guillopé [6] and Epstein [2]; see also [7, Appendix] and Fig.2(b)).

We begin by recalling some results of [7, Section 3 and Appendix]. The Laplacian $\Delta^{Y_{0l}} = D_r^2 - i \tanh r D_r + \Delta^{\mathbb{R}/l\mathbb{Z}}(\cosh r)^{-2}$ on the hyperbolic half-cylinder Y_{0l} is, through conjugation by $\cosh^{1/2} r$, equivalent to the operator

$$D_r^2 + \frac{\Delta^{\mathbb{R}/l\mathbb{Z}} + 1/4}{\cosh^2 r} + \frac{1}{4}$$

on $L^2(\mathbb{R}_+ \times \mathbb{R}/l\mathbb{Z}, drd\theta)$. We can expand this in terms of the eigenfunctions on $\mathbb{R}/l\mathbb{Z}$ to obtain

$$\bigoplus_{m\in\mathbb{Z}} D_r^2 + \frac{(2\pi m/l)^2 + 1/4}{\cosh^2 r} + \frac{1}{4}$$

on $l^2(\mathbb{Z}, L^2(\mathbb{R}_+, dr))$.

Modifying slightly the notation of [7], we have a generalized eigenfunction satisfying the Dirichlet boundary condition for the one-dimensional problem $D_r^2 + \frac{(2\pi m/l)^2+1/4}{\cosh^2 r} - k^2$

$$\tilde{E}_{\nu_m}(r,k) = \tilde{a}(\nu_m,k) \sinh r \cosh^{1+\nu_m} r \, {}_2F_1((\nu_m - ik + 2)/2, (\nu_m + ik + 2)/2, 3/2; -\sinh^2 r) \tag{3.1}$$

where $\nu_m = -1/2 + i(2\pi m/l)$ and

$$\tilde{a}(\nu_m, k) = \frac{2^{-ik}\Gamma((\nu_m - ik + 2)/2)\Gamma((-\nu_m - ik + 1)/2)}{\Gamma(-ik)\Gamma(3/2)}.$$

Then, for $r, r' > 0$,

$$\delta(r - r') = \frac{1}{4\pi} \int_{-\infty}^{\infty} \tilde{E}_{\nu_m}(r,k)\tilde{E}_{\nu_m}(r', -k)dk.$$

In order to be consistent with our first example, we shall use as the variable $s = 1/2 - ik$ and set $E_{\nu_m}(r, s) = \tilde{E}_{\nu_m}(r, k)$. The scattering matrix $S_{0l}(s)$ for the hyperbolic half-cylinder with Dirichlet boundary conditions is

$$S_{0l}(s) = \bigoplus_{m\in\mathbb{Z}} s(\overline{H}_{0,-1/2+2im\pi/l})(k), \;\; s = 1/2 - ik$$

