CORRECTION TO "BREIT WIGNER APPROXIMATION AND THE DISTRIBUTION OF RESONANCES", COMM. MATH. PHYS. 204(1999), 329-351.

VESSELIN PETKOV AND MACIEJ ZWORSKI

In Lemmas 2 and 4 in [4], the Harnack inequality is applied incorrectly at the end of the proofs. The crucial inequality

$$|\operatorname{Re} g_{\rho,\tau}(\lambda)| \leq C|\lambda|^n \tag{1}$$

still holds however, as will be explained here. The main result of the paper [4] (the high energy justification of the Breit-Wigner approximation) remains valid:

$$s(\lambda+\delta) - s(\lambda-\delta) = 2\pi \sum_{|\lambda_j - \lambda|<1} \omega_{\mathbb{C}_+}(\lambda_j, [\lambda-\delta, \lambda+\delta]) + \mathcal{O}(\delta)\lambda^{n-1}\,, \tag{2}$$

and so do other statements made in [4].

One of the consequencies of (1) was the following bound on the number of resonances

$$\#\,\{\lambda_j \ : \ |\lambda - \lambda_j| \leq R\} \leq CR\lambda^{n-1}\,, \quad R > 1\,. \tag{3}$$

We remark here that this inequality was recently generalized by Bony [1], who using a different method considered the semi-classical case and non-compactly supported perturbations.

To see that (1) holds, we first observe that if $\theta_0 < \pi/n$ then for any $\delta > 0$,

$$\log|\det S(-re^{i\theta})| > -C_\delta r^n\,, \ r > r_0\,, \ \theta \in (0, \theta_0) \setminus \Sigma(r)\,, \ |\Sigma(r)| < \delta\,, \tag{4}$$

which follows from [4, Lemma 1] and the minimum modulus estimates for functions of finite order in an angle (see [2, Theorem 56] where it is consequence of the Carleman inequality recalled in [4, (4.2)]; see also [5, Sect.7] for a direct argument leading to a similar estimate).

As in the proof of [4, Lemma 2], Cartan's lemma gives that

$$\begin{aligned}
&|\operatorname{Re} g_{\rho,\tau}(re^{i\theta})| \leq C_\delta \tau^n\,, \quad \theta \in (0,\theta_0) \setminus \Sigma(r)\,, \ |\Sigma(r)| < \delta\,, \ |re^{i\theta} - \tau| < \tau/B\,,\\
&\operatorname{Re} g_{\rho,\tau}(\lambda) \leq C\tau^n\,, \ |\tau - \lambda| < \tau/B\,, \ \operatorname{Im}\lambda \geq 0\,,
\end{aligned}$$

where we used (4) to obtain the first inequality. We can now apply the Harnack inequality *correctly* to conclude that

$$|\operatorname{Re} g_{\rho,\tau}(\lambda)| \leq C\gamma^{-1}\tau^n\,, \quad \operatorname{Im}\lambda > \gamma\tau\,, \quad |\lambda - \tau| \leq \tau/2B\,.$$

From the unitarity of the scattering matrix we also have that

$$\operatorname{Re} g_{\rho,\tau}(\lambda) = -\operatorname{Re} g_{\rho,\tau}(\bar\lambda)\,.$$

To obtain a lower bound everywhere we apply the following lemma which was kindly pointed out to us by W.K. Hayman:

Lemma. *Suppose that u is harmonic in $D(0,1)$, and*

$$|u(z)| \leq \frac{K}{|\operatorname{Im} z|}\,, \quad u(z) = -u(\bar z)\,, \quad z \in D(0,1)\,.$$

Then, for every $\epsilon > 0$, there exists $C = C(\epsilon)$ such that

$$|u(z)| \leq CK|\operatorname{Im} z|, \quad z \in D(0, 1-\epsilon).$$

Proof: We use the Poisson formula and the symmetry (and we can assume that the hypotheses hold in a slightly bigger disc):

$$\begin{aligned} u(re^{i\theta}) =& \frac{1}{2\pi}\int_0^{2\pi} \frac{(1-r^2)u(e^{i\varphi})}{1-2r\cos(\theta-\varphi)+r^2} d\varphi \\ =& \frac{1}{\pi}\int_0^{\pi} \frac{8(1-r^2)r\sin\theta\sin\varphi u(e^{i\varphi})}{(1-2r\cos(\theta-\varphi)+r^2)(1-2r\cos(\theta+\varphi)+r^2)} d\varphi . \end{aligned}$$

Since we know that

$$|u(e^{i\varphi})| \leq K/\sin\varphi, \quad 0 \leq \varphi \leq \pi,$$

we conclude that for $r < 1-\epsilon$,

$$|u(z)/\operatorname{Im} z| \leq 8\epsilon^{-4}K.$$

$\square$

To get (1) we apply the lemma with $u(z) = \tau^{-n}\operatorname{Re} g_{\rho,\tau}(\tau + 2B\tau z)$. We point out that in the factorization defining $g_{\rho,\tau}$ in [4, Lemma 2], it only matters that we take a product over resonances in $|\tau - \lambda| \leq \tau/C$, for some C. That makes $g_{\rho,\tau}$ defined in $|\tau-\lambda| \leq \tau/2C$, with (1) holding there. Cauchy estimates then imply that we gain decay in λ when derivatives are applied.

