PSEUDOSPECTRA IN AUTOMORPHIC SCATTERING

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1. Introduction and statement of results

Following the work of Faddeev and Pavlov on automorphic scattering, Lax and Phillips [4] constructed a natural operator, B, which, in the case of modular surface, has the cuspidal eigenvalues and the zeros of the Riemann zeta function as its spectrum. That operator arises as the generator of the Lax-Phillips semi-group [3] – see Section 2 for a review in the automorphic setting, and [7] for a presentation of a generalized theory.

The unbounded operator B is not normal and hence it is natural to consider its *pseudospectral* behaviour, that is, the behaviour of the norm of its resolvent. For normal operators that norm decays away from the spectrum but its behaviour for non-normal operators has structure independent of the spectrum. The importance of this well known fact has been emphasized recently by Trefethen [8] and in the context of semi-classics has been discussed by the second author in [10].

In this note we show that in a strip containing the non-trivial zeros of the zeta function, the behaviour of the resolvent of B is close to the behaviour of a resolvent of a normal operator. More precisely, for $\operatorname{Re} s < 1/2$, the spectrum of $B + 1/2$ is given by the nontrivial zeros of $\zeta(2s)$, and the norm of the resolvent of $B + 1/2$ is comparable to $|\zeta(2s)|^{-1}$:

Theorem 1. *Let B be the generator of the Lax-Phillips semigroup for the modular surface, $\mathbb{H}^2/PSL(2;\mathbb{Z})$, with a in the definition of B chosen sufficiently large. Then in the critical strip, $0 < \epsilon \leq \operatorname{Re} s \leq \frac{1}{2} - \epsilon$, we have*

$$\langle (B + 1/2 - s)^{-1} \rangle \sim_\epsilon \langle s \rangle^{1-2\operatorname{Re} s} \frac{1}{|\zeta(2s)|} . \tag{1.1}$$

Here $\langle \bullet \rangle = (1 + \| \bullet \|)^{\frac{1}{2}}$, and $f \sim_\epsilon g$ if $f/C_\epsilon \leq g \leq C_\epsilon f$, for some $C_\epsilon > 0$ (depending on ϵ).

A more precise statement can be given using the function ξ – see Section 4. In practical terms, (1.1) means that the zeros of $\zeta(s)$ could probably be successfully computed using the non-self-adjoint operator B.

To formulate the general result we have to recall the class of surfaces: we say that a smooth two-dimensional Riemannian manifold X is a manifold with cusp

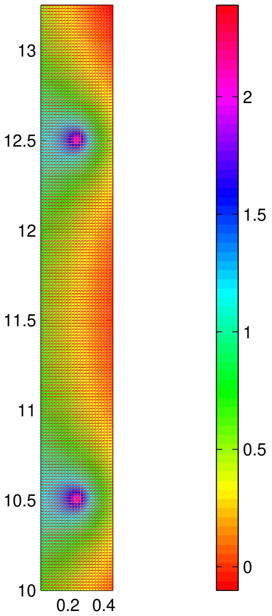

FIGURE 1. Approximate level sets of the norm of the resolvent $(B + 1/2 - s)^{-1}$. The shadings corresponding to a logarithmic scale are shown on the right.

ends if X can be written as

$$X = X_0 \cup_1^m X_i \tag{1.2}$$

where X_0 is a compact manifold (with boundary) and each X_i is isomorphic to

$$[a_i, \infty)_y \times \mathbb{S}^1_\theta, \tag{1.3}$$

with $a_i > 0$ and with metric $y^{-2}((dy)^2 + (d\theta)^2)$. As mentioned above, for such a manifold we can define the Lax-Phillips semigroup as in [4, 6], and we call its generator B. The scattering matrix $\Phi(s)$ can be defined via the 0th Fourier coefficient of the Eisenstein series. In Section 2 we briefly recall the definition of these objects, noting in particular the dependence of B on the choice of a. Theorem 1 is an immediate consequence of

Theorem 2. *For $\epsilon > 0$,*

$$\operatorname{Re} s \leq \frac{1}{2} - \epsilon \quad \textit{and} \quad \left| s - \frac{1}{2} \right| > 1 \,,$$

and by choosing a sufficiently large in the definition of K, there exists $C_\epsilon > 0$ such that

$$\frac{1}{C_\epsilon}\|\Phi(s)\|\frac{a^{1-2\operatorname{Re}s}}{1-2\operatorname{Re}s} \leq \langle (B+1/2-s)^{-1}\rangle \leq C_\epsilon \frac{\langle a+3\rangle^{1-2\operatorname{Re}s}}{1-2\operatorname{Re}s}\langle\Phi(s)\rangle\,, \tag{1.4}$$

where $\langle \bullet \rangle = (1+\|\bullet\|)^{\frac{1}{2}}$.