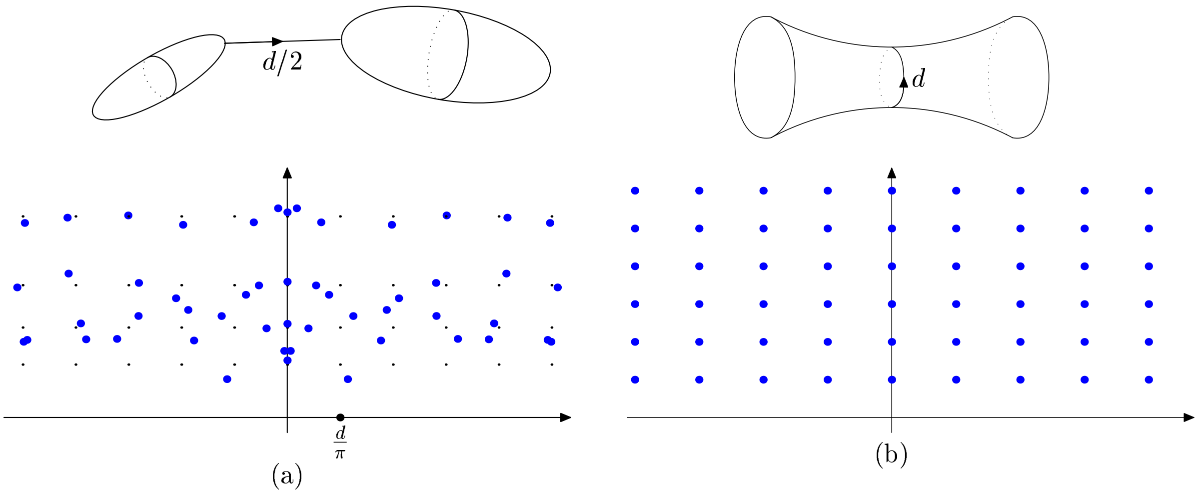

FIGURE 2. (a) Resonances associated to two strictly convex bodies: in every fixed strip, the resonances become closer to points on the lattice as the real part increases. (b) Resonances for a hyperbolic cylinder: all resonances lie exactly on a lattice. The underlying dynamical structure, exactly one hyperbolic closed orbit, is the same in the two examples.

where $\overline{H}_{0,-1/2+2im\pi/l}$ is $D_r^2 + \frac{(2\pi m/l)^2+1/4}{\cosh^2 r}$ with Dirichlet boundary conditions. This gives us ([7, Lemmas 3.3, 3.4]) that

$$S_{0l}(s) = \bigoplus_{m\in\mathbb{Z}} s_{lm}(s) \tag{3.2}$$

with

$$s_{lm}(s) = \frac{2^{2s-1}\Gamma(1/2-s)\Gamma((1+s-i2\pi m/l)/2)\Gamma((1+s+i2\pi m/l)/2)}{\Gamma(s-1/2)\Gamma((2-s-i2\pi m/l)/2)\Gamma((2-s+i2\pi m/l)/2)} \tag{3.3}$$

and the resonances of the Dirichlet Laplacian associated to $s_{lm}(s)$ are $\pm i2\pi m/l - n$, $n \in 2\mathbb{N}-1$. Notice that this means that for any $\beta \in \mathbb{R}$, corresponding to each eigenfunction on $\mathbb{R}/l\mathbb{Z}$ there are only a finite number of resonances with real part greater than β. Note too that for $m \neq 0$, the resonance $i2\pi m/l - n$ has multiplicity two as a resonance of $S_{0l}(s)$ but only one as a resonance of $s_{lm}(s)$ and $s_{l(-m)}(s)$. For this reason, when $m \neq 0$ we can rule out the possibility of needing terms with t dependence in the expansion someplace other than the exponential.

If X is a manifold with boundary, we use the notation $\dot{C}_c^\infty(X)$ to denote the smooth, compactly supported functions on X that vanish to infinite order at the boundary.

Theorem 2. *Let $f \in \dot{C}_c^\infty(Y_{0l})$ and let $\chi \in C_c^\infty(Y_{0l})$. Then there exist $v_{m,n}, w_n \in C_c^\infty(Y_{0l})$ such that as $t \to \infty$,*

$$\chi \frac{\sin(t\sqrt{\Delta^{Y_{0l}} - 1/4})}{\sqrt{\Delta^{Y_{0l}} - 1/4}} f = \sum_{\substack{0<n<\beta \\ m\in\mathbb{Z},\, n\in 2\mathbb{N}-1}} e^{(i2m\pi/l-n-1/2)t} v_{m,n} + \sum_{\substack{0<n<\beta \\ n\in 2\mathbb{N}-1}} e^{(-n-1/2)t} w_n t + \mathcal{O}(e^{(-\beta-1/2)t})$$

if $\beta \notin 2\mathbb{N}-1$. The same conclusion holds for $f \in C_c^\infty(Y_l)$ and Δ_l with the resonance set replaced by the resonance set of the full hyperbolic cylinder $\pm 2\pi im/l - n$, $m \in \mathbb{N}_0$, $n \in \mathbb{N}_0$.

Let $\{\phi_m\}$, $m \in \mathbb{Z}$ be an orthonormal set of eigenfunctions of $\Delta^{\mathbb{R}/l\mathbb{Z}}$, with corresponding eigenvalue $(2\pi m/l)^2$. As in the first example, we shall use the spectral representation

$$\frac{\sin(t\sqrt{\Delta-1/4})}{\sqrt{\Delta-1/4}} \tag{3.4}$$
$$= \frac{1}{8\pi}\sum_{m\in\mathbb{Z}}\phi_m(\theta)\overline{\phi}_m(\theta')\int_{\operatorname{Re}s=1/2}\frac{e^{(s-1/2)t}-e^{-(s-1/2)t}}{s-1/2}E_{\nu_m}(r,s)E_{\nu_m}(r',1-s)\cosh^{-1/2}r\cosh^{-1/2}r'ds.$$

As for the first example, the proof involves a contour deformation and the difficulty lies in justifying it, bounding the residues, and bounding the integral over $\operatorname{Re}s=\beta$. We prove the proposition by proving a series of lemmas which justify the contour deformation and bound the remainder.