The estimate (3) follows from the argument of [4, Proposition 1]:

$$\begin{aligned} \# \{\lambda_j \ : \ |\lambda-\lambda_j| \leq R\} &\leq C\int_{\lambda-3R}^{\lambda+3R} \sum_{\substack{|\lambda_j-\lambda|\leq\lambda/2 \\ \lambda_j\in\Lambda_\rho}} \frac{\operatorname{Im}\lambda_j}{|t-\lambda_j|^2} dt \\ &= C(s(\lambda+3R) - s(\lambda-3R) - i(g_{\rho,\lambda}(\lambda+3R) - g_{\rho,\lambda}(\lambda-3R))) \\ &= \mathcal{O}(R)\lambda^{n-1}, \quad R \leq \lambda/6 \end{aligned} \tag{5}$$

by (1) and Cauchy inequalities. For $R > \lambda/C$ the estimate follows from the global bound on the number of resonances. We stress here that the estimate on $g_{\rho,\tau}$ is only needed on the real axis, and hence, in odd but not even dimensions, it follows directly from Melrose's argument in [3].

To get (2), we proceed as in [4, (4.3)], and hence need to find a bound for

$$\begin{aligned} \frac{1}{\rho}\int_0^{\pi} \log|\det S(-\lambda-\rho e^{i\theta})|\sin\theta d\theta = -\frac{1}{\rho^2}\operatorname{Re}\int_{\Gamma_{\lambda,\rho}} \log|\det S(z)|dz \\ = \frac{1}{\rho^2}\operatorname{Re}\int\!\!\int_{\Omega_{\lambda,\rho}} 2i\,\bar\partial_z \log|\det S(z)|\mathcal{L}(dz) \\ = -\frac{1}{\rho^2}\operatorname{Re}\int\!\!\int_{\Omega_{\lambda,\rho}} i\,\partial_z \log\det S(z)\mathcal{L}(dz), \end{aligned}$$

where $\Omega_{\lambda,\rho} = \{z : \operatorname{Im} z \geq 0, |z+\lambda| \leq \rho\}$, $\Gamma_{\lambda,\rho}$, its boundary, $\mathcal{L}(dz)$, the Lebesgue measure on $\mathbb{C}$, and where we used Green's formula. We now write

$$\partial_z \log\det S(z) = g'_{\rho,\tau}(z) + \sum_{\substack{|\lambda-\lambda_j|<2\rho \\ \lambda_j\in\Lambda_\rho}} \left(\frac{1}{z-\bar\lambda_j} - \frac{1}{z-\lambda_j}\right) + \sum_{\substack{2\rho\leq|\lambda-\lambda_j|<\lambda/B \\ \lambda_j\in\Lambda_\rho}} \left(\frac{1}{z-\bar\lambda_j} - \frac{1}{z-\lambda_j}\right),$$

where we can take $\tau = \lambda$. The first two terms are estimated using (1) and (3). To estimate the last term we will use the equality in the second line of (5), and to apply it, we rewrite it as

$$\sum_{\substack{2\rho \leq |\lambda - \lambda_j| < \lambda / B, \\ \lambda_j \in \Lambda_\rho}} \left(\frac{2i \operatorname{Im} \lambda_j}{|\operatorname{Re} z - \lambda_j|^2} + i \int_0^{\operatorname{Im} z} \Big(\frac{1}{(\operatorname{Re} z + iy - \overline{\lambda_j})^2} - \frac{1}{(\operatorname{Re} z + iy - \lambda_j)^2} \Big) dy \right) ,$$

$|z - \lambda| < \rho$. Using (5), the integral of the first term is bounded by $C\lambda^{n-1}$. It remains to estimate the second term, and for that we write

$$\begin{aligned} \sum_{2\rho \leq |\lambda - \lambda_j| < \lambda / B} \frac{1}{|\lambda - \lambda_j|^2} &\leq \frac{1}{\rho} \sum_{1 \leq k \leq C \log \lambda} 2^{-2k} \# \left\{ \lambda_j \ : \ |\lambda - \lambda_j| \leq 2^{k+1} \rho \right\} \\ &\leq C \sum_{1 \leq k \leq C \log \lambda} \frac{\lambda^{n-1}}{2^k} = \mathcal{O}(\lambda^{n-1}) . \end{aligned}$$

The method described here can also be used to show that [4, Lemma 4] remains valid. We chose a direct approach for the sake of brevity.

References

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Departement de Mathematiques Apliquees, Universite de Bordeaux I,, 351, Cours de la Liberation, 33405 Talence, FRANCE
E-mail address: petkov@math.u-bordeaux.fr

Mathematics Department, University of California,, Evans Hall, Berkeley, CA 94720, USA
E-mail address: zworski@math.berkeley.edu