The line $\operatorname{Re}s = 1/2$ corresponds the continuous spectrum of the Laplacian Δ. We note that the eigenvalues of $B+1/2$ include ν_j such that $\nu_j(1-\nu_j)$ is an eigenvalue of Δ. Thus the resolvent of $B+1/2$ blows up at points corresponding to the embedded eigenvalues of Δ (if there are any), while $\Phi(s)$ is regular on $\operatorname{Re}s = 1/2$. We also remark that the proof of Theorem 2 shows that the upper bound in (1.4) holds regardless of the size of a, though the constant depends on a.

We conclude this section with some remarks concerning (1.4). The operator B can be defined for any compactly supported *black box* perturbation of the Laplacian on odd dimensional Euclidean space, $\mathbb{R}^n$, [7]. The case of cusps corresponds to the dimension one, where the black box includes all the non-zero Fourier modes and the compact part, X_0. We also have the scattering matrix $S(\lambda) : L^2(\mathbb{S}^{n-1}) \to L^2(\mathbb{S}^{n-1})$, where now we use the usual scattering theoretical convention in which $S(\lambda)$ is unitary for λ real. Generically, or under some mild assumptions, we do not have any embedded eigenvalues (corresponding to cusp forms in the arithmetic examples), and hence it is natural to ask if in general

$$\langle (B-\lambda)^{-1}\rangle \sim \langle S(\lambda)\rangle\,, \quad \text{for } |\operatorname{Im}\lambda| \leq C\,, \tag{1.5}$$

where, as mentioned above, we now used the spectral convention of [7].

We remark that the black box formalism allows us to see that X_0 may be allowed to have mild singularities, for example, an elliptic point.

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2. Preliminaries

In this section we recall the definition of the resonance set and of B, the generator of the Lax-Phillips semigroup. In particular, we are interested in the relationship between the spectrum of B and the resonance set.

For $j = 1, ..., m$, the Eisenstein series $E_j(z,s)$ are defined by

$$(\Delta - s(1-s))E_j(z,s) = 0$$

and, on the ith cusp X_i, the 0th Fourier coefficient of E_j is

$$\delta_{ij}y^s + \Phi_{ji}(s)y^{1-s}. \tag{2.1}$$

The scattering matrix is

$$\Phi(s) = (\Phi_{ij}(s))_{1\leq i,j\leq m}.$$

Let

$$\phi(s) = \det(\Phi(s)).$$

To define the resonance set of X, we first define the multiplicity $m(\eta)$ for $\eta \in \mathbb{C}$. We follow definition 5.16 of [5] (see also [6]).

- If $\mathrm{Re}(\eta) \geq 1/2$, $\eta \neq 1/2$, define $m(\eta)$ to be the dimension of the eigenspace of Δ for the eigenvalue $\eta(1-\eta)$.
- If $\mathrm{Re}(\eta) < 1/2$, then $m(\eta)$ is the dimension of the eigenspace for the eigenvalue $\eta(1-\eta)$ plus the order of the pole, or minus the order of the zero, of $\phi(s)$ at η.
- $m(1/2)$ is the dimension of the eigenspace with eigenvalue $1/4$ plus

$$(\mathrm{Tr}(\Phi(1/2)) + m)/2\,.$$

Then the resonance set $\sigma(X)$ is the set $\{\eta \in \mathbb{C}$ such that $m(\eta) > 0\}$ (Definition 5.16 of [5]; compare the singular set of [6]). For general background and references on resonances we refer the reader to [9].

We recall the definition of the Lax-Phillips semigroup in this setting. For further details, see [4] or [6].