Let $\chi,\tilde{\chi}\in C_c^\infty(\mathbb{R}_+)$. We will need bounds in s and m on $\chi(r)E_{\nu_m}(r,s)E_{\nu_m}(r',1-s)\tilde{\chi}(r')$ for fixed $\operatorname{Re}s$ and bounds, in m, on its residues. We note that polynomial bounds will be sufficient for our purposes, since we shall be pairing with a function $f\in C_c^\infty(Y_{0l})$. Since $(2\pi m/l)^2$ are the eigenvalues on the cross section $\mathbb{R}/l\mathbb{Z}$, and since $\Delta^{\mathbb{R}/l\mathbb{Z}}f\in C_c^\infty(Y_{0l})$ when $f\in C_c^\infty(Y_{0l})$, a polynomial bound in m suffices. Similarly, because f vanishes to infinite order at $r=0$, repeated integration by parts shows that a polynomial bound in s is enough.

We first concern ourselves with the residues.

Lemma 3.1. *Let $\chi,\tilde{\chi}\in C_c^\infty(\mathbb{R}_+)$. If $0<n<N_0$, the residues of $\chi(r)E_{\nu_m}(r,s)E_{\nu_m}(r',1-s)\tilde{\chi}(r')$ at $s=\pm 2\pi im/l-n$ are bounded by $C\langle m\rangle^{N_0+1/2}$.*

Proof. We will use that

$$E_{\nu_m}(r,s)=s_{lm}(s)E_{\nu_m}(r,1-s)$$

so that we need only bound $s_{lm}(s)$ and its residues and bound $E_{\nu_m}(r,1-s)$ in the good half plane $\operatorname{Re}(1-s)>1/2$.

At the pole $s=\pm i2\pi m/l-n$, $n\in 2\mathbb{N}-1$,

$$E_{\nu_m}(r,1-s)_{|s=\pm i2\pi m/l-n}=\tilde{a}(\nu_m,i/2\pm 2\pi m/l+in)\sinh r(\cosh r)^{1/2+2\pi im/l} \tag{3.5}$$
$$\times\,{}_2F_1(1+\frac{n}{2}+\frac{\pi im}{l}\mp\frac{\pi im}{l},\frac{1}{2}+\frac{\pi im}{l}\pm\frac{\pi im}{l}-\frac{n}{2},\frac{3}{2},-\sinh^2 r).$$

From [3, Sect. 2.3.2], for bounded n and r,

$$|{}_2F_1(1+\frac{n}{2}+\frac{\pi im}{l}\mp\frac{\pi im}{l},\frac{1}{2}+\frac{\pi im}{l}\pm\frac{\pi im}{l}-\frac{n}{2},\frac{3}{2},-\sinh^2 r)|\le C\langle m\rangle^{n/2-1/2}. \tag{3.6}$$

Using Stirling's formula, we find that

$$|\tilde{a}(\nu_m,i/2\pm 2\pi m/l+in)|\le C\langle m\rangle^{-n/2+1/2}. \tag{3.7}$$

For $m\neq 0$, the residue of s_{lm} at $s=\pm i2\pi m/l-n$ is

$$\frac{(-1)^{(1-n)/2}2^{\pm i4\pi m/l-2n}\Gamma(1/2\mp i2\pi m/l+n)\Gamma((1-n\pm i4\pi m/l)/2)}{\Gamma(\pm i2\pi m/l-n-1/2)\Gamma(2\mp i4\pi m/l+n)/2)\Gamma((2+n)/2)\Gamma((n+1)/2)}$$

which, for fixed n, is easily seen to be bounded by $C\langle m\rangle^{n+1/2}$ using the relation $\Gamma(z+1)=z\Gamma(z)$ and Stirling's formula. This, along with (3.5), (3.6) and (3.7), shows that the residues are bounded as claimed. $\square$

The bound given by this lemma is enough to show that the sum over the $v_{m,n}$ in Theorem 2, which corresponds to a sum over residues, converges absolutely, since we may integrate by parts in the tangential variable to obtain decay in $|m|$.