The energy form for the wave equation

$$\left(-\frac{\partial^2}{\partial t^2} - (\Delta - 1/4)\right) u(t) = 0,$$

$u(0) = f_1$, $u_t(0) = f_2$, is

$$E(f) = \|df_1\|^2 - \frac{1}{4}\|f_1\|^2 + \|f_2\|^2, \tag{2.2}$$

where $f = \{f_1, f_2\}$. This is not positive definite. However, by some further manipulations (see [6] for details) one can see that

$$G(f) = E(f) + c\int_M |f_1|^2 d\,\mathrm{vol}_X$$

is positive definite for some compact set $M \subset X$ and for sufficiently large c. We fix such M and c. We denote by H_G the completion of $C_0^\infty(X) \times C_0^\infty(X)$ with respect to the G norm.

Let

$$A = \begin{pmatrix} 0 & I \\ -\Delta + 1/4 & 0 \end{pmatrix}.$$

Then $U(t) = e^{At}$ is unitary with respect to the energy form E but grows exponentially in the G norm.

Choose an $a>0$ so that $a>a_i$ for $i=1,...,m$ (a_i from (1.3)) and $M\cap\{(y,\theta)\in X_i:\ y>a\}=\emptyset$. Following [6], for $f=\{f_1,f_2\}\in H_G$ we define the projection π onto the interaction space $K\subset H_G$ by

$$\begin{aligned}\pi f =& f \text{ except for the zero Fourier coefficients in each cusp}\\ (\pi f)_j^{(0)}(y) =& \{(f_1)_j^{(0)}(a)(y/a)^{1/2},0\} \text{ for } \ y>a \text{ in the cusp } X_j.\end{aligned}$$

Here $g_j^{(0)}(y)=\frac{1}{2\pi}\int_{(y,\theta)\in X_j} g(y,\theta)d\theta$ is the 0th Fourier coefficient of g in the cusp X_j. Then set

$$Z(t)=\pi U(t)\pi \text{ for } \ t\geq 0.$$

The operators $Z(t)$ form a strongly continuous semigroup of operators on K [4]; we call its generator B. Note that B depends (mildly) on our choice of a, since K does. We have ([6, Corollary 3.4, Corollary 4.2], [5, Theorem 5.22])

$$\sigma(X)=\operatorname{Spec}(B+1/2),$$

and, for $\eta\in\sigma(M)$, $m(\eta)$ is equal to the dimension of the generalized eigenspace of B with eigenvalue $\eta-1/2$.

We shall use C, C_ϵ, et cetera, to denote positive constants whose value may change from line to line and which may depend on the subscripts.

3. Estimate on the resolvent

We will obtain estimates on the resolvent of B using estimates on the resolvent of the cut-off Laplacian. For these, in turn, we need estimates on the Eisenstein series, and the following definition of a cut-off operator.

Let $R(s)=(\Delta-s(1-s))^{-1}:L^2(X)\to L^2(X)$ for $\operatorname{Re}s>1/2$, except for finitely many values of s corresponding to eigenvalues of Δ. Let $b>a_i$, $i=1,...,m$. Then for $f\in H_G$, define

$$\begin{aligned}\chi_{b,0}f =& f \text{ except for the zero Fourier coefficients in each cusp}\\ (\chi_{b,0}f)_j^{(0)}(y) =& \{0,0\} \text{ for } \ y>b \text{ in the cusp } X_j.\end{aligned}$$

As is well known, $\chi_{b,0}R(s)\chi_{b,0}$ has a meromorphic continuation to $\mathbb{C}$ – see for instance [1].

We shall need some bounds on the Eisenstein series.

Lemma 3.1. *For $1/2+\epsilon\leq\operatorname{Re}s$ and $|s-1/2|>1$,*

$$\|\chi_{b,0}E_j(\cdot,s)\|\leq C_\epsilon\frac{\langle b\rangle^{\operatorname{Re}s-1/2}}{\sqrt{2\operatorname{Re}s-1}}$$

for $j=1,...,m$.