We state the following lemma for the slightly easier case of $e^{t(s-1/2)}$ rather than for $e^{t(s-1/2)}/(s-1/2)$ in order to avoid the difficulty at $s=1/2$. However, this can be easily treated as indicated in (2.8) by considering at the same time the term of (3.4) with $e^{-t(s-1/2)}/(s-1/2)$. As the main difficulty here is the behaviour of the integrand at infinity, we avoid the question of the behaviour near $s = 1/2$. This lemma, combined with the previous one and the remarks above, justify the contour deformation to $\mathrm{Re}\, s = -\beta$.

Lemma 3.2. *Let* $\chi \in C_c^\infty(\mathbb{R}_+)$, $f \in \dot{C}_c^\infty(Y_{0l})$, *and*

$$h_m(r,s;f) = \chi(r)\cosh^{-1/2} r\phi_m(\theta) \times \int_{\mathbb{R}_+}\int_{\mathbb{R}/l\mathbb{Z}} e^{t(s-1/2)} E_{\nu_m}(r,s)E_{\nu_m}(r',1-s)\cosh^{1/2} r' f(r',\theta')\overline{\phi}_m(\theta')d\theta' dr'.$$

Then if $\beta > 0$, $\beta \notin 2\mathbb{N}-1$, *then*

$$\int_{\mathrm{Re}\, s=1/2} h_m(r,s;f)ds = \int_{\mathrm{Re}\, s=-\beta} h_m(r,s;f)ds + 2\pi i \sum_{\substack{n<\beta \\ n\in 2\mathbb{N}-1}} \mathrm{Res}_{s=\pm i2\pi m/l-n}\, h_m(r,s;f).$$

Proof. In order to prove the lemma, we need only show that

$$\lim_{R\to\infty} \int_{\substack{|\,\mathrm{Im}\, s|=R \\ 1/2>\mathrm{Re}\, s>-\beta}} h_m(r,s;f)ds = 0.$$

Since

$$(s(1-s))^{l'} h_m(r,s;f) = h_m\left(r,s;\left(D_r^2 + \frac{(2\pi m/l)^2+1/4}{\cosh^2 r}\right)^{l'} f\right) \tag{3.8}$$

and the right-hand side has the same properties as $h_m(r,s;f)$, we need only show that $\chi(r)E_{\nu_m}(r,s)E_{\nu_m}(r',1-s)\chi(r')$ is polynomially bounded by $\langle s\rangle$ as $|\,\mathrm{Im}\, s| \to \infty$ with $\mathrm{Re}\, s$ bounded. Using [13, 7.2.8], we see that

$$|\chi(r)_2F_1((s+i2\pi m/l+1)/2,(i2\pi m/l-s+2)/2,3/2;-\sinh^2 r)| \le C$$

when s is in such a region and $|s|$ is sufficiently large. It only remains to bound $\tilde{a}(\nu_m, is-i/2)\tilde{a}(\nu_m, i/2-is)$ which is easily done using Stirling's formula, to get that

$$|\tilde{a}(\nu_m, is-i/2)\tilde{a}(\nu_m, i/2-is)| \le C\langle s\rangle$$

when $\mathrm{Re}\, s$ is in a compact set and $\mathrm{Re}\, s \notin -(2\mathbb{N}-1)$, $\mathrm{Re}\, s \notin 2\mathbb{N}$. This completes the proof. $\square$

Notice that the previous proof is made simple by the fact that we do not need uniform bounds in m. This is not the case in the next lemma, which is why its proof is more involved than the earlier ones.

Lemma 3.3. *Let* $\chi \in C_c^\infty(Y_{0l})$, $f \in \dot{C}_c^\infty(Y_{0l})$, *and let* $\beta > 0$, $\beta \notin (\mathbb{N}\cup(\mathbb{N}-1/2))$. *Then*

$$\sum_{m\in\mathbb{Z}} \phi_m(\theta)\chi(r)\int_{\mathrm{Re}\, s=-\beta}\int_{\mathbb{R}_+}\int_{\mathbb{R}/l\mathbb{Z}} \frac{e^{t(s-1/2)}}{s-1/2} E_{\nu_m}(r,s)E_{\nu_m}(r',1-s)f(r',\theta')\overline{\phi}_m(\theta')\left(\frac{\cosh r'}{\cosh r}\right)^{1/2} dy'dr'ds = \mathcal{O}(e^{-t(\beta+1/2)})$$

as $t\to\infty$.