Proof. Let $\Psi_j \in C^\infty(X)$ be a function supported on the end X_j which is 1 for $y > a_j + 1$. Then

$$E_j(z,s) = \Psi_j y^s - R(s)(\Delta - s(1-s))\Psi_j y^s. \tag{3.1}$$

Since

$$\|(\Delta - s(1-s))\Psi_j y^s\| \le C\langle s\rangle \frac{\langle a_j\rangle^{\operatorname{Re} s - 1/2}}{\sqrt{2\operatorname{Re} s - 1}}$$

and

$$\|R(s)\| \le \frac{1}{\operatorname{dist}(s(1-s), \operatorname{Spec}(\Delta))} \le C_\epsilon \langle s\rangle^{-1}$$

for such values of s, we have

$$\|R(s)(\Delta - s(1-s))\Psi_j y^s\| \le C_\epsilon \frac{\langle a_j\rangle^{\operatorname{Re} s}}{\sqrt{2\operatorname{Re} s - 1}}. \tag{3.2}$$

Since

$$\|\chi_{b,0}\Psi_j y^s\| \le \frac{C}{\sqrt{2\operatorname{Re} s - 1}}\langle b\rangle^{\operatorname{Re} s - 1/2},$$

we obtain the upper bound. □

For the lower bound on the resolvent, we will need finer information. We stress that in the case of a single cusp, in particular for the modular surface, the situation is simpler.

Lemma 3.2. *Let $a_M = \max_{1\le j\le m} a_i$. For $1/2 + \epsilon \le \operatorname{Re} s$ and $|s - 1/2| > 1$, for any $\delta > 0$ we can choose $b > 2a_M$ sufficiently large so that*

$$\left| \frac{b^{1-2\operatorname{Re} s}(2\operatorname{Re} s - 1)}{2\pi(1-(1/2)^{2\operatorname{Re} s - 1})} \int_{z\in X_k,\, b/2<y<b} E_j(y,\theta,s) y^{\overline{s}} y^{-2} dy d\theta - \delta_{jk} \right| \le \delta$$

for $j, k = 1, \dots, m$. We may also choose b sufficiently large so that, for $j = 1, \dots, m$,

$$\int_{z\in X_j,\, a_M\le y\le b} |E_j(y,\theta,s) - \sum_{k\ne j} d_k E_k(y,\theta,s)|^2 y^{-2} dy d\theta > \frac{1}{10(2\operatorname{Re} s - 1)} b^{2\operatorname{Re} s - 1}$$

for any $d_i \in \mathbb{C}$ with $|d_i| < 2$.

Proof. The inequality (3.2) together with (3.1) and (2.1), the definition of $\Phi_{ji}(s)$, implies that

$$|\Phi_{ji}(s)|^2 \le \frac{1}{2\pi} C_\epsilon^2 a_i^{2\operatorname{Re} s - 1} \langle a_j\rangle^{2\operatorname{Re} s} \le C_\epsilon C^{2\operatorname{Re} s} \tag{3.3}$$

for $i, j = 1, ..., m$. A straightforward computation shows, for $b > 2a_M$, $\operatorname{Im} s \neq 0$,

$$\begin{aligned}
&\frac{b^{1-2\operatorname{Re} s}(2\operatorname{Re} s - 1)}{2\pi(1-(1/2)^{2\operatorname{Re} s-1})}\int_{z\in X_k,\, b/2<y<b} E_j(y,\theta,s)y^{\overline{s}}y^{-2}dyd\theta \\
&= \frac{b^{1-2\operatorname{Re} s}(2\operatorname{Re} s - 1)}{2\pi(1-(1/2)^{2\operatorname{Re} s-1})}\int_{z\in X_k,\, b/2<y<b} (\delta_{kj}y^{2\operatorname{Re} s} + \Phi_{jk}(s)y^{1-2i\operatorname{Im} s})y^{-2}dyd\theta \\
&= \delta_{jk} + \frac{b^{1-2\operatorname{Re} s}(2\operatorname{Re} s - 1)}{(1-(1/2)^{2\operatorname{Re} s-1})}\frac{\Phi_{jk}(s)}{-2i\operatorname{Im} s}(b^{-2i\operatorname{Im} s} - (b/2)^{-2i\operatorname{Im} s}).
\end{aligned}$$

Using (3.3), and the fact that $\operatorname{Re} s \geq 1/2 + \epsilon$, we can choose b sufficiently large that

$$\left|\frac{b^{1-2\operatorname{Re} s}(2\operatorname{Re} s - 1)}{(1-(1/2)^{2\operatorname{Re} s-1})}\frac{\Phi_{jk}(s)}{-2i\operatorname{Im} s}(b^{-2i\operatorname{Im} s} - (b/2)^{-2i\operatorname{Im} s})\right| < \delta.$$

A similar argument handles the case $\operatorname{Im} s = 0$, proving the first part of the lemma.