Proof. Here we need uniform polynomial bounds on $E_{\nu_m}(r,s)$ and $E_{\nu_m}(r,1-s)$ when r is in a compact set and as $|m|\to\infty$, $|s|\to\infty$ with $\mathrm{Re}\, s = \beta$.

Recall that

$$E_{\nu_m}(r,s) = \tilde{a}(\nu_m, is-i/2)\sinh r(\cosh r)^{1/2+2\pi im/l} \times {}_2F_1((s+2\pi im/l+1)/2,(2\pi im/l-s+2)/2,3/2;-\sinh^2 r).$$

It is relatively easy to bound the hypergeometric function either in s (as was done in the previous lemma) or in m, but we are unaware of a bound in both independently. Instead we will take the approach of writing

$${}_2F_1((s+2\pi im/l+1)/2,(2\pi im/l-s+2)/2,3/2;-\sinh^2 r) \tag{3.9}$$

as a sum of derivatives of ${}_2F_1((s'+2\pi im/l+1)/2,(2\pi im/l-s'+2)/2,3/2;-\sinh^2 r)$ with $1/2<\operatorname{Re}s'<5/2$, where we can obtain bounds on it using properties of the resolvent.

We will use

$$(c-a){}_2F_1(a-1,b,c;z)+(2a-c-az+bz){}_2F_1(a,b,c;z)+a(z-1){}_2F_1(a+1,b,c;z)=0 \tag{3.10}$$

([13, 3.4.19]). Let γ be the greatest integer strictly less than $(5/2+\beta)/2$ and let $s=-\beta+i\tau$, $\tau\in\mathbb{R}$. We can write ${}_2F_1((i\tau-\beta+2\pi im/l+1)/2,(2\pi im/l-i\tau+\beta+2)/2,3/2;-\sinh^2 r)$ as a sum of

$${}_2F_1((i\tau-\beta+2\gamma+\frac{2\pi im}{l}+1)/2+j_1,(\frac{2\pi im}{l}-i\tau+\beta-2\gamma+2)/2+j_2,\frac{3}{2};-\sinh^2 r) \tag{3.11}$$

where $j_1,j_2\in\{0,1\}$ and the coefficients are rational functions in $2\pi im/l$, $\tau=\operatorname{Im}s$, and $\sinh^2 r$, the degree of which does not exceed 2γ, and whose denominators are bounded away from 0 when $\beta\notin 2\mathbb{N}-2$. These coefficients can be bounded by $C\langle m\rangle^{2\gamma}\langle s\rangle^{2\gamma}$, where the constant depends on β. There are four functions (3.11) to bound now. We use the notation

$$(b)_j=b(b+1)\cdots(b+j-1).$$

If at least one of $j_1,j_2\neq 0$, then we use

$$\frac{d^j}{dz^j}[z^{a+j-1}{}_2F_1(a,b,c;z)]=(a)_j z^{a-1}{}_2F_1(a+j,b,c;z) \tag{3.12}$$

for positive integers j ([13, 3.4.4]) to write (3.11) as a sum of

$${}_2F_1\left((i\tau-\beta+2\gamma+\frac{2\pi im}{l}+1)/2,(\frac{2\pi im}{l}-i\tau+\beta-2\gamma+2)/2,3/2;-\sinh^2 r\right) \tag{3.13}$$

and its derivative in r (and second derivative, if $j_1=j_2=1$) with coefficients which are rational functions of order no greater than two in $\sinh r$, $\cosh r$, m, and τ, with the denominator bounded away from 0. Note that $1/2<2\gamma-\beta<5/2$ when $\beta\notin 2\mathbb{N}-1/2$, $\beta>0$. The derivatives are not a difficulty, since once we have a polynomial bound on the function (3.13) we get a similar bound on the derivative using the fact that (3.13) solves the equation

$$(D_r^2+\frac{(2\pi m/l)^2+1/4}{\cosh^2 r}+1/4-s'(1-s'))h=0.$$

Let $s'=i\tau-\beta+2\gamma$, and note that (3.13) is equal to

$$(a(\nu_m,is'-i/2))^{-1}(\sinh r)^{-1}(\cosh r)^{-1/2-2\pi im/l}E_{\nu_m}(m,s')$$

with $1/2<\operatorname{Re}s'<5/2$.