On the end X_j,

$$E_j(s) - \sum_{k\neq j} d_k E_k(s) = y^s + \alpha y^{1-s} + \beta,$$

where β is orthogonal to the zero mode. Therefore, when $\operatorname{Im} s \neq 0$,

$$\begin{aligned}
&\int_{z\in X_j,\, a_M\leq y\leq b} |E_j(y,\theta,s) - \sum_{k\neq j} d_k E_k(y,\theta,s)|^2 y^{-2}dyd\theta \\
&\qquad \geq \frac{2\pi}{2\operatorname{Re} s - 1}\left(b^{2\operatorname{Re} s-1} - a_M^{2\operatorname{Re} s-1} + |\alpha|^2(a_M^{-2\operatorname{Re} s+1} - b^{-2\operatorname{Res} +1})\right) \\
&\qquad\qquad - 4\pi\operatorname{Re}\left(\frac{\alpha}{2i\operatorname{Im} s}(b^{-2i\operatorname{Im} s} - a_M^{-2i\operatorname{Im} s})\right).
\end{aligned}$$

If $|d_k| \leq 2$, then using (3.3), $|\alpha| \leq C_\epsilon C^{\operatorname{Re} s}$. Therefore, we may choose b sufficiently large that

$$\begin{aligned}
&\frac{1}{2\operatorname{Re} s - 1}(b^{2\operatorname{Re} s-1} - a_M^{2\operatorname{Re} s-1}) \geq \\
&\qquad 2|\alpha|\left|\frac{1}{2i\operatorname{Im} s}(b^{-2i\operatorname{Im} s} - a_M^{-2i\operatorname{Im} s})\right| + \frac{1}{10(2\operatorname{Re} s - 1)}b^{2\operatorname{Re} s-1}
\end{aligned}$$

for $1/2 + \epsilon \leq \operatorname{Re} s$.

A similar analysis works if $\operatorname{Im} s = 0$. □

The bounds on the cut-off resolvent $\chi_{b,0}R(s)\chi_{b,0}$ will be instrumental in bounding the resolvent of B.

Lemma 3.3. *For* $\operatorname{Re} s \leq 1/2 - \epsilon$ *and* $|s - 1/2| > 1$, *for some* $C_\epsilon > 0$

$$\|\chi_{b,0}R(s)\chi_{b,0}\| \leq C_\epsilon \frac{\langle b\rangle^{1-2\operatorname{Re} s}}{\langle s\rangle(1-2\operatorname{Re} s)}(1 + \|\Phi(s)\|).$$

Moreover, for sufficiently large b and for the same s values

$$(1 + \|\chi_{b,0} R(s) \chi_{b,0}\|) \geq \frac{1}{C_\epsilon} \frac{\langle b \rangle^{1-2\operatorname{Re} s}}{\langle s \rangle (1 - 2\operatorname{Re} s)} \|\Phi(s)\|.$$

Proof. We use

$$R(s) - R(1-s) = \frac{1}{1-2s} \sum_{j=1}^{m} E_j(s) \otimes E_j(1-s) \tag{3.4}$$

where

$$(E_j(s) \otimes E_j(1-s) f)(z) = E_j(z,s) \int_X E_j(z', 1-s) f(z') d\operatorname{vol}_X .$$

Our restrictions on s mean that $1/2 + \epsilon \leq \operatorname{Re}(1-s)$, and we have

$$\|R(1-s)\| \leq C_\epsilon \langle s \rangle^{-1}. \tag{3.5}$$

We may write

$$\sum_{j=1}^{m} E_j(s) \otimes E_j(1-s) = \sum_{i,j=1}^{m} \Phi_{ji}(s) E_i(1-s) \otimes E_j(1-s). \tag{3.6}$$

Using Lemma 3.1,

$$\|\chi_{b,0} E_i(1-s) \otimes \chi_{b,0} E_j(1-s)\| \leq C_\epsilon \frac{\langle b \rangle^{1-2\operatorname{Re} s}}{1 - 2\operatorname{Re} s}. \tag{3.7}$$

Then using (3.4), (3.5), (3.6), and (3.7), we obtain the upper bound on the cut-off resolvent.

To obtain the lower bound, let $v = (v_1, ..., v_m) \in \mathbb{C}^m$ be a vector with $\|v\| = 1$ and $\|\Phi(s)v\| = \|\Phi(s)\|$. (Of course, v depends on s.) For $\delta = \frac{1}{5m}$, let b be the number whose existence is guaranteed by Lemma 3.2 for $1-s$ in place of s. Now let

$$f = \frac{b^{2\operatorname{Re} s - 1}(1 - 2\operatorname{Re} s)}{2\pi(1 - (1/2)^{1-2\operatorname{Re} s})} \sum_j \chi_{[b/2,b],j} y^{1-\overline{s}} v_j$$

where $\chi_{[b/2,b],j}$ is the function supported on the jth cusp which is one for $b/2 \leq y \leq b$ and 0 otherwise. Note that

$$\|f\|^2 \leq C b^{2\operatorname{Re} s - 1}(1 - 2\operatorname{Re} s), \tag{3.8}$$

and, using Lemma 3.2,

$$\left| \int_X E_j(z', 1-s) f(z') d\operatorname{vol}_X - v_j \right| \leq \delta \sum |v_k| = \frac{1}{5m} \sum |v_k| \leq \frac{1}{5\sqrt{m}}.$$

We wish to obtain a lower bound on

$$\| \sum_{i,j} \Phi_{ji}(s) \chi_{b,0} E_i(1-s) \otimes \chi_{b,0} E_j(1-s) f \|$$

in terms of $\|\Phi(s)\|$. Let $\tilde{v}_j = \int_X E_j(z', 1-s) f(z') d\operatorname{vol}_X$ and let $u_i = \sum \Phi_{ji}\tilde{v}_j$. Notice that

$$(\sum |u_i|^2)^{1/2} = \|\Phi(v + (\tilde{v} - v))\| \geq \|\Phi v\| - \|\Phi(v - \tilde{v})\| \geq \|\Phi\| - \frac{1}{5}\|\Phi\| = \frac{4}{5}\|\Phi\|.$$

Now let $|u_{i_0}| = \max_i |u_i|$, and note that $|u_{i_0}| \geq \frac{4}{5\sqrt{m}}\|\Phi\|$. Now consider

$$\begin{aligned}
&\|\sum_{ij} \Phi_{ji}(s)\chi_{b,0}E_i(1-s) \otimes \chi_{b,0}E_j(1-s)(f)\|^2 \\
&= \|\sum_i u_i \chi_{b,0} E_i(1-s)\|^2 \\
&\geq \int_{z\in X_{i_0}, a_M \leq y \leq b} \left|\sum_i u_i E_i(1-s)\right|^2 y^{-2} dy d\theta \\
&= |u_{i_0}|^2 \int_{z\in X_{i_0}, a_M \leq y \leq b} \left| E_{i_0}(1-s) + \sum_{i\neq i_0} \frac{u_i}{|u_{i_0}|} E_i(1-s)\right|^2 y^{-2} dy d\theta.
\end{aligned}$$

Now, using the second part of Lemma 3.2, we have that

$$\begin{aligned}
&\|\sum_{ij} \Phi_{ji}(s)\chi_{b,0}E_i(1-s) \otimes \chi_{b,0}E_j(1-s)(f)\|^2 \\
&\qquad\qquad \geq |u_{i_0}|^2 \frac{b^{1-2\operatorname{Re} s}}{10(1-2\operatorname{Re} s)} \geq \frac{b^{1-2\operatorname{Re} s}}{C(1-2\operatorname{Re} s)}\|\Phi\|^2.
\end{aligned}$$

Using (3.8), the bound on $\|f\|$, we obtain the desired lower bound on $\|\chi_{b,0}R(s)\chi_{b,0}\|$. □