In order to bound $E(\nu_m,s')$ when $1/2<\operatorname{Re}s'<5/2$, we use, in analogy with (2.3),

$$\begin{aligned} E(\nu_m,s') &= 2\sinh((s'-1/2)r) \\ &\quad -\left(D_r^2+\frac{(2\pi m/l)^2+1/4}{\cosh^2 r}+\frac{1}{4}-s'(1-s')\right)^{-1}\frac{(2\pi m/l)^2+1/4}{\cosh^2 r}2\sinh((s'-1/2)r)\end{aligned} \tag{3.14}$$

when $1/2<\operatorname{Re}s'<5/2$ and we recall that we have chosen the convention that $(D_r^2+\frac{(2\pi m/l)^2+1/4}{\cosh^2 r}+\frac{1}{4}-s'(1-s'))^{-1}$ is bounded on $L^2_0(\mathbb{R}_+)$ when $\operatorname{Re}s'>1/2$. We have

$$\left\|\left(D_r^2+\frac{(2\pi m/l)^2+1/4}{\cosh^2 r}+\frac{1}{4}-s'(1-s')\right)^{-1}\right\|\leq\frac{C}{|\operatorname{Im}s'||\operatorname{Re}s'-1/2|} \tag{3.15}$$

when $\operatorname{Re}s'>1/2$. Since

$$\left\|\frac{(2\pi m/l)^2+1/4}{\cosh^2 r}(e^{(s'-1/2)r}-e^{-(s'-1/2)r})\right\|_{L^2(\mathbb{R}_+)}\leq C\frac{\langle m\rangle^2}{|\operatorname{Re}s'-5/2|^{1/2}} \tag{3.16}$$

when $1/2 \leq \operatorname{Re} s' < 5/2$, we find that for $1/2+\epsilon < \operatorname{Re} s' < 5/2-\epsilon$, $|E_{\nu_m}(r,s')| \leq C\langle m\rangle^2 \langle s'\rangle^{1+\epsilon} e^{(s'-1/2)r}$ where the constant depends on ϵ and we have used (3.14), (3.15), (3.16) as well as the Sobolev embedding theorem.

It remains to bound $(\tilde{a}(\nu_m, i(s+2\gamma)-i/2))^{-1}$ and $\tilde{a}(\nu_m, is-i/2)$ when $\operatorname{Re} s = -\beta$ – actually, we need only bound their product. We have

$$\begin{aligned}
&\left|\frac{\tilde{a}(\nu_m, is-i/2)}{\tilde{a}(\nu_m, i(s+2\gamma)-i/2)}\right| \\
&= \left|2^{-2\gamma}\frac{\Gamma(s+2\gamma-1/2)}{\Gamma(s-1/2)}\frac{\Gamma((2\pi i m/l+s+1)/2)}{\Gamma(2\pi i m/l+s+2\gamma+1)/2)}\frac{\Gamma(-2\pi i m/l+s+1)/2)}{\Gamma((-2\pi i m/l+s+2\gamma+1)/2)}\right| \\
&= \left|\frac{2^{-2\gamma}(s-1/2)_{2\gamma}}{((2\pi i m/l+s+1)/2)_\gamma(-2\pi i m/l+s+1)/2)_\gamma}\right| \\
&\leq C\langle s\rangle^{2\gamma}
\end{aligned}$$

when $\operatorname{Re} s \notin -(2\mathbb{N}-1)$. Finally, this shows that

$$|\chi(r)E_{\nu_m}(r,s)| \leq C\langle m\rangle^{2\gamma+4}\langle s\rangle^{4\gamma+7}$$

when $\operatorname{Re} s = -\beta$, $\beta > 0$, $\beta \notin \mathbb{N}\cup\mathbb{N}-1/2$.

A polynomial bound for $E_{\nu_m}(r, 1-s)$ is found in a similar way. These polynomial bounds are enough then to show that the integral in the statement of the lemma is of order $\mathcal{O}(e^{-t(\beta+1/2)})$. □

We point out that the same method applies to the case of the Neumann Laplacian on Y_{0l}, using the generalized eigenfuctions as in [5]. The Laplacian on the full cylinder Y_l can be built from the Dirichlet and Neumann Laplacians on the half-cylinder (by decomposing $L^2(Y_l)$ into subspaces of odd and even functions) and hence we obtain the second part of Theorem 2.

As a final remark we point out that a more complex example with the same dynamical structure is given by two convex obstacles – see Fig.2(a). The resonances are shown to be asymptotic to a lattice of points, [9], [4]. The estimates on the resolvent given in [9] seem sufficient for obtaining an analogue of Theorem 2 above ([10]).

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