We can now give

Proof of Theorem 2: From [7, (2.6)] or [4], we have

$$\pi(A-\lambda)^{-1}\pi = (B-\lambda)^{-1},$$

and

$$(A-\lambda)^{-1} = -\begin{pmatrix} \lambda & I \\ -\Delta + \frac{1}{4} & \lambda \end{pmatrix} (\Delta - \frac{1}{4} + \lambda^2)^{-1}$$

where $(\Delta - \frac{1}{4} + \lambda^2)^{-1}$ is bounded on $L^2(X)$ for all but a finite number of λ with $\operatorname{Re}\lambda > 0$ (compare [7, (2.5)]). This gives us

$$\left(B + \frac{1}{2} - s\right)^{-1} = \pi \begin{pmatrix} -s + \frac{1}{2} & -I \\ \Delta - \frac{1}{4} & -s + \frac{1}{2} \end{pmatrix} R(s)\pi. \tag{3.9}$$

Let $\{f_1, f_2\} \in H_G$. We note that

$$\pi f = \chi_{a,0} f + \sum_j \chi_{[a,\infty),j}\{(f_1)_j^{(0)}(a)(y/a)^{1/2}, 0\}, \tag{3.10}$$

where $\chi_{[a,\infty),j} = \chi_{[a,\infty),j}(y)$ is the characteristic function for $[a,\infty)$ on the jth cusp X_j and is 0 elsewhere. Therefore,

$$
\begin{aligned}
R(s)\pi f =& R(s)\left(\chi_{a,0}f + \sum_{j=1}^{m}\{v_{j,s},0\}\right) \\
&+ \frac{1}{1/4 - s(1-s)}\sum_{j=1}^{m}\Psi_j\{(f_1)_j^{(0)}(a)(y/a)^{1/2},0\}
\end{aligned}
\tag{3.11}
$$

where Ψ_j is smooth function supported on the jth cusp X_j which is 0 for $y \leq a$ and 1 for $y > a+2$ and

$$
v_{j,s} = \chi_{[a,\infty),j}\{(f_1)_j^{(0)}(a)(y/a)^{1/2},0\} - \frac{(\Delta - s(1-s))}{1/4 - s(1-s)}\Psi_j\{(f_1)_j^{(0)}(a)(y/a)^{1/2},0\}.
\tag{3.12}
$$

Note that $v_{j,s}$ is supported on the jth cusp X_j, with $a \leq y \leq a+2$, and $\|v_{j,s}\| \leq C|(f_1)_j^{(0)}(a)|$, if $|s-1/2| > 1$. In fact, we have, for $|s-1/2|>1$,

$$
\|\Psi(\chi_{a,0}f_1 + \sum_{j=1}^{m} v_{j,s})\|_{H^1} \leq C_\Psi \|f\|_{H_G}
\tag{3.13}
$$

for any $\Psi \in C_c^\infty(X)$.

It is immediate from (3.9), (3.10), and Lemma 3.3 that

$$
\|(B+1/2-s)^{-1}\{0,f_2\}\|_{H_G} \leq C_\epsilon \frac{\langle a\rangle^{1-2\operatorname{Re} s}}{1-2\operatorname{Re} s}\langle \Phi(s)\rangle \|\{0,f_2\}\|_{H_G}.
$$

It remains to bound $\|(B+1/2-s)^{-1}\{f_1,0\}\|_{H_G}$. If $g=\{g_1,g_2\}$, we introduce the notation $\pi g = \{\pi_1 g_1, \pi_2 g_2\}$. Note that

$$
\|d\pi_1 g_1\|^2 - \frac{1}{4}\|\pi_1 g_1\|^2 = \|\chi_{a,0} d g_1\|^2 - \frac{1}{4}\|\chi_{a,0} g_1\|^2,
$$

and that $\chi_{a,0}\Psi_j = 0$. Therefore, for some $\Psi \in C_c^\infty(X)$,

$$
\begin{aligned}
&\left| \|d\pi_1 R(s)\pi_1 f_1\|^2 - \frac{1}{4}\|\pi_1 R(s)\pi_1 f_1\|^2 \right| \\
&= \left| \|\chi_{a,0} dR(s)\pi_1 f_1\|^2 - \frac{1}{4}\|\chi_{a,0}R(s)\pi_1 f\|^2 \right| \\
&\leq C\|\chi_{a,0}R(s)\chi_{a+3,0}\|^2_{L^2\to L^2}\|\Psi(\chi_{a,0}f_1 + \sum_{j=1}^{m} v_{j,s})\|^2_{H^1} \\
&\leq C_\epsilon \langle s\rangle^{-2}\langle a+3\rangle^{2-4\operatorname{Re} s}(1-2\operatorname{Re} s)^{-2}(\|\Phi(s)\|+1)^2\|\{f_1,0\}\|^2_{H_G},
\end{aligned}
$$

using (3.11), (3.13), and Lemma 3.3. We also have

$$\int_M |\pi_1 R(s)(\chi_{a,0} f_1 + \sum_{j=1}^m v_{j,s})|^2 d\,\mathrm{vol}_X$$
$$\leq C_\epsilon \langle s\rangle^{-2} \langle a+3\rangle^{2-4\,\mathrm{Re}\,s}(1-2\,\mathrm{Re}\,s)^{-2}(1+\|\Phi(s)\|)^2 \|f\|^2_{H_G},$$

so that

$$\|\{\pi_1(-s+1/2)R(s)\pi_1 f_1, 0\}\|_{H_G} \leq C_\epsilon \langle a+3\rangle^{1-2\,\mathrm{Re}\,s}(1-2\,\mathrm{Re}\,s)^{-1}\langle\Phi(s)\rangle \|f\|_{H_G}.$$

A similar argument bounds $\|\{0, \pi_2(\Delta - 1/4)R(s)\pi_1 f_1\}\|_{H_G}$, finishing the proof of the upper bound.

By looking at the operator in the lower right corner of (3.9) and using (3.10), we see that $\|(s-1/2)\chi_{a,0}R(s)\chi_{a,0}\|_{L^2\to L^2}$ provides a lower bound on $\|(B+1/2-s)^{-1}\|$. Thus, applying Lemma 3.3 we obtain the lower bound. □

4. Proof of Theorem 1

Consider the special case of $X = \mathbb{H}^2/PSL(2;\mathbb{Z})$, the quotient of the hyperbolic upper half plane by $PSL(2;\mathbb{Z})$. As is well known (see for instance [4]), this has scattering matrix

$$\Phi(s) = \sqrt{\pi}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\frac{\zeta(2s-1)}{\zeta(2s)} \tag{4.1}$$

where Γ is the Euler Γ-function and ζ is the Riemann ζ-function. The functional equation of the scattering matrix, $\Phi(1-s) = \Phi(s)^{-1}$, corresponds to the functional equation for the ζ function, best given in terms of the classical ξ function:

$$\begin{gathered}\xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)\,, \quad \xi(s) = \xi(1-s)\,,\\ \Phi(s) = \frac{2s\xi(2s-1)}{(2s-2)\xi(2s)}\,.\end{gathered} \tag{4.2}$$

The nice feature of $\xi(2s)$ is that it is an entire function of order one with zeros given by the resonances of X, that is, the non-trivial zeros of $\zeta(2s)$. Applying Theorem 2 we obtain a slightly more precise version of Theorem 1. To see the formulation in terms of the zeta function in the critical strip we recall (see (4.2) above) that

$$\zeta(2s-1) = -2^{2s-1}\pi^{2s-2}\cos(\pi s)\Gamma(2-2s)\zeta(2-2s).$$

Using Stirling's formula,

$$\left|\frac{1}{\zeta(s)}\right| \leq \frac{\zeta(\mathrm{Re}\,s)}{\zeta(2\,\mathrm{Re}\,s)} \text{ for } \quad \mathrm{Re}\,s > 1$$

([2, Theorem 2, Chapter 8]), and the fact that $\zeta(s)$ is bounded in any half-plane $\operatorname{Re} s \geq 1 + \delta > 1$, we obtain, for $\epsilon < \operatorname{Re} s < 1/2 - \epsilon$

$$\frac{1}{C_\epsilon} \frac{\langle \operatorname{Im} s \rangle^{1-2\operatorname{Re} s}}{|\zeta(2s)|} < |\Phi(s)| < C_\epsilon \frac{\langle \operatorname{Im} s \rangle^{1-2\operatorname{Re} s}}{|\zeta(2s)|}.$$

Combining this with Theorem 2, we obtain Theorem 1. Again we do not need to demand that a is large for the upper bound.